The Maltitude Construction in a Convex Noncyclic Quadrilateral

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Abstract. This note is linked to a recent paper of O. Radko and E. Tsukerman. We consider the maltitude construction in a convex noncyclic quadrilateral and we determine a point that can be viewed as a generalization of the anticenter.

1. Introduction

In [5] it is investigated the perpendicular bisector construction in a noncyclic quadrilateral \( Q = Q^{(0)} = ABCD \). The perpendicular bisectors of the sides of \( Q \) determine a noncyclic quadrilateral \( Q^{(1)} = A_1B_1C_1D_1 \), whose vertices are the centers of the triad circles, i.e., the circles passing through three vertices of \( Q \). This process can be iterated to obtain a sequence of noncyclic quadrilaterals: \( Q^{(0)}, Q^{(1)}, Q^{(2)}, \ldots \).

All even generation quadrilaterals are similar, and all odd generation quadrilaterals are similar. Further, there is a point \( W \) that serves as the center of the spiral similarity for any pair of quadrilaterals \( Q^{(n)}, Q^{(n+2)} \). If \( Q \) is a convex noncyclic quadrilateral, the quadrilaterals \( Q^{(n)}, Q^{(n+2)} \) are homotetic, the ratio of similarity is a negative constant and the quadrilaterals in the iterated perpendicular bisectors construction converge to \( W \). In a convex noncyclic quadrilateral the limit point \( W \) can be viewed as a generalization of the circumcenter.

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2. Characteristic and affinity

In [3] it is proved that if \( Q \) is a convex quadrilateral, then \( Q^{(1)} \) is affine to \( Q \). It follows that, for any \( n \), \( Q^{(n+1)} \) is affine to \( Q^{(n)} \).

For the convenience of the reader, we give a proof of this property. In [2] it is defined the characteristic of a quadrilateral \( Q \) as follows. Let \( E \) be the common point of the diagonals \( AC \) and \( BD \) of \( Q \). For the ratios \( \frac{AE}{EC} \) and \( \frac{CE}{EA} \), let \( h \) be the one not greater than 1. Also for the ratios \( \frac{BE}{ED} \) and \( \frac{DE}{EB} \), let \( k \) be the one not greater than 1. The pair \( \{h, k\} \) is the characteristic of \( Q \). In [2] it is proved that two convex quadrilaterals are affine if and only if they have the same characteristic. We consider now the quadrilateral \( Q^{(1)} = A_1B_1C_1D_1 \). The line \( A_1C_1 \) is perpendicular to the radical axis \( BD \) of the circle passing through \( B, C, D \) and the circle passing through \( A, B, D \). Similarly, the line \( B_1D_1 \) is perpendicular to the line \( AC \). Further, the lines \( A_1B_1, B_1C_1, C_1D_1, D_1A_1 \) are perpendicular to the lines \( DC, AD, BA, CB \), respectively. It follows that, if \( E_1 \) is the common point of diagonals \( A_1C_1 \) and \( B_1D_1 \) of \( Q^{(1)} \), the triangle pairs \( ABE \) and \( C_1D_1E_1 \), \( BCE \) and \( A_1D_1E_1 \), \( CDE \) and \( A_1B_1E_1 \) are similar. Therefore we have

\[
\frac{AE}{BE} = \frac{E_1D_1}{E_1C_1}, \quad \frac{BE}{EC} = \frac{A_1E_1}{E_1D_1}, \quad \frac{EC}{ED} = \frac{B_1E_1}{A_1E_1},
\]

from which

\[
\frac{AE}{EC} = \frac{A_1E_1}{E_1C_1}, \quad \frac{BE}{ED} = \frac{B_1E_1}{E_1D_1}.
\]

Thus, \( Q \) and \( Q^{(1)} \) have the same characteristic and are affine.

3. Maltitudes

In [3] it is considered also the quadrilateral \( Q_m \) determined by the maltitudes of a convex noncyclic quadrilateral \( Q \). A maltitude of \( Q \) is the perpendicular line
The maltitude construction in a convex noncyclic quadrilateral through the midpoint of a side to the opposite side [1]. In [4] it is proved that the maltitudes are concurrent in a point, called anticenter, if and only if $Q$ is cyclic.

In [3] it is proved that the quadrilateral $Q_m = A'_1B'_1C'_1D'_1$ is the symmetric of $Q^{(1)}$ with respect to the centroid $G$ of $Q$. This property follows from the fact that the maltitudes of $Q$ are transformed into the perpendicular bisectors of $Q$ in the half-turn about $G$.

The existence of the point $W$, as the limit point in the iterated perpendicular bisectors construction, implies that the symmetric $W'$ of $W$ with respect to $G$ is the limit point in the iterated maltitudes construction. Furthermore, in a convex noncyclic quadrilateral the limit point $W'$ can be viewed as a generalization of the anticenter.

We observe that in a cyclic quadrilateral the circumcenter and the anticenter are symmetric with respect to the centroid. If $Q$ is a convex noncyclic quadrilateral, in analogy with the case of a cyclic quadrilateral, we call the line containing $G$, $W$ and $W'$ the Euler line of $Q$.

References


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