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Using Complex Weighted Centroids to Create Homothetic Polygons

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Abstract. After first defining weighted centroids that use complex arithmetic, we then make a simple observation which proves Theorem 1. We next define complex homothety. We then show how to apply this theory to triangles (or polygons) to create endless numbers of homothetic triangles (or polygon). The first part of the paper is fairly standard. However, in the final part of the paper, we give two examples which illustrate that examples can easily be given in which the simple basic underpinning is so disguised that it is not at all obvious. Also, the entire paper is greatly enhanced by the use of complex arithmetic.

1. Introduction to the basic theory

Suppose A, B, C, x, y are complex numbers that satisfy $xA+yB=C$, $x+y=1$. It easily follows that $A+y\,(B-A)=C$ and $x\,(A-B)+B=C$. This simple observation with its geometric interpretation is the basis of this paper.

Definition. Suppose $M_1, M_2, \ldots, M_m$ are points in the complex plane and k_1, k_2, …, k_m are complex numbers that satisfy $\sum\limits_{i=1}^{m} k_i = 1$. Of course, each complex point M_i is also a complex number. The weighted centroid of these complex points $\{M_1, M_2, \ldots, M_m\}$ with respect to $\{k_1, k_2, \ldots, k_m\}$ is a complex point G_M defined by $G_M = \sum\limits_{i=1}^{m} k_i M_i$.

The complex numbers k_1, k_2, …, k_m are called weights and in the notation G_M it is always assumed that the reader knows what these weights are.

If k_1, k_2, …, k_m, $\overline{k_1}$, $\overline{k_2}$, …, $\overline{k_n}$ are complex numbers, we denote the sums $S_k = \sum\limits_{i=1}^{m} k_i, S_{\overline{k}} = \sum\limits_{i=1}^{n} \overline{k_i}$.

Suppose M_1, M_2, …, M_m, $\overline{M_1}$, $\overline{M_2}$, …, $\overline{M_n}$ are points in the complex plane. Also, k_1, k_2, …, k_m, $\overline{k_1}$, $\overline{k_2}$, …, $\overline{k_n}$ are complex numbers that satisfy $\sum\limits_{i=1}^{m} k_i + \sum\limits_{i=1}^{n} \overline{k_i} = 1$. Thus, $S_k + S_{\overline{k}} = 1$.

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Denote $G_{M\cup\overline{M}}=\sum\limits_{i=1}^{m}k_iM_i+\sum\limits_{i=1}^{n}\overline{k_i}\,\overline{M_i}$.

Thus, $G_{M\cup\overline{M}}$ is the weighted centroid of $\left\{M_1,\ldots,M_m,\overline{M}_1,\ldots,\overline{M}_n\right\}$ with respect to the weights $\left\{k_1,\ldots,k_m,\overline{k_1},\ldots,\overline{k_n}\right\}$.

It is obvious that $\sum\limits_{i=1}^{m}\frac{k_i}{S_k}=1$ and $\sum\limits_{i=1}^{n}\frac{\overline{k_i}}{S_{\overline{k}}}=1$.

Denote $G_M=\sum\limits_{i=1}^{m}\frac{k_i}{S_k}M_i$ and $G_{\overline{M}}=\sum\limits_{i=1}^{n}\frac{\overline{k_i}}{S_{\overline{k}}}\overline{M_i}$.

Thus, G_M is the weighted centroid of $\{M_1,M_2,\ldots,M_m\}$ with respect to the weights $\left\{\frac{k_1}{S_k},\frac{k_2}{S_k},\ldots,\frac{k_m}{S_k}\right\}$ and $G_{\overline{M}}$ is the weighted centroid of $\left\{\overline{M_1},\overline{M_2},\ldots,\overline{M_n}\right\}$ with respect to the weights $\left\{\frac{\overline{k_1}}{S_{\overline{k}}},\frac{\overline{k_2}}{S_{\overline{k}}},\ldots,\frac{\overline{k_n}}{S_{\overline{k}}}\right\}$.

As always, these weights are understood in the notation $G_M, G_{\overline{M}}$.

Since $G_{M\cup\overline{M}}=\sum\limits_{i=1}^{m}k_iM_i+\sum\limits_{i=1}^{n}\overline{k_i}\,\overline{M}_i=S_k\cdot\sum\limits_{i=1}^{m}\frac{k_i}{S_k}M_i+S_{\overline{k}}\cdot\sum\limits_{i=1}^{n}\frac{\overline{k_i}}{S_{\overline{k}}}\overline{M}_i$ it is obvious that $(*)$ is true.

$(*)$ $S_k\cdot G_M+S_{\overline{k}}\cdot G_{\overline{M}}=G_{M\cup\overline{M}}$ where $S_k+S_{\overline{k}}=1$.

From equation $(*)$ and $S_k+S_{\overline{k}}=1$ it is easy to see that (1) and (2) are true.

(1) $G_M+S_{\overline{k}}\left(G_{\overline{M}}-G_M\right)\equiv G_{M\cup\overline{M}}$.

(2) $G_{\overline{M}}+S_k\left(G_M-G_{\overline{M}}\right)\equiv G_{M\cup\overline{M}}$.

2. Basic theorem

The identity $(*)$ and the formula (1) of § 1 proves the following Theorem 1.

Theorem 1. *Suppose M_1, M_2, …, M_m, $\overline{M_1}$, $\overline{M_2}$, …, $\overline{M_n}$ are points in the complex plane. Also, suppose $P=\sum\limits_{i=1}^{m}k_iM_i+\sum\limits_{i=1}^{n}\overline{k_i}\,\overline{M_i}$ where k_1, …, k_m, $\overline{k_1}$, …, $\overline{k_n}$ are complex numbers that satisfy $\sum\limits_{i=1}^{m}k_i+\sum\limits_{i=1}^{n}\overline{k_i}=1$. Then there exists complex numbers $x_1,x_2,\ldots,x_m$ where $\sum\limits_{i=1}^{m}x_i=1$ and there exists complex numbers $y_1,y_2,\ldots,y_n$ where $\sum\limits_{i=1}^{n}y_i=1$ and there exists a complex number z such that the following are true.*

(1). *$x_1,\ldots,x_m,y_1,\ldots,y_n,z$ are rational function of k_1, …, k_m, $\overline{k_1}$, …, $\overline{k_n}$.*

(2). *$P=Q+z\,(R-Q)$ where Q,R are defined by $Q=\sum\limits_{i=1}^{m}x_iM_i, R=\sum\limits_{i=1}^{n}y_i\overline{M_i}$.*

As we illustrate in Section 6, the values of x_1, …, x_m, y_1, …, y_n, z as rational functions of k_1, k_2, …, k_m, $\overline{k_1}$, $\overline{k_2}$, …, $\overline{k_n}$ can be computed adhoc from any specific situation that we face in practice. We observe that Q is the weighted centroid of the complex points M_1, M_2, …, M_m using the weights $x_1,x_2,\ldots,x_m$ and R is the weighted centroid of the complex points $\overline{M_1}$, $\overline{M_2}$, . . . , , $\overline{M_n}$ using the weights $y_1, y_2, \ldots, y_n$. Of course, Theorem 1 is completely standard.

3. Complex homothety

If A, B are points in the complex plane, we denote $AB = B - A$. This also means that AB is the complex vector from A to B. Also, we define $|AB|$ to be the length of this vector AB. If k is any complex number, then $k = r(\cos\theta + i\sin\theta)$, $r \geq 0$, is the polar form of k. It is assumed that the reader knows that

$$[r(\cos\theta + i\sin\theta)] \cdot [\overline{r}(\cos\phi + i\sin\phi)] = r \cdot \overline{r}(\cos(\theta + \phi) + i\sin(\theta + \phi)).$$

Suppose $S, P, \overline{P}$ where $S \neq P, S \neq \overline{P}$ are points in the complex plane and $k = r(\cos\theta + i\sin\theta)$, $r > 0$, is a non-zero complex number. Also, suppose $S\overline{P} = k(SP)$ whereas always $S\overline{P} = \overline{P} - S$ and $SP = P - S$. Since

$$S\overline{P} = k(SP) = [r(\cos\theta + i\sin\theta)] \cdot (SP) = (\cos\theta + i\sin\theta) \cdot [r \cdot (SP)],$$

we see that the complex vector $S\overline{P}$ can be constructed from the complex vector SP in the following two steps.

First, we multiply the vector SP by the positive real number (or scale factor) r to define a new vector, $SP' = r \cdot (SP)$. Since $SP' = P' - S$, the new point P' is collinear with S and P with P, P' lying on the same side of S and $|SP'| = r \cdot |SP|$.

Next, we rotate the vector SP' by θ radians counterclockwise about the origin O as the axis to define the final vector $S\overline{P}$. Of course, the final point $\overline{P}$ itself is computed by rotating the point P' by θ radians counterclockwise about the axis S. If A, B, C, x, y are complex and $xA + yB = C$, $x + y = 1$, then $A + y(B - A) = C$. Therefore, $AC = y \cdot AB$ and if $y = r(\cos\theta + i\sin\theta)$,, $r \geq 0$, we see how to construct the point C.

From this construction, the following is obvious. Suppose $S \neq P$ are arbitrary variable points in the complex plane and $S\overline{P} = k \cdot (SP)$ where $k \neq 0$ is a fixed complex number.

Then the triangles $\triangle SP\overline{P}$ will always have the same geometric shape (up to similarity) since $\angle PS\overline{P} = \theta$ and $\left|S\overline{P}\right| : |SP| = r : 1$ when $k = r(\cos\theta + i\sin\theta)$, $r > 0$. Next, let us suppose that the complex triangles $\triangle ABC$ and $\triangle\overline{ABC}$ and the complex point S are related as follows:

$$S\overline{A} = k \cdot (SA), \ S\overline{B} = k \cdot (SB), \ S\overline{C} = k \cdot (SC),$$

where $k \neq 0$ is some fixed complex number.

We call this relation complex homothety (or complex similitude). Also, S is the center of homothety (or similitude) and k is the homothetic ratio (or ratio of similitude). When k is real we have the usual homothety of two triangle. Of course, for both real or complex k, it is fairly obvious that $\triangle ABC$, and $\triangle\overline{ABC}$ are always geometrically similar and $\frac{|\overline{AB}|}{|AB|} = \frac{|\overline{AC}|}{|AC|} = \frac{|\overline{BC}|}{|BC|} = |k|$.

Of course, this same definition of complex homothety also holds for two polygons $ABCDE, \ldots$ and $\overline{A}\ \overline{B}\ \overline{C}\ \overline{D}\ \overline{E}, \ldots$.

4. Using Theorem 1 to create endless homothetic triangles

Let M_1, M_2, $\ldots$, M_m, $\overline{M_{a1}}$, $\overline{M_{a2}}$, $\ldots$, $\overline{M_{an}}$, $\overline{M_{b1}}$, $\overline{M_{b2}}$, $\ldots$, $\overline{M_{bn}}$, $\overline{M_{c1}}$, $\overline{M_{c2}}$, $\ldots$, $\overline{M_{cn}}$ be any points in the plane.

As a specific example of this, we could start with a triangle $\triangle ABC$ and let M_1, $M_2, \ldots, M_m$ be any fixed points in the plane of $\triangle ABC$ such as the centroid, orthocenter, Lemoine point, incenter, Nagel point, etc.

Also, $\overline{M_{a1}}, \ldots, \overline{M_{an}}$ are fixed points that have some relation to side BC. $\overline{M_{b1}}$, $\ldots$, $\overline{M_{bn}}$ are fixed points that have some relation to side AC and $\overline{M_{c1}}, \ldots, \overline{M_{cn}}$ are fixed points that have some relation to side AB.

Let k_1, k_2, $\ldots$, k_m, $\overline{k_1}$, $\overline{k_2}$, $\ldots$, $\overline{k_n}$ be arbitrary but fixed complex numbers that satisfy $\sum\limits_{i=1}^{m} k_i + \sum\limits_{i=1}^{n} \overline{k_i} = 1$.

Define points P_a, P_b, P_c as follows.

(1) $P_a = \sum\limits_{i=1}^{m} k_i M_i + \sum\limits_{i=1}^{n} \overline{k}_i \overline{M}_{ai}$.

(2) $P_b = \sum\limits_{i=1}^{m} k_i M_i + \sum\limits_{i=1}^{n} \overline{k}_i \overline{M}_{bi}$.

(3) $P_c = \sum\limits_{i=1}^{m} k_i M_i + \sum\limits_{i=1}^{n} \overline{k}_i \overline{M}_{ci}$.

Note that these points P_a, P_b, P_c are being defined in an analogous way. From Theorem 1, there exists complex numbers $x_1, x_2, \ldots, x_m$ where $\sum\limits_{i=1}^{m} x_i = 1$, y_1, y_2, $\ldots$, y_n where $\sum\limits_{i=1}^{n} y_i = 1$, and z such that the following statements are true.

(1) $x_1, \ldots, x_m, y_1, y_2, \ldots, y_n, z$ are rational functions of $k_1, \ldots, k_m, \overline{k_1}, \ldots, \overline{k_n}$.

(2) $P_a = Q + z\,(R_a - Q)$, $P_b = Q + z\,(R_b - Q)$, $R_c = P + z\,(R_c - Q)$, where $Q = \sum\limits_{i=1}^{m} x_i M_i$, and $R_a = \sum\limits_{i=1}^{n} y_i \overline{M}_{ai}$, $R_b = \sum\limits_{i=1}^{n} y_i \overline{M}_{bi}$, $R_c = \sum\limits_{i=1}^{n} y_i \overline{M}_{ci}$.

(3) $QP_a = z \cdot (QR_a)$, $QP_b = z \cdot (QR_b)$, $QP_c = z \cdot (QR_c)$.

(3) follows from (2) since, for example, $P_a - Q = QP_a$.

From (3) it also follows that $\triangle P_a P_b P_c$ is homothetic to $\triangle R_a R_b R_c$ with a center of homothety Q and a ratio of homothety $\frac{QP_a}{QR_a} = \frac{QP_b}{QR_b} = \frac{QP_c}{QR_c} = z$. Also, of course, $\triangle P_a P_b P_c \sim \triangle R_a R_b R_c$ with a ratio of similarity $\frac{|P_a P_b|}{|R_a R_b|} = \frac{|P_a P_c|}{|R_a R_c|} = \frac{|P_b P_c|}{|R_b R_c|} = |z|$.

In the above construction, we could lump some (but not all) of the points in $\{M_1,\ M_2,\ \ldots,\ M_m\}$ with each of the three sets of points $\{\overline{M_{a1}},\ \ldots,\ \overline{M_{an}}\}$, $\{\overline{M_{b1}}, \ldots,\ \overline{M_{bn}}\}$, $\{\overline{M_{c1}}, \ldots,\ \overline{M_{cn}}\}$. For example, we could deal with the four sets $\{M_2, \ldots, M_m\}$, $\{M_1, \overline{M_{a1}}\ , \ldots, \overline{M_{an}}\}$, $\{M_1, \overline{M_{b1}}, \ldots, \overline{M_{bn}}\}$, $\{M_1, \overline{M_{c1}}, \ldots, \overline{M_{cn}}\}$. We then use the same formulas as above and we have

$$QP_a = z \cdot (QR_a)\,, \quad QP_b = z \cdot (QR_b)\,, \quad QP_c = z \cdot (QR_c)\,,$$

where $Q = \sum_{i=2}^{m} x_i M_i$, $R_a = \left(\sum_{i=1}^{n} y_i \overline{M_{ai}}\right) + y_{n+1} M_1$, $R_b = \left(\sum_{i=1}^{n} y_i \overline{M_{bi}}\right) + y_{n+1} M_1$, $R_c = \left(\sum_{i=1}^{n} y_i \overline{M_{ci}}\right) + y_{n+1} M_1$ with $\sum_{i=2}^{m} x_i = 1$ and $\sum_{i=1}^{n+1} y_i = 1$.

As we illustrate in Section 7, by redefining our four sets $\{M_i\}$, $\{\overline{M_{ai}}\}$, $\{\overline{M_{bi}}\}$, $\{\overline{M_{ci}}\}$ in different ways, we can vastly expand our collections of homothetic triangles.

5. Two specific examples

5.1. *Problem 1.* Suppose $\triangle ABC$ lies in the complex plane. In $\triangle ABC$ let AD, BE, CF be the altitudes to sides BC, AC, AB respectively, where the points D, E, F lie on sides BC, AC, AB. The $\triangle DEF$ is called the orthic triangle of $\triangle ABC$. The three altitudes AD, BE, CF always intersect at a common point H which is called the orthocenter of $\triangle ABC$. Also, let O be the circumcenter of $\triangle ABC$ and let A', B', C' denote the midpoints of sides BC, AC, AB respectively. The line HO is called the Euler line of $\triangle ABC$. Define the points P_a, P_b, P_c as follows where k, e, m, n, r are fixed real numbers.

(1) $AP_a = k \cdot AH + e \cdot HD + m \cdot AO + n \cdot AA' + r \cdot OA'$.
(2) $BP_b = k \cdot BH + e \cdot HE + m \cdot BO + n \cdot BB' + r \cdot OB'$.
(3) $CP_c = k \cdot CH + e \cdot HF + m \cdot CO + n \cdot CC' + r \cdot OC'$.

Show that there exists a point Q on the Euler line HO of $\triangle ABC$, a point R_a on side BC, a point R_b on side AC, a point R_c on side AB, and a real number z such that $\triangle P_a P_b P_c$ and $\triangle R_a R_b R_c$ are homothetic with center of homothety Q and real ratio of homothety $\frac{QP_a}{QR_a} = \frac{QP_b}{QR_b} = \frac{QP_c}{QR_c} = z$.

We can also show that there exists a point S on the Euler line OH such that this $\triangle R_a R_b R_c$ is the pedal triangle of S formed by the feet of the three perpendiculars from S to sides BC, AC, BC.

Solution. We first deal with equation (1) given in Problem 1. Equations (2), (3) give analogous results.

Since $AP_a = P_a - A, AH = H - A, HD = D - A$, etc, we see that equation (1) is equivalent to

$$P_a - A = k\,(H - A) + e\,(D - H) + m\,(O - A) + n\,\left(A' - A\right) + r\,\left(A' - O\right).$$

This is equivalent to $(**)$.

$(**)$ $\quad P_a = (1 - k - m - n)A + (k - e)\,H + eD + (m - r)\,O + (n + r)\,A'$.

From geometry, we know that $AH = 2 \cdot OA', BH = 2 \cdot OB', CH = 2 \cdot OC'$. Thus, $H - A = 2\,(A' - O)$ and $A = H + 2\,(O - A')$.

Substituting this value for A in $(**)$ we have

$$\begin{aligned} P_a = {} & (1 - k - m - n)\left(H + 2O - 2A'\right) + (k - e)\,H + eD \\ & + (m - r)\,O + (n + r)\,A'. \end{aligned}$$

This is equivalent to the following.

$$\begin{aligned} P_a = &\ (1-m-n-e)\,H + (2-2k-m-2n-r)\,O + eD \\ &+ (-2+2k+2m+3n+r)\,A'. \end{aligned}$$

Calling $1-m-n-e=\theta$, $2-2k-m-2n-r=\phi$, $e=\lambda$, and $-2+2k+2m+3n+r=\psi$, we have

$$P_a = \theta H + \phi O + \lambda D + \psi A',$$

where $\theta+\phi+\lambda+\psi=1$.

As in Theorem 1, we now lump H, O together and lump D, A' together. Therefore,

$$\begin{aligned} P_a = &\ [\theta H + \phi O] + \left[\lambda D + \psi A'\right] \\ = &\ (\theta+\phi)\left[\frac{\theta H}{\theta+\phi} + \frac{\phi O}{\theta+\phi}\right] + (\lambda+\psi)\left[\frac{\lambda D}{\lambda+\psi} + \frac{\psi A'}{\lambda+\psi}\right]. \end{aligned}$$

Calling $\frac{\theta H}{\theta+\phi} + \frac{\phi O}{\theta+\phi} = Q$, and $\frac{\lambda D}{\lambda+\psi} + \frac{\psi A'}{\lambda+\psi} = R_a$, we have

$$\begin{aligned} P_a = &\ (\theta+\phi)\,Q + (\lambda+\psi)\,R_a \\ = &\ Q + (\lambda+\psi)\,(R_a - Q) \\ = &\ Q + z\,(R_a - Q) \end{aligned}$$

where $z = \lambda+\psi = -2+2k+2m+3n+r+e$.

Of course, Q lies on the Euler line HO and R_a lies on the side BC since $\theta, \phi, \lambda, \psi$ are real.

By symmetry, equations (2), (3) yield the following analogous results.

$$P_b = Q + z\,(R_b - Q) \quad \text{and} \quad P_c = Q + z\,(P_c - Q)\,,$$

where $R_b = \frac{\lambda E}{\lambda+\psi} + \frac{\psi B'}{\lambda+\psi}$, nd $R_c = \frac{\lambda F}{\lambda+\psi} + \frac{\psi C'}{\lambda+\psi}$.

Of course, Q lies on the Euler line HO, R_a lies on side $BC,\ R_b$ lies on side AC and R_c lies on side AB.

Since $QP_a = (\lambda+\psi)\,(QR_a) = z\cdot QR_a$, $QP_b = (\lambda+\psi)\,(QR_b) = z\cdot QR_b$, and $QP_c = (\lambda+\psi)\,(QR_c) = z\cdot QR_c$, we see that $\triangle R_aR_bR_c \sim \triangle P_aP_bP_c$ are homothetic with ratio of homothety $\frac{QP_a}{QR_a} = \frac{QP_b}{QR_b} = \frac{QP_c}{QR_c} = z$.

Also, $\triangle R_aR_bR_c \sim \triangle P_aP_bP_c$ with ratio of similarity $\frac{|P_aP_b|}{|R_aR_b|} = \frac{|P_aP_c|}{|R_aR_c|} = \frac{|P_bP_c|}{|R_bR_c|} = |z|$.

Since D, E, F lie at the feet of the perpendiculars HD, HE, HF and since A', B', C' lie at the feet of the perpendiculars OA', OB', OC', it is easy to see that there exists a point S on the Euler line HO such that $\triangle R_aR_bR_c$ is the pedal triangle of S with respect to $\triangle ABC$.

We now deal with a special case of Problem 1. Let $k=e, m=n=r=0$. Then $\theta = 1-e = 1-k, \phi = 2-2k, \lambda = k, \psi = -2+2k$. Also, $\theta+\phi = 3-3k, \lambda+\psi = -2+3k$. Therefore, $Q = \frac{\theta H}{\theta+\phi} + \frac{\phi O}{\theta+\phi} = \frac{1}{3}H + \frac{2}{3}O$.

From geometry, we see that the center of homothety is $Q = G$ where G is the centroid of $\triangle ABC$. Also, G is still the center of homothety of $\triangle P_aP_bP_c$ and $\triangle R_aR_bR_c$ even for the case where k is complex.

Also, we see that $R_a = \frac{kD}{-2+3k} + \frac{(-2+2k)A'}{-2+3k}$, and the ratio of homothety is $z = -2 + 3k$.

If we let $k = e = 2, m = n = r = 0$, we see that $R_a = \frac{1}{2}D + \frac{1}{2}A', R_b = \frac{1}{2}E + \frac{1}{2}B', R_c = \frac{1}{2}F + \frac{1}{2}C'$.

From geometry we know that the nine point center N of $\triangle ABC$ lies at the mid point of the line segment HO.

Therefore, if $k = e = 2, m = n = r = 0$, we see that $\triangle R_aR_bR_c$ is the pedal triangle of the nine point center N. Also, when $k = e = 2, m = n = r = 0$, we see that $\triangle P_aP_bP_c$ is geometrically just the (mirror) reflections of vertices A, B, C about the three sides BC, AC, AB respectively. Also, the ratio of homothety z is $z = -2 + 3k = 4$. Thus, $\triangle P_aP_bP_c$ is four times bigger than $\triangle R_aR_bR_c$.

5.2. *Problem 2.* Suppose $\triangle ABC$ lies in the complex plane. As in Problem 1, let AD, BE, CF be the altitudes for sides BC, AC, AB respectively where D, E, F lie on sides AB, AC, BC. Let I be the incenter of $\triangle ABC$ and let the incircle (I, r) be tangent to the sides AB, AC, BC at the points X, Y, Z respectively.

Define the points P_a, P_b, P_c as follows.

(1) $P_a = D + i\,(IX)$,
(2) $P_b = E + i\,(IY)$,
(3) $P_c = F + i\,(IZ)$, where i is the unit imaginary.

We wish to find $\triangle R_aR_bR_c$ and a complex number z such that $\triangle P_aP_bP_c$ and $\triangle R_aR_bR_c$ are homothetic with a center of homothety I and a complex ratio of homothety $z = \frac{IP_a}{IR_a} = \frac{IP_b}{IR_b} = \frac{IP_b}{IR_b}$.

Solution.

We first study what $\triangle P_aP_bP_c$ is geometrically. First, we note that $i \cdot IX$, $i \cdot IY$, $i \cdot IZ$ simply rotates the vectors IX, IY, IZ by 90° in the counterclockwise direction about the origin O as the axis. Also, we note that $|IX| = |X - I| = |IY| = |Y - I| = |IZ| = |Z - I| = r$ where r is the radius of the inscribed circle $I(r)$.

Therefore, the points P_a, P_b, P_c lie on sides AB, AC, BC respectively and the distance from D to P_a is r (going in the counterclockwise direction), the distance from E to P_b is r (going counterclockwise) and the distance from F to P_c is r (going counterclockwise).

We next analyze equation (1) in the problem. The analysis of equations (2), (3) is analogous.

Now equation (1) is equivalent to

$$P_a = D + i\,(X - I) = -i \cdot I + [iX + D] = -i \cdot I + (1 + i)\left[\frac{iX}{1+i} + \frac{D}{1+i}\right].$$

Observe that $-i + (1 + i) = 1$ and $\frac{i}{1+i} + \frac{1}{1+i} = 1$.

Define $R_a = \frac{iX}{1+i} + \frac{D}{1+i} = D + \frac{i}{1+i}(X - D) = D + \frac{i}{1+i}(DX)$ since $X - D = DX$.

Therefore, $DR_a = \frac{i}{1+i}(DX) = \left(\frac{1+i}{2}\right)(DX)$ since $R_a - D = DR_a$.

Also, $P_a = -iI + (1+i)R_a = I + (1+i)(R_a - I)$. Therefore, $IP_a = (1+i)(IR_a)$ since $P_a - I = IP_a$ and $R_a - I = IR_a$.

Therefore, by symmetry, we have the following equations.

(1) $DR_a = \left(\frac{1+i}{2}\right)(DX), ER_b = \left(\frac{1+i}{2}\right)(EY), FR_c = \left(\frac{1+i}{2}\right)(FZ)$.

(2) $IP_a = (1+i)(IR_a), IP_b = (1+i)(IR_b), IP_c = (1+i)(IR_c)$.

Equation (1) tells us how to construct $\triangle R_aR_bR_c$ from the points $\{D, X\}$,$\{E, Y\}$, $\{F, Z\}$.

Also, $\triangle P_aP_bP_c$ and $\triangle R_aR_bR_c$ are homothetic with center of homothety I and complex ratio of homothety $z = 1 + i = \frac{IP_a}{IR_a} = \frac{IP_b}{IR_b} = \frac{IP_c}{IR_c}$.

Also, $\triangle P_aP_bP_c \sim \triangle R_aR_bR_c$ and $\frac{|IP_a|}{|IR_a|} = \frac{|IP_b|}{|IR_b|} = \frac{|IP_c|}{|IR_c|} = |1+i| = \sqrt{2}$. Also, $\frac{|P_aP_b|}{|R_aR_b|} = \frac{|P_aP_c|}{|R_aR_c|} = \frac{|P_bP_c|}{|R_bR_c|}$.

6. Discussion

For a deeper understanding of the many applications of Theorem 1, we invite the reader to consider the following alternative form of Problem 1 in §5.1.

Problem 1 (alternate form) The statement of the definitions P_a, P_b, P_c is the same as in Problem 1.

However, we now define A'', B'', C'' to be the (mirror) reflections of O about the sides AB, AC, BC respectively. Therefore, $OA'' = 2 \cdot OA'$, $OB'' = 2 \cdot OB'$, $OC'' = 2 \cdot OC'$. We now substitute A'', B'', C'' for A', B', C' in the problem by using $A'' - O = 2(A' - O)$, etc. and ask the reader to solve the same problem when we deal with A, B,C, H, D, E, F, O, A'', B'' ,C'' instead of A, B, C, H, D, E, F, O, A', B', C'. Also, we show that R_a, R_b, R_c will lie on lines DA'', EB'', FC'' instead of lying on sides AB, AC, BC. The pedal triangle part of the problem is ignored. The center of homothety Q will still lie on the Euler line HO. This illustrates the endless way that Theorem 1 can be used to create homothetic triangles (and polygons).

Reference

[1] N. A. Court, *College Geometry*, Barnes and Noble, Inc., New York, 1963.

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