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An Elementary View on Gromov Hyperbolic Spaces

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Abstract. In the most recent decades, metric spaces have been studied from a variety of viewpoints. One of the important characterizations developed in the study of distances is Gromov hyperbolicity. Our goal here is to provide two approachable, but also intuitive examples of Gromov hyperbolic metric spaces. The authors believe that such examples could be of interest to readers interested in advanced Euclidean geometry; such examples are in fact a familiar introduction into coarse geometries. They are both elementary and fundamental. A scholar familiar with concepts like Ptolemy's cyclicity theorem or various geometric loci in the Euclidean plane could find a familiar environment by working with the concepts presented here.

1. Motivation

The reader familiar with the advanced Euclidean geometry will already have a major advantage when she/he pursues the study of specialized themes in metric geometry. On certain topics, the insight into some ideas developed historically within the triangle geometry or alongside classes of fundamental inequalities serves as a great aid in understanding the profound phenomena in metric spaces. Additionally, from a mathematical standpoint, it is of particular interest to find connections of advanced Euclidean geometry with other areas of mathematics.

One of the most accessible introductions into metric geometry is D. Burago, Y. Burago, and S. Ivanov's monograph [2]. In this well-written monograph, section 8.4 (pp. 284–288) is dedicated to the study of Gromov hyperbolic spaces. The chapter is particularly detailed, but we feel that some more elementary examples would serve the exposition well.

Our motivation in writing this note is to provide the reader who is familiar with advanced Euclidean geometry with an idea of a possible research topic in a more advanced context.

2. Gromov hyperbolic spaces: definition, notations, brief guidelines among references

Following M. Gromov's influential work [5], in recent years several investigators have been interested in showing that metrics, particularly in the area of geometric function theory, are Gromov hyperbolic (to mention here with a few examples,

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see [1, 7, 8, 9]). In the classical theory, an important class of examples of Gromov hyperbolic spaces are the CAT(κ) spaces, with $\kappa < 0$ (see [4], p.106). The reader's ultimate goal is to understand the fundamental monograph [6], which serves as guidelines to many researchers and attracts major interest.

For a formal definition, consider a metric space (M, d) where d satisfies the usual definition of a distance. Given $X, Y, Z \in M$, the quantity $(X|Y)_Z = \frac{1}{2}[d(X,Z) + d(Y,Z) - d(X,Y)]$ is called the *Gromov product* of X and Y with respect to Z. Denote $a \wedge b = \min\{a, b\}$. The metric space (M, d) is called *Gromov hyperbolic* (see Definition 8.4.6, p. 287 in [2]) if there exists some constant $\delta \geq 0$ such that

$$(X|Y)_W \geq (X|Z)_W \wedge (Z|Y)_W - \delta,$$

for all $X, Y, W, Z \in M$.

Sometimes it is more convenient to study the pointwise characterization of Gromov hyperbolic spaces. Using the fact that $a \vee b = \max\{a, b\}$, the Gromov hyperbolic condition can be rewritten in the following way:

(M, d) is a Gromov hyperbolic metric space if there exists a constant $\delta \geq 0$ such that

$$d(X,Z) + d(Y,W) \leq [d(Z,W) + d(Y,Z)] \vee [d(X,Y) + d(Z,W)] + 2\delta,$$
$$\forall X, Y, W, Z \in M.$$

The geometric idea is best captured in Mikhail Gromov's description from [6, p.19], where he writes: "It is hardly possible to find a convincing definition of the curvature (tensor) for an arbitrary metric space X, but one can distinguish certain classes of metric spaces corresponding to Riemannian manifolds with curvatures of a given type. This can be done, for example, by imposing inequalities between mutual distances of finite configurations of points in X".

3. Examples of Gromov hyperbolic spaces

In this section we present two examples of Gromov hyperbolic spaces.

Proposition 1. *Let* $A(-1,0), B(0,1),$ *and* $D(0,-1)$ *be points in the Cartesian plane endowed with the Euclidean distance* d*. Let* $M \subset \mathbb{R}^2$ *be the set*

$$M = \{A, B, D\} \cup \{C | C(x,0), x \geq 0\}.$$

Then the metric space (M, d) *is Gromov hyperbolic with* $\delta \in \left[\frac{3-\sqrt{2}}{2}, \frac{4-\sqrt{2}}{2}\right]$.

Proof. We check that there exists a constant $\delta \geq 0$ such that

$$d(X,Z) + d(Y,W) \leq [d(Z,W) + d(Y,Z)] \vee [d(X,Y) + d(Z,W)] + 2\delta,$$

for all X, Y, $Z \in M$. Note that $d(B,D) = 2$, $d(A,C) = x+1$, $d(A,B) = d(A,D) = \sqrt{2}$, and

$$d(C,D) = d(C,B) = \sqrt{x^2+1}.$$

In order to determine our constant $\delta > 0$, we require the following condition:

$$d(A,C) + d(B,D) \leq [d(A,B) + d(C,D)] \vee [d(A,D) + d(C,B)] + 2\delta.$$

However, $d(A,B) + d(C,D) = d(A,D) + d(C,B)$, thus finding δ reduces to the following:

$$x + (3 - \sqrt{2}) - 2\delta \leq \sqrt{x^2 + 1}, \qquad \forall x \geq 0.$$

An inequality such as $x + b \leq \sqrt{x^2+1}$, for all $x \geq 0$ leads to $\delta \geq \frac{3-\sqrt{2}}{2}$ when $b \leq 0$ and $\delta \leq \frac{4-\sqrt{2}}{2}$ when $b \geq -1$. In all the other cases, the basic inequality holds for $\delta \geq 0$. That is, the metric space (M,d) is Gromov hyperbolic with $\delta \in \left[\frac{3-\sqrt{2}}{2}, \frac{4-\sqrt{2}}{2}\right]$. □

Proposition 2. *Let $A(0,1)$, $B(-1,0)$, $C(0,-1)$ $D(a,0)$, with $a \in (0,2)$ be points in the interior of the disk centered at the origin of radius 2, endowed with the Cayley distance (see* [3]*)*

$$d(X,Y) = \frac{1}{2} \ln \frac{SX}{SY} : \frac{sX}{sY}, \tag{1}$$

where $\{s,S\} = \overline{XY} \cap C((0,0),2)$. Then the set

$$M = \{A,B,C\} \cup \{D | D(a,0), a \in (0,2)\}$$

endowed with the metric space induced by Cayley's distance is a Gromov hyperbolic metric space if

$$\delta > \frac{1}{4} \cdot \ln 27\sqrt{3} \left(\frac{\sqrt{7}+1}{\sqrt{7}-1} \right)^2.$$

Proof. A direct computation shows that

$$d(A,D) = d(C,D) = \frac{1}{2} \ln \left[\frac{\sqrt{3a^2+4}+1}{\sqrt{3a^2+4}-1} \cdot \frac{\sqrt{3a^2+4}+a^2}{\sqrt{3a^2+4}-a^2} \right]$$

$$d(A,B) = d(B,C) = \frac{1}{2} \ln \left[\frac{\sqrt{7}+1}{\sqrt{7}-1} \right]^2$$

$$d(A,C) = \frac{1}{2} \ln 9, \qquad d(B,D) = \frac{1}{2} \ln \frac{3(2+a)}{2-a}.$$

In order to determine $\delta > 0$, we require the condition:

$$d(A,C) + d(B,D) \leq [d(A,B) + d(C,D)] \vee [(d(A,D) + d(C,B)] + 2\delta.$$

On the other hand, $d(A,B) + d(C,D) = d(A,D) + d(C,B)$, thus determining δ reduces to

$$\ln \frac{27(2+a)}{2-a} \leq \ln \left[\left(\frac{\sqrt{7}+1}{\sqrt{7}-1} \right)^2 \cdot \frac{\sqrt{3a^2+4}+1}{\sqrt{3a^2+4}-1} \cdot \frac{\sqrt{3a^2+4}+a^2}{\sqrt{3a^2+4}-a^2} \cdot e^{4\delta} \right]$$

for any $a \in (0,2)$. In fact, the inequality

$$\frac{27(2+a)}{2-a} \leq \left(\frac{\sqrt{7}+1}{\sqrt{7}-1}\right)^2 \cdot \frac{\sqrt{3a^2+4}+1}{\sqrt{3a^2+4}-1} \cdot e^{4\delta}$$

holds exactly when

$$\left(\frac{\sqrt{7}+1}{\sqrt{7}-1}\right)^2 \cdot e^{4\delta} > 27\sqrt{3}.$$

Therefore

$$\delta > \frac{1}{4} \cdot \ln 27\sqrt{3} \left(\frac{\sqrt{7}+1}{\sqrt{7}-1}\right)^2.$$

$\square$

In all the other cases one should consider in this proof, we obtain similar computations; these computations have not been included here, to preserve the quality of our presentation. Our goal is to underline the fundamental geometric core of Gromov hyperbolic metric spaces by the use of these examples.

Note that in the second example, the order of the points in the Cayley distance in (1) is chosen so that the cross-ratio yields a value greater than 1.

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