Forum Geometricorum
Volume 12 (2012) 287–300.

FORUM GEOM
ISSN 1534-1178

On Tripolars and Parabolas

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Abstract. Starting with an analysis of the configuration of chords of contact points with two lines, defined on conics circumscribing a triangle and tangent to these lines, we prove properties relating to the case the conics are parabolas and a resulting method to construct the parabola tangent to four lines.

1. Introduction

It is well known ([3, p. 42], [10, p. 184], [7, II, p. 256]), that given three points A, B, C and two lines in general position, there are either none or four conics passing through the points and tangent to the given lines. A light simplification of Chasles notation ([2, p. 304]) for these curves is $3p2t$ *conics*. The conics exist if either the two lines do not intersect the interior of the triangle ABC or the two lines intersect the interior of the same two sides of ABC. In all other cases there are no conics satisfying the above requirements. In this article, we obtain a formal condition (Theorem 6) for the existence of these conics, relating to the geometry of the triangle ABC. In addition we study the configuration of a triangle and two lines satisfying certain conditions. In §2 we introduce the *middle-tripolar*, which plays a key role in the study. In §3 we review the properties of generalized quadratic transforms, which are relevant for our discussion. In §§4, 5 we relate the classical result of existence of $3p2t$ conics to the geometry of the triangle ABC. In the two last sections we prove related properties and construction methods for parabolas.

2. The middle-tripolar

If a parabola circumscribes a triangle ABC and is tangent to a line l (at a point different from the vertices), then l does not intersect the interior of ABC. In this section we obtain a characterization of such lines. For this, we start with a point P on the plane of triangle ABC and consider its traces A_1, B_1, C_1 and their harmonic conjugates A_2, B_2, C_2, with respect to the sides BC, CA, AB, later lying on the tripolar $tr(P)$ of P (See Figure 1). By applying Newton's theorem ([5, p. 62]) on the diagonals of the quadrilateral $A_1B_1B_2A_2$ we see that the middles A', B', C' respectively of the segments A_1A_2, B_1B_2, C_1C_2 are on a line, which I call the *middle-tripolar* of the point P and denote by m_P. In the following discussion a crucial role plays a certain symmetry among the four lines defined by the sides of the cevian $A_1B_1C_1$ of P and the tripolar $tr(P)$, in relation to the *harmonic associates* ([13, p. 100]) P_1, P_2, P_3 of P. It is, namely, readily seen that for each of these four points the corresponding sides of cevian triangle and tripolar define the same set of four lines. A consequence of this fact is that all four points P, P_1, P_2, P_3 define the same middle-tripolar, which lies totally in the exterior of

Publication Date: December 17, 2012. Communicating Editor: Paul Yiu.

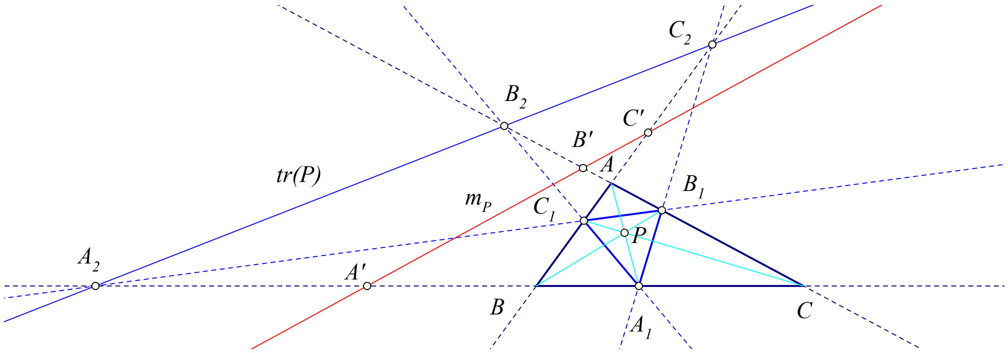

Figure 1. The middle-tripolar m_P of P

the triangle ABC. Combining these two properties, we see that for every point P of the plane, not coinciding with the side-lines or the vertices of the triangle, the corresponding middle-tripolar m_P lies always outside the triangle. It is easy to see that all these properties are also consequences of the following algebraic relation, which is proved by a trivial calculation.

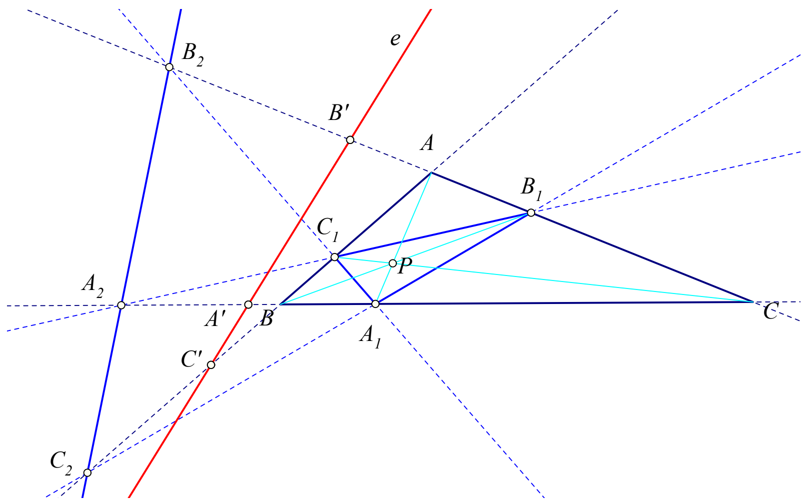

Figure 2. Given e find P such that $e = m_P$

Lemma 1. *If the point P defines through its trace A_1 the ratio $\frac{A_1B}{A_1C} = k$, then the corresponding middle-tripolar m_P defines on the same side of the triangle ABC the ratio $\frac{A'B}{A'C} = k^2$.*

Using this lemma, we can see that every line e exterior to the triangle and not coinciding with a side-line or vertex, defines a point P, interior to the triangle, such that $e = m_P$. It suffices for this to take the ratios defined by e on the side lines

$$k_1 = \frac{A'B}{A'C}, \qquad k_2 = \frac{B'C}{B'A}, \qquad k_3 = \frac{C'A}{C'B}$$

and define the points A_1, B_1, C_1 with corresponding ratios

$$\frac{A_1B}{A_1C} = -\sqrt{k_1}, \qquad \frac{B_1C}{B_1A} = -\sqrt{k_2}, \qquad \frac{C_1A}{C_1B} = -\sqrt{k_3}.$$

A simple application of Ceva's theorem implies that these points define cevians through the required point P, and proves the following lemma.

Lemma 2. *Every line e not intersecting the interior of the triangle ABC and not coinciding with a side-line or vertex of the triangle is the middle-tripolar m_P of a unique point P in the interior of the triangle ABC.*

3. Quadratic transform associated to a base

If a conic circumscribes a triangle ABC and is tangent to two lines l, l' (at points different from the vertices), then it is easily seen that either the lines do not intersect the interior of the triangle or they intersect the interior of the same couple of sides of the triangle. In this section we obtain a characterization of such lines. For this we start with a *base* $A(1,0,0), B(0,1,0), C(0,0,1), D(1,1,1)$ of the projective plane ([1, I, p. 95]). To this base is associated a quadratic transform f, described in the corresponding coordinates through the formulas

$$x' = \frac{1}{x}, \qquad y' = \frac{1}{y}, \qquad z' = \frac{1}{z}.$$

This generalizes the *Isogonal* and the *Isotomic* transformations of a given triangle ABC and has analogous to them properties ([9]). The most simple of them are, that f is involutive $(f^2 = I)$, fixes D and its three harmonic associates, and maps lines to conics through the vertices of ABC. In addition, the harmonic associates of D define analogously the same transformation. Of interest in our study is also the induced transformation f^* of the dual space $(P^2)^*$, consisting of all lines of the projective plane P^2. The transformation f^* can be defined by the requirement of making the following diagram of maps commutative $(f^* \circ tr = tr \circ f)$.

$$\begin{array}{ccc} P^2 & \xrightarrow{\ f\ } & P^2 \\ {\scriptstyle tr}\big\downarrow & & \big\downarrow{\scriptstyle tr} \\ (P^2)^* & \xrightarrow[\ f^*\ ]{} & (P^2)^* \end{array}$$

Here tr denotes the operation $l_P = tr(P)$ of taking the tripolar line of a point with respect to ABC. For every line l the line $l' = f^*(l)$ is found by first taking the tripole P_l of l, then taking $P' = f(P_l)$ and finally defining $l' = tr(P')$. It is easily seen that $(f^*)^2 = I$ and that f^* fixes the sides of the cevian triangle and the tripolar of P. The next lemma follows from a simple computation, which I omit (See Figure 3).

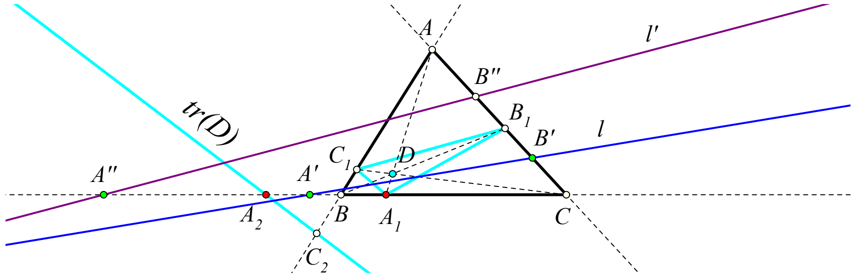

Figure 3. $l' = f^*(l)$ intersects BC on $A'' = A'(A_1A_2)$

Lemma 3. *Let A_1, B_1, C_1 be the traces of D on BC, CA, AB and A_2, B_2, C_2 their harmonic conjugates with respect to these side-endpoints. For every line l intersecting these sides, correspondingly, at A', B', C', the line $l' = f^*(l)$ intersects these sides at the corresponding harmonic conjugates $A'' = A'(A_1A_2), B'' = B'(B_1B_2), C'' = C'(C_1C_2)$.*

Lemma 4. *Let A, B, C, D be a projective base and f the corresponding quadratic transform. For every line l not coinciding with a side-line or vertex of ABC, the lines $l, l' = f^*(l)$ satisfy the following property: either both do not intersect the interior of ABC or both intersect the interior of the same pair of sides of ABC.*

The proof is again an easy calculation in coordinates, which I omit. The next theorem, a sort of converse of the preceding one, shows that this construction charactetrizes the lines tangent to a conic circumscribing a triangle.

Theorem 5. *Let ABC be a triangle and l, l' be a pair of lines having the property of the previous lemma. Then there is a point D, such that A, B, C, D is a projective base with quadratic transformation f and such that $l' = f^*(l)$.*

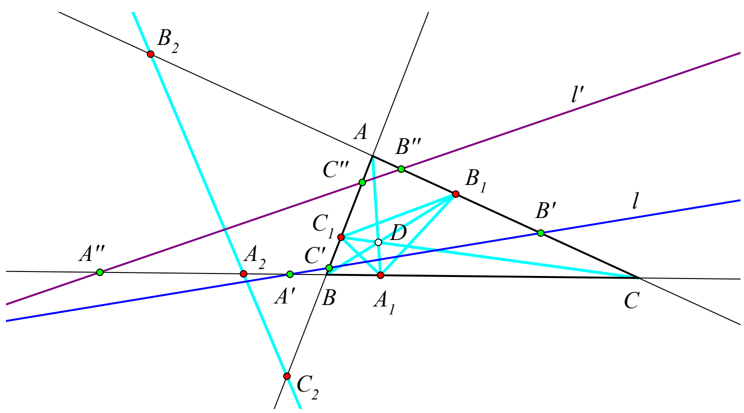

Figure 4. The common harmonics defined by ABC and the two lines

To prove the theorem consider first the intersection points A', B', C' of l, and A'', B'', C'' of l' correspondingly with the sides BC, CA, AB of the triangle. By the hypothesis follows that the pairs of segments $A'A''$, BC either do not intersect or one of them contains the other. The same is true for the pairs $B'B'', CA$ and $C'C'', AB$. It follows that there are exactly two real points A_1, A_2 on line BC, which are common harmonics with respect to (B, C) and (A', A'') i.e. (A_1, A_2) are simultaneously harmonic conjugate with respect to (B, C) and (A', A''). Analogously there are defined the common harmonics (B_1, B_2) of (C, A) and (B', B'') and the common harmonics (C_1, C_2) of (A, B) and (C', C'') (See Figure 4). To prove the theorem, it is sufficient to show that three points out of the six A_1, A_2, B_1, B_2, C_1, C_2 are on a line. This can be done by a calculation or, more conveniently, by reducing it to lemma 2 (see also [6, p. 232]). In fact, consider the projecitvity g fixing A, B, C and sending line l' to the line at infinity $m' = g(l')$. Then line l maps to a line $m = g(l)$. Since projectivities preserve cross ratios, the common harmonic points of l, l' map to corresponding common harmonic points of m, m'. By Lemma 2 line m is the middle-tripolar of some point and three of these harmonic points are on a line. Consequently, their images under g^{-1} are also on a line.

4. $3p2t$ conics

The structure of a triangle ABC and two lines l, l', studied in the preceding section, is precisely the one for which we have four solutions to the problem of constructing a conic passing through three points and tangent to two lines (a $3p2t$ conic). The standard proof of this classical theorem ([3, p. 42], [10, p. 184], [7, II, p. 256], [4], [12]) relies on a consequence of the theorem of Desargues ([11, p. 127]).

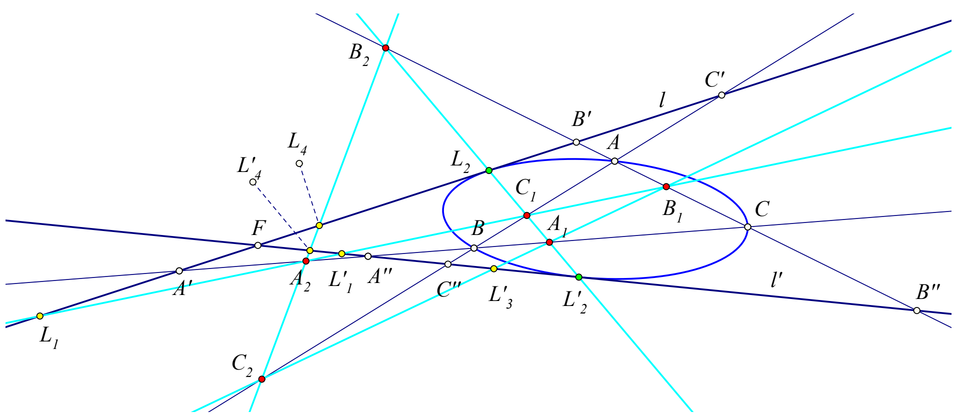

Figure 5. A_1, A_2 fixed points of the involution interchanging $(B, C), (A', A'')$

By this, all conics, tangent to two fixed lines l, l' at two fixed points, determine through their intersections with a fixed line an involution ([11, p. 102]) on the points of this line. Such an involution is completely defined by giving two pairs of corresponding points, such as (B, C) and (A', A'') in Figure 5. The chord of contact points contains the fixed points of the involution, characterized by the fact to be simultaneous harmonic conjugate with respect to the two pairs defining the involution. In Figure 5, the fixed points of the involution on line BC are A_1, A_2. Analogously are defined the fixed points of the involutions operating on the two other sides of the triangle ABC. Thus, there are obtained three pairs of points $(A_1, A_2), (B_1, B_2), (C_1, C_2)$ on respective sides of the given triangle.

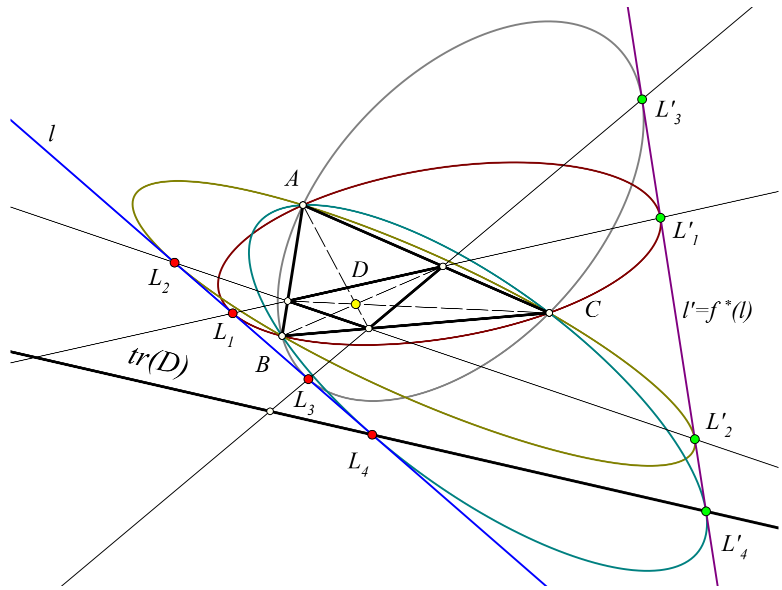

Figure 6. The four circumconics of ABC tangent to $l, l' = f^*(l)$

By the analysis made in the previous sections we see that these six points lie, by three, on four lines, whose intersections with l, l' define the contact points with the conics. The ingredient added to this proof by our remarks is that these four lines are the sides of a cevian triangle and the associated tripolar of a certain point D, defined directly by the triangle ABC and the two lines l, l' (See Figure 6). Thus, the theorem can be formulated in the following way, which brings into the play the geometry of the triangle involved.

Theorem 6. *Let A, B, C, D be a projective base and l a line not coinciding with the side-lines or vertices of triangle ABC. Let also $L_i, L'_i, (i = 1, 2, 3, 4)$ be the intersections of lines $l, l' = f^*(l)$ with the side-lines of the cevian triangle of D and the tripolar $tr(D)$. The four conics, passing, each, through $(A, B, C, L_i, L'_i$ $(i = 1, 2, 3, 4))$, are tangent to l and l'. Conversely, every conic circumscribing ABC and tangent to two lines l, l' is part of such a configuration for an appropriate point D.*

Remarks. (1) The transformation f^* is a sort of *dual* of f and operates in $(P^2)^*$ in the same way f operates in P^2. As noticed in §3, f^* is an involutive quadratic transformation, which fixes the sides of the cevian triangle of D and the tripolar $tr(D)$. Analogously to f, which maps lines to circumconics of ABC, f^* maps the lines of the pencil through a fixed point Q, representing a *line* of $(P^2)^*$, to the tangents of the conic inscribed in ABC, whose perspector ([13, p. 115]) is $f(Q)$. The theorem identifies points (L_i, L_i') with the *lines* of $(P^2)^*$ joining the *fixed points* of this transformation, correspondingly, with the *points* l, l' of $(P^2)^*$.

(2) In the converse part of the theorem the point D is not unique. The structures, though, defined by it and which are relevent for the problem at hand, are indeed unique. Any one of the harmonic associates D_1, D_2, D_3 of D will define the same f and f^* and the same four lines, intersecting the lines l, l' in the same pairs of points (L_i, L_i'). In each case, three of the lines will be the side-lines of the associated cevian triangle and the fourth will be the associated tripolar. Thus, in the last theorem, one can always select the point D in the interior of the triangle ABC, and this choice makes it unique.

Corollary 7. *Given the triangle ABC, the pairs of lines l, l' for which there is a corresponding $3p2t$ conic, are precisely the pairs $l, l' = f^*(l)$, where l is any line not coinciding with the side-lines or vertices of ABC and f^* is defined by a point D lying in the interior of the triangle.*

5. Four parabolas and a hyperbola

If one of the two lines of the last theorem, l' say, is the line at infinity, then it is easily seen that the other line can be identified with the middle-tripolar of some point D. This leads to the following theorem.

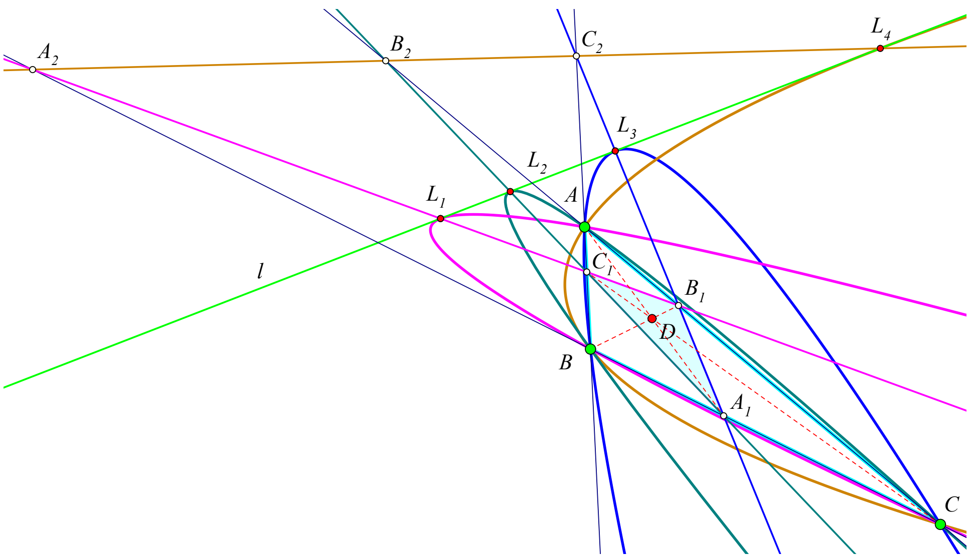

Figure 7. The four parabolas through A, B, C tangent to line $l = m_D$

Theorem 8. *For every point D in the interior of the triangle ABC the sides of its cevian triangle and its tripolar are parallel to the axes of the four parabolas circumscribing the triangle and tangent to its middle-tripolar m_D. The intersections* *of these four lines with m_D are the contact points of the parabolas with m_D. Conversely, every parabola through the vertices of a triangle ABC, touching a line l is member of a quadruple of parabolas constructed in this way.*

Figure 7 shows a complete configuration of three points A, B, C, a line $l = m_D$ and the four parabolas passing through the points and tangent to the line. By the analysis made in §2, line l contains the middles of segments A_1A_2, B_1B_2 and C_1C_2.

The theorem implies that if a parabola c circumscribes a triangle ABC, then for each tangent l to the parabola, at a point different from the vertices, there are precisely three other parabolas circumscribing the same triangle and tangent to the same line. These three parabolas can be then determined by first locating the corresponding point D. The possibility to have D lying in the interior of the triangle, shows that one of the lines drawn parallel to the axes of these parabolas from the corresponding contact point does not intersect the interior of the triangle, whereas the other three do intersect the interior, defining the cevian triangle of point D. Point D is the tripole of that parallel, which does not intersect the interior. This rises the interest for finding the locus of D in dependence of the tangent to the parabola. The next theorem lists some of the properties of this locus and its relations to the parabola.

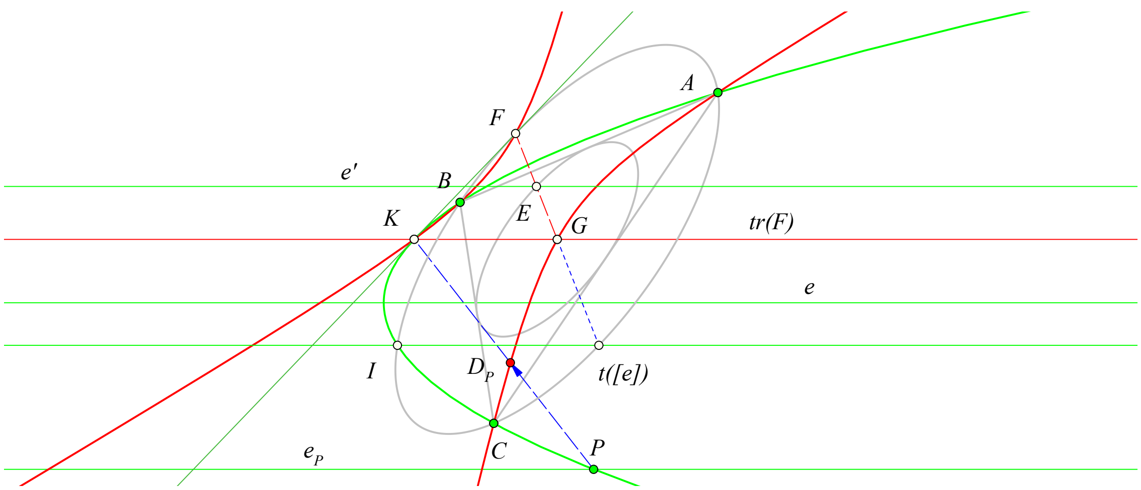

Figure 8. The hyperbola locus

Theorem 9. *Let c be a parabola with axis e circumscribing the triangle ABC. The locus of tripoles D_P of lines e_p, which are the parallels to the axis from the points P of the parabola, is a hyperbola circumscribing the triangle and has, among others, the properties:*

(1) *The hyperbola passes through the centroid G and has its perspector at the point at infinity $[e]$ determined by the direction of e. The perspector E of the parabola is on the inner Steiner ellipse of ABC and coincides with the center of the hyperbola.*

(2) *Line EG passes through the fourth intersection point F of the hyperbola with the outer Steiner ellipse. This line contains also the isotomic conjugate $t([e])$ of $[e]$. The tripole of this line is the fourth intersection point I of the parabola with the outer Steiner ellipse.*
(3) *The fourth intersection point K of the parabola and the hyperbola is the tripole of e', where e' the parallel to e through E. Line KG is parallel to the axis e and is also the tripolar of F. It is also $D_K = F$ and line FK is a common tangent to the parabola and the outer Steiner ellipse. The tangents to the hyperbola at F, K intersect on the parabola at its intersection point with line e'.*
(4) *The hyperbola is the image $g(c)$ of the parabola under the homography g, which fixes A, B, C and sends K to F.*
(5) *All lines joining P to D_P pass through K.*

Most of the properties result by applying theorems on general conics circumscribing a triangle, adapted to the case of the parabola.

In (1) the result follows from the general property of circumconics to be generated by the tripoles of lines rotating about a fixed point (the *perspector* of the conic). In our case the fixed point is the point at infinity $[e]$, determined by the direction of the axis of the parabola, and the lines passing through $[e]$ are all lines parallel to e. That the conic is a hyperbola follows from the existence of two tangents to the inner Steiner ellipse, which are parallel to the axis e. These two parallels have their tripoles at infinity, as do all tangents to the inner Steiner ellipse, implying that the conic is a hyperbola. That this hyperbola passes through the centroid G results from its definition, since G is the tripole of the line at infinity, which is a line of the pencil generating the conic. The claim on the perspector E follows also from a well known property for circumscribed conics, according to which the center C and the perspector P of a circumconic are *cevian quotients* ($C = G/P$, [13, p. 109]). This is a reflexive relation, and since the perspector $[e]$ of the hyperbola coincides with the center of the parabola, their quotients will be also identical.

In (2) point F is the symmetric of G w.r. to E. It belongs to the outer Steiner ellipse, which is homothetic to the inner one and lies also to the hyperbola, since E is its center. That points $E = G/[e], G$ and $t([e])$ are collinear follows by the vanishing of a simple determinant in barycentrics. The tripole I of line EG is the claimed intersection, since E, G are the respective perspectors of these conics.

In (3) line e' contains both the perspector of the parabola and the perspector of the hyperbola, so its tripole belongs to both corresponding conics.

In $(4, 5)$ and the rest of (3) the statements follow by an easy computation, and the fact, that the matrix of g^{-1} in barycentrics is

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix},$$

where (a, b, c) are the coordinates of the point at infinity of line e. This is a homography mapping the outer Steiner ellipse to the hyperbola, by fixing A, B, C and sending F to K.

6. Relations to parabolas tangent to four lines

The two next theorems explore some properties of the parabolas tangent to four lines, which are the sides of a triangle together with the tripolar of a point with respect to that triangle. The focus is on the role of the middle tripolar m_D.

Theorem 10. *Let $A_1B_1C_1$ be the cevian triangle of point D with respect to triangle ABC. The parabola tangent to the sides of $A_1B_1C_1$ and the tripolar of D has its axis parallel to line $l = m_D$. In addition, the triangle ABC is self-polar with respect to the parabola.*

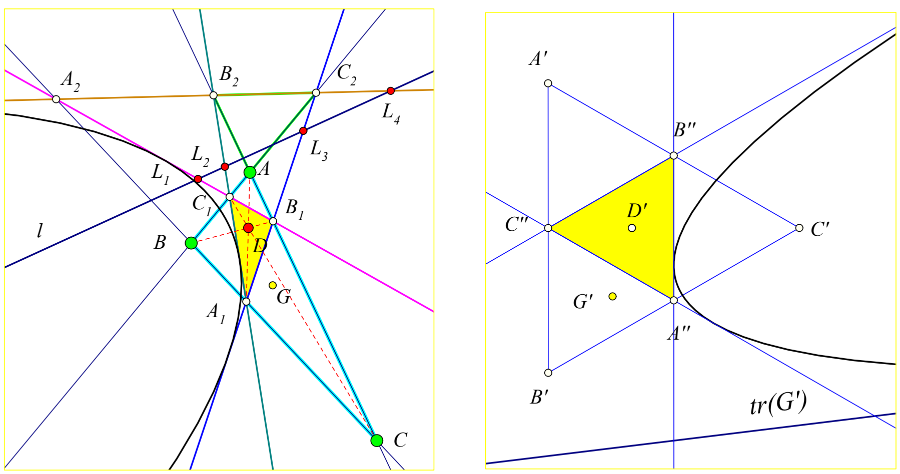

Figure 9. Reduction to the equilateral

The proof of the first part is a consequence of the theorem of Newton ([3, p. 208]), according to which, the centers of the conics which are tangent to four given lines is the line through the middles of the segments joining the diagonal points of the quadrilateral defined by the four lines (the *Newton line* of the quadrilateral [5, p. 62]). The parabola c tangent to the four lines has its center at infinity, thus later coincides with the point at infinity of this line and this proves the first part of the theorem. The second part results from a manageable calculation, but it can be given also a proof, by reducing it to a special configuration via an appropriate homography. In fact, consider the homography f, which maps the vertices of the triangle ABC and point D, correspondingly, to the vertices of the equilateral $A'B'C'$ and its centroid D'. Since homographies preserve cross ratios, they preserve the relation of a line, to be the tripolar of a point. Thus, the line at infinity, which is the tripolar of the centroid G, maps to the tripolar $tr(G')$ of point $G' = f(G)$ (See Figure 9). It follows that the image conic $c' = f(c)$ of the parabola c is also a parabola, since it is tangent to five lines $A''B''$, $B''C''$, $C''A''$, $tr(G')$, $f(A_2B_2)$, one of which is the line at infinity ($f(A_2B_2)$). Here $A'' = f(A_1), B'' = f(B_1), C'' = f(C_1)$ denote the middles of the sides of the equilateral. The proof of the second part results then from the following lemma.

Lemma 11. *If a parabola is inscribed in a triangle, then the anticomplementary triangle is self-polar with respect to the parabola.*

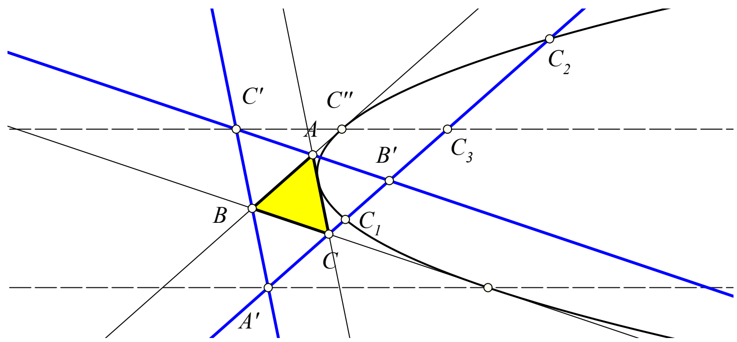

Figure 10. $A'B'C'$ is self-dual w.r. to the parabola inscribed in ABC

To prove the lemma consider a parabola c inscribed in a triangle ABC. Consider also its anticomplementary $A'B'C'$ and the point C'' of tangency with side AB (See Figure 10). The parallel to AB through C, which is a side of the anticomplementary, intersects the parabola at two points C_1, C_2 and by a well known property of parabolas ([8, p. 58]), the tangents at C_1, C_2 meet at the symmetric C' of the middle C_3 of C_1C_2 with respect to C''. Thus C' coincides with a vertex of the anticomplementary, being also the pol of line C_1C_2, as claimed.

Remark. The converse is also true: *If a conic is inscribed in a triangle, such that the anticomplementary is self-polar, then the conic is a parabola.*

Theorem 12. *Let the parabola c be tangent to the sides of the triangle ABC and to the tripolar $tr(D)$ of a point D. Then its contact point with $tr(D)$ is the intersection point of this line with the middle-tripolar $l = m_D$.*

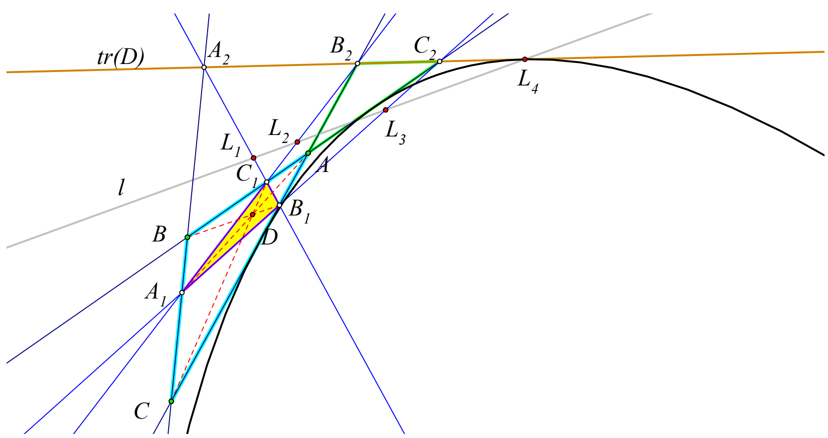

Figure 11. The contact point L_4 with the tripolar

This is proved by an argument similar to that, used in the preceding theorem. In fact, define the homography f mapping triangle ABC to an equilateral $A'B'C'$ and point D to the centroid of $A'B'C'$. Then, see, as in the preceding theorem, that the image conic $c' = f(c)$ of the parabola c is again a parabola. Let then P be the pole of line $l = m_D$ with respect to c. Since P is on line $A_2B_2 = tr(D)$ (See Figure 11), which maps under f to the line at infinity, its image $P' = f(P)$ is at infinity. Hence the image-line $l' = f(l)$ is parallel to the axis of c'. Thus, l' intersects the parabola c' at its point at infinity, which is the image $f(Q)$, where Q is the contact point of c with the line A_2B_2. From this follows that point Q coincides with the intersection point of lines l and A_2B_2, as claimed.

7. The points of tangency

Four lines in general position define a complete quadrilateral $ABCDEF$, four triangles ADE, ABF, BCE, CDF, the diagonal triangle HIJ and four points $ADEp, ABFp, BCEp$ and $CDFp$, which are correspondingly the tripoles of one of these lines with respect to the triangle of the remaining three (See Figure 11). The notation is such, that the tripolar of each of these four points, with respect to the triangle appearing in its label, is the remaining line out of the four, carrying the missing from the label letters (e.g. triangle ABF, tripole $ABFp$ and tripolar DCE). The harmonic associates of each of these points with respect to the corresponding triangle are the vertices of the diagonal triangle HIJ. It is easily seen that the harmonic associates of any of the four points $ADEp, ABFp, BCEp$ and $CDFp$, with respect to HIJ, are the remaining three points.

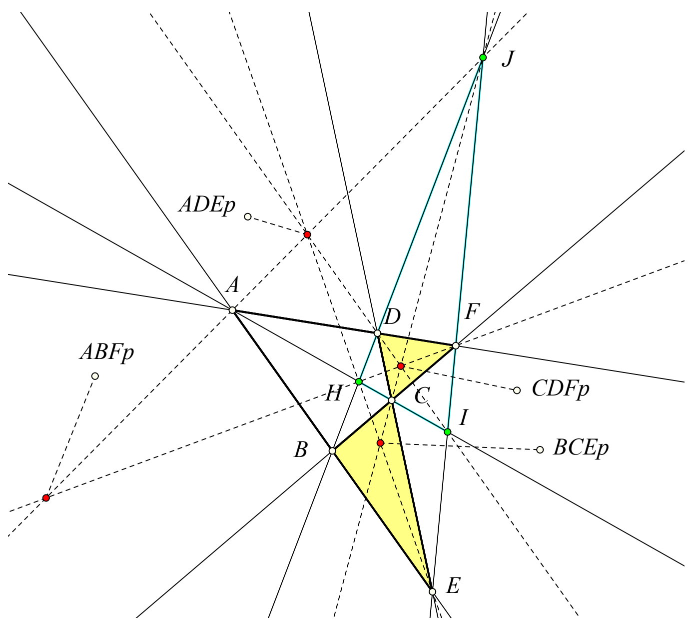

Figure 12. Four lines, four triangles, four points

Applying theorem-12 to each one of the four triangles and the corresponding tripole we obtain four middle-tripolars $ADEn, ABFn, BCEn, CDFn$, which intersect the corresponding lines BCF, CDE, ADF, ABE at corresponding points of tangency $ADEq$, $ABFq$, $BCEq, CDFq$ with the parabola tangent to the four given lines (See Figure 12). This remark leads to a construction method of the parabola tangent to four given lines. The method is not more complicated than the classical one ([8, p. 57]), which uses the circumcircles and orthocenters of the triangles defined by the four lines. In fact, once the middle-tripolars are found, the method uses only intersections of lines. The determination of the middle-tripolars, on the other side, requires either the construction of the harmonic conjugate of a point w.r. to two other points, or the construction of points on lines having a given ratio of distances to two other points of the same line. For example, referring to the last Figure 12, if the ratio $\frac{BA}{BE} = k$, then the corresponding ratio of the intersection point B' of lines $ADEn$ and ABE is $\frac{B'A}{B'E} = k^2$. Point B' is also the middle of segment $B''B$, where $B'' = B(A, E)$ is the harmonic conjugate of B w.r. to (A, E). Once the four contact points are found, one can easily construct a fifth point on the parabola and define it as a conic passing through five points. For this it suffices to find the middle M of a chord, e.g. the one joining $BCEq, CDFq$ and take the middle of MA.

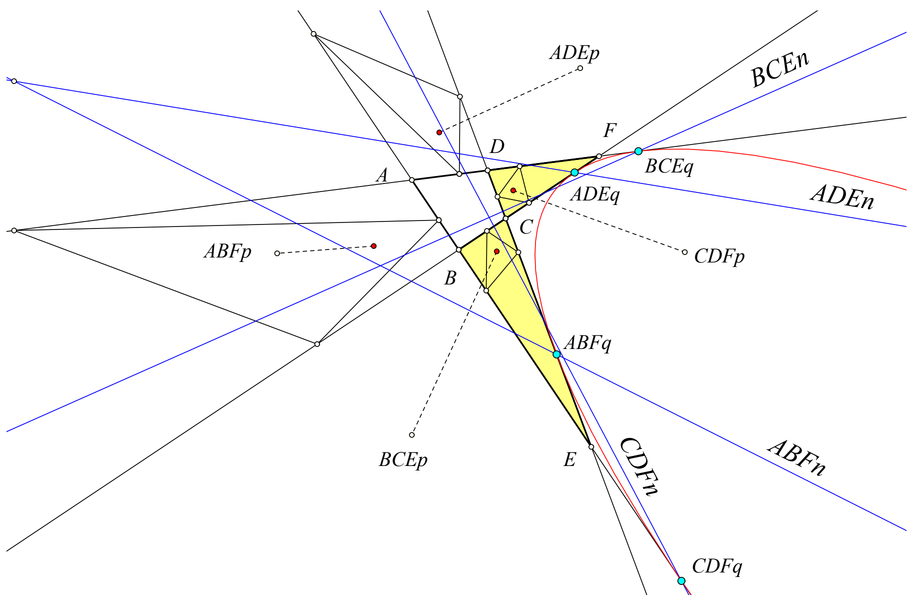

Figure 13. The contact points of the parabola tangent to four lines

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