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Bicentric Quadrilaterals through Inversion

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Abstract. We show that inversion is a delightful tool for making some recent and some older results on bicentric quadrilaterals more transparent and to smoothen their proofs. As a main result we give an illustrative interpretation of Yun's inequality and derive a sharper form.

1. Introduction

Figure 1 shows a bicentric quadrilateral $ABCD$, its circumcircle $\mathcal{C}$ with center O and radius R, and its incircle $\mathscr{C}$ with center Z and radius r, $OZ = d$. The sides of $ABCD$ are tangent to $\mathscr{C}$ at E, F, G, H. Apply an inversion with respect to $\mathscr{C}$.

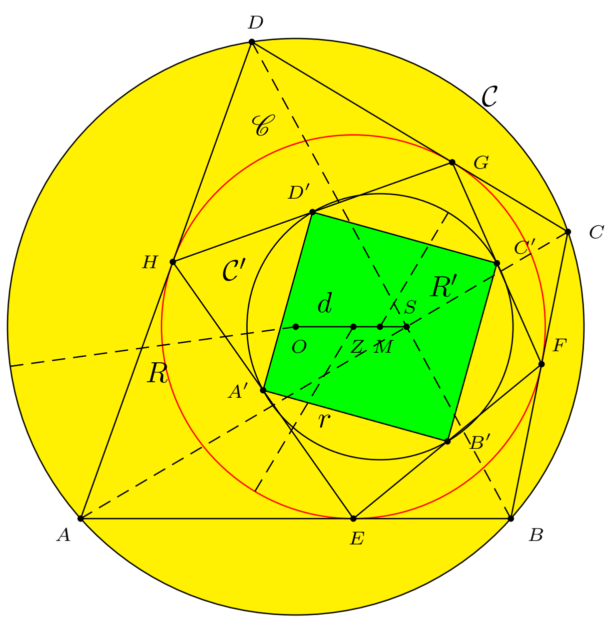

Figure 1

The images A', B', C', D' of the vertices lie on the circle $\mathcal{C}'$ with center M and radius R', $MZ = d'$. The image A' lies on the polar of A with respect to $\mathscr{C}$ and is therefore the midpoint of EH. The same applies to the other images. $A'B'C'D'$ is a rectangle, because being the quadrilateral of the midpoints of $EFGH$ it is a cyclic parallelogram. The diagonals EG and HF are orthogonal, since they are parallel to the sides of $A'B'C'D'$. Cf. [14, step 2] and [7, 837 ff.].

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2. Orthogonality of Newton lines

Theorem 1 (9, Theorem 6)**.** *A tangential quadrilateral $ABCD$ - without axes of symmetry - is cyclic if and only if its Newton line is perpendicular to the Newton line of its contact quadrilateral.*

The restriction is included, since Newton lines do not exist in tangential quadrilaterals with several axes of symmetry, for isosceles tangential trapezoids the theorem is obvious and it is false for kites. Let I and J be the points of intersection of AB and CD, respectively. BC and AD. The midpoints M_{AC}, M_{BD}, M_{IJ} are collinear in any quadrilateral. The line passing through these points is called the Newton line. To prove the collinearity one could use barycentric coordinates. For a visual proof, connect some midpoints of the quadrilateral sides and the appearing parallelograms will guide you. More information about Newton lines can be found in [1, pp. 116–118] and [2]. The points X on the Newton line have a special property: The sum of the signed areas of AXB and CXD equals the sum of the signed areas of AXD and BXC. This can be seen easily from the equivalence of both

$$\overrightarrow{XM_{AC}} \times (\overrightarrow{AB} + \overrightarrow{CD}) = 0 \quad \text{and} \quad \overrightarrow{XM_{BD}} \times (\overrightarrow{AB} + \overrightarrow{CD}) = 0$$

to

$$\overrightarrow{XA} \times \overrightarrow{XB} + \overrightarrow{XC} \times \overrightarrow{XD} = \overrightarrow{XD} \times \overrightarrow{XA} + \overrightarrow{XB} \times \overrightarrow{XC}.$$

If $ABCD$ is a tangential quadrilateral its consecutive sides a, b, c and d satisfy $a + c = b + d$, and therefore the center Z of its incircle share the property that the sum of the areas of AZB and CZD equals the sum of the areas of AZD and BZC. Hence Z belongs to the Newton line.

Proof of Theorem 1. Suppose that the Newton line of $ABCD$, *i.e.*, the line n_1 through M_{AC}, Z, M_{BD}, M_{IJ}, and the Newton line of $EFGH$, *i.e.*, the line n_2 through M_{EG}, M_{FH}, are perpendicular. Apply the inversion with respect to the incircle $\mathcal{C}$. The images of I and J of M_{EG} and M_{FH} lie on the image of n_2, which is a circle through Z orthogonal to n_1, whose center lies on n_1. If $M_{IJ} \in n_1$ is not the center of this circle, then I and J are symmetrical with respect to n_1 and $ABCD$ is a kite, which was excluded. Hence M_{IJ} is the center of the image of n_2, $\angle IZJ = 90^\circ$, $EG \perp FH$, $A'B'C'D'$ is a rectangle and $ABCD$ cyclic. This argument can be reversed easily.

3. Fuss' formula

We derive Fuss' theorem (cf. [3], [7, 837 ff.], [8, Theorem 125], [11, p.1],) by inversion. I found no other place in literature, except the quoted book [7], where Fuss' theorem is proved with inversion. But the calculations in F. G.-M.'s book are somewhat cumbersome.

Observe - with Thales' theorem or angle chasing - that $B'SD'Z$ is a parallelogram. M being the midpoint of ZS, the parallelogram law says

$$4R'^2 + 4d'^2 = 4MD'^2 + 4d'^2 = 2ZD'^2 + 2SD'^2 = 2r^2.$$

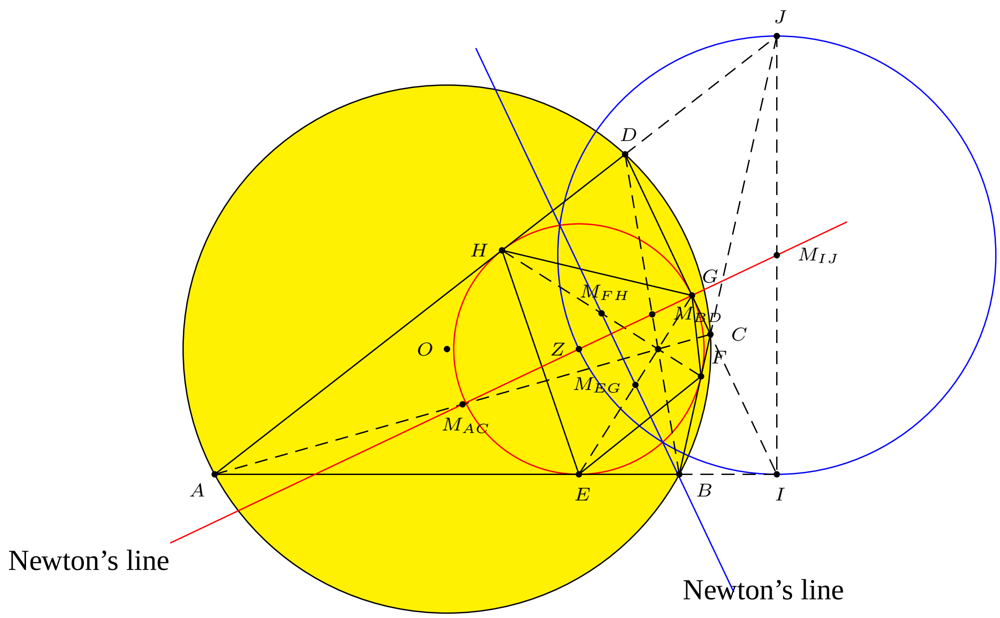

Figure 2

The formulae for radius and midpoint distance of an inverted circle $R' = \frac{r^2 R}{R^2-d^2}$ and $d' = \frac{r^2 d}{R^2-d^2}$, [8, p. 51], substituted into $2R'^2 + 2d'^2 = r^2$ lead to Fuss' formula

$$\frac{1}{(R-d)^2} + \frac{1}{(R+d)^2} = \frac{1}{r^2}.$$

4. Poncelet's porism

Theorem 2. *$ABCD$ is a bicentric quadrilateral with circumcircle $\mathcal{C}$ and incircle $\mathscr{C}$. Then bicentric quadrilaterals with circumcircle $\mathcal{C}$ and incircle $\mathscr{C}$ can be constructed starting from any point of the circumcircle $\mathcal{C}$* (cf. [4], [6], [12], [13]).

Proof. If $ABCD$ is bicentric (see Figure 1), R, r and d obey Fuss' formula. Using inversion with respect to $\mathscr{C}$, the circumcircle $\mathcal{C}$ of $ABCD$ is mapped onto the circle $\mathcal{C}'$ with center M and $2R'^2 + 2d'^2 = r^2$, just reverse the substitutions above. Let S be a point such that M is the midpoint between the center Z of the incircle and this point S. Choose any point A' on $\mathcal{C}'$. This point A' and its diametrically opposite point C' form with Z and S a parallelogram. From the parallelogram law follows that A' is the midpoint of a chord HE of $\mathscr{C}$ which forms together with S a right triangle. G and F are the endpoints of the chords from E and H through S and B', C' and D' the midpoints of the corresponding chords. Inversion with respect to $\mathscr{C}$ converts the circles with diameters ZE, ZF, ZG, ZH into the sides of the bicentric quadrilateral whose vertices are the images of A', B', C' and D'. $\square$

5. Carlitz' inequality

Furthermore, from $2R'^2 + 2d'^2 = r^2$ we get $\sqrt{2}R' \leq r$. Substituted into $R = \frac{r^2 R'}{R'^2 - d'^2} \geq \frac{r^2}{R'} \geq \sqrt{2}r$, Carlitz' inequality [5] is obtained.

6. Coaxial system of circles

Writing $2R'^2 + 2d'^2 = r^2$ as $\frac{R'^2}{d'} + d' = \frac{r^2}{2d'}$, we see that in a bicentric quadrilateral $ABCD$ the image S' of the point S of intersection of GE and FH - and also of AC and BD by Pascal's theorem applied to a degenerated hexagon - is the same when inverted with respect to $\mathscr{C}$ or when inverted with respect to $\mathcal{C}'$. This means that the circle with diameter SS' is orthogonal to $\mathscr{C}$ and to $\mathcal{C}'$ - and also to $\mathcal{C}$ by inversion with respect to $\mathscr{C}$. This reveals $\mathcal{C}, \mathscr{C}$ and $\mathcal{C}'$ as members of a coaxial system of circles with limiting points S and S'. The perpendicular bisector of SS' is the radical axis of this coaxial system, [8, chapter III].

7. Yun's inequality revisited

With $\frac{A+B}{2} = E$, $\frac{B+C}{2} = F$ and the law of sines $2r\sin E = FH$, $2r\sin F = EG$, Yun's inequality

$$\frac{\sqrt{2}r}{R} \leq \frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2} + \sin\frac{B}{2}\cos\frac{C}{2} + \sin\frac{C}{2}\cos\frac{D}{2} + \sin\frac{D}{2}\cos\frac{A}{2}\right) \leq 1,$$

[10], [15], is converted by multiplication with $2r$ into $\frac{2\sqrt{2}r^2}{R} \leq \frac{EG+FH}{2} \leq 2r$. The right hand side is obvious. We increase the left hand side applying the formula for the radius of an inverted circle $R = \frac{r^2 R'}{R'^2 - d^2} \geq \frac{r^2}{\sqrt{R'^2 - d'^2}}$ to $\frac{2\sqrt{2}r^2}{R} \leq 2\sqrt{2R'^2 - 2d'^2}$. From $2R'^2 + 2d'^2 = r^2$ we get $\frac{2\sqrt{2}r^2}{R} \leq 2\sqrt{r^2 - (2d')^2}$. But $2\sqrt{r^2 - (2d')^2}$ is the length of the minimum chord of the circle $\mathscr{C}$ through S and $\frac{EG+FH}{2}$ is the mean of any two orthogonal chords through S, which is obviously greater, equality occurs only for squares $ABCD$ when $S = Z$.

Comparing one chord instead of the mean of two orthogonal chords with the minimum chord we get the inequality

$$\frac{\sqrt{2}r}{R} \leq \sin\frac{A+B}{2} = \sin\frac{A}{2}\sin\frac{D}{2} + \sin\frac{B}{2}\sin\frac{C}{2},$$

of which Yun's inequality is a consequence.

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