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Five Proofs of an Area Characterization of Rectangles

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Abstract. We prove in five different ways a necessary and sufficient condition for a convex quadrilateral to be a rectangle regarding its area expressed in terms of its sides.

There are a handful of well known characterizations of rectangles, most of which concerns one or all four of the angles of the quadrilateral (see [8, p.34]). One example is that a parallelogram is a rectangle if and only if it has (at least) one right angle. Here we shall prove that *a convex quadrilateral with consecutive sides* a, b, c, d *is a rectangle if and only if its area* K *satisfies*

$$K = \tfrac{1}{2}\sqrt{(a^2+c^2)(b^2+d^2)}. \tag{1}$$

We give five different proofs of this area characterization.

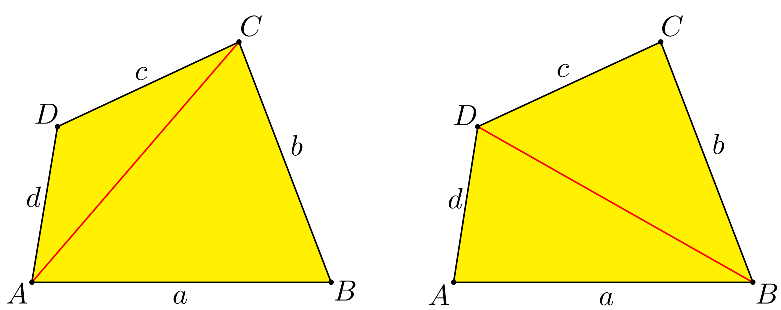

Figure 1. Dividing a quadrilateral into two triangles

First proof. For the area of a convex quadrilateral, we have (see the left half of Figure 1)

$$K = \tfrac{1}{2}ab\sin B + \tfrac{1}{2}cd\sin D \le \tfrac{1}{2}(ab+cd),$$

where there is equality if and only if $B = D = \frac{\pi}{2}$. Using the following algebraic identity due to Diophantus of Alexandria

$$(ab+cd)^2 + (ad-bc)^2 = (a^2+c^2)(b^2+d^2)$$

directly yields the two dimensional Cauchy-Schwarz inequality

$$ab+cd \le \sqrt{(a^2+c^2)(b^2+d^2)}$$

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with equality if and only if $ad = bc$. Hence the area of a convex quadrilateral satisfies

$$K \leq \tfrac{1}{2}\sqrt{(a^2+c^2)(b^2+d^2)} \tag{2}$$

with equality if and only if $B = D = \frac{\pi}{2}$ and $ad = bc$. The third equality is equivalent to $\frac{a}{c} = \frac{b}{d}$, which together with $B = D$ yields that triangles ABC and CDA are similar. But these triangles have the side AC in common, so they are in fact congruent right triangles (since $B = D = \frac{\pi}{2}$). Then the angles at A and C in the quadrilateral must also be right angles, so $ABCD$ is a rectangle. Conversely it is trivial, that in a rectangle $B = D = \frac{\pi}{2}$ and $ad = bc$. Hence there is equality in (2) if and only if the quadrilateral is a rectangle.

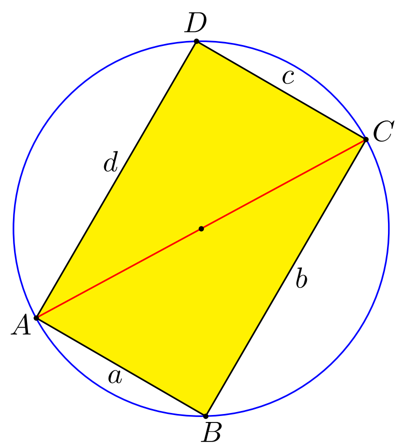

Figure 2. Congruent right triangles ABC and CDA

Second proof. A diagonal can divide a convex quadrilateral into two triangles in two different ways (see Figure 1). Adding these four triangle areas yields that the area K of the quadrilateral satisfies

$$\begin{aligned} 2K &= \tfrac{1}{2}ab\sin B + \tfrac{1}{2}bc\sin C + \tfrac{1}{2}cd\sin D + \tfrac{1}{2}da\sin A \\ &\leq \tfrac{1}{2}ab + \tfrac{1}{2}bc + \tfrac{1}{2}cd + \tfrac{1}{2}da = \tfrac{1}{2}(a+c)(b+d) \end{aligned}$$

where there is equality if and only if $A = B = C = D = \frac{\pi}{2}$. Thus

$$K \leq \tfrac{1}{4}(a+c)(b+d), \tag{3}$$

which is a known inequality for the area of a quadrilateral (see [2, p.129]), with equality if and only if it is a rectangle.[1] According to the AM-GM inequality,

$$(a+c)^2 = a^2 + c^2 + 2ac \leq 2(a^2+c^2)$$

[1]An interesting historical remark is that the formula $K = \frac{a+c}{2} \cdot \frac{b+d}{2}$ (this is another area characterization of rectangles) was used by the ancient Egyptians to calculate the area of a quadrilateral, but it's only a good approximation if the angles of the quadrilateral are close to being right angles. In all quadrilaterals but rectangles the formula gives an overestimate of the area, which the tax collectors probably didn't mind.

with equality if and only if $a = c$. Similarly, $(b+d)^2 \le 2(b^2+d^2)$. Using these two inequalities in (3), which we first rewrite, we get

$$\begin{aligned} K &\le \tfrac{1}{4}\sqrt{(a+c)^2(b+d)^2} \\ &\le \tfrac{1}{4}\sqrt{2(a^2+c^2)\cdot 2(b^2+d^2)} = \tfrac{1}{2}\sqrt{(a^2+c^2)(b^2+d^2)}. \end{aligned}$$

There is equality if and only if $a = c$, $b = d$, and $A = B = C = D = \frac{\pi}{2}$, that is, only when the quadrilateral is a rectangle.

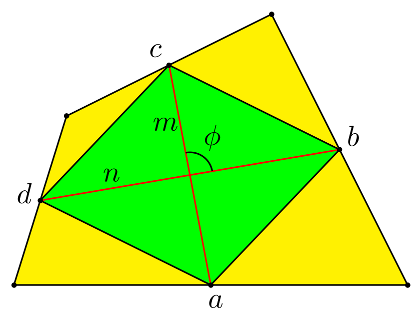

Figure 3. The Varignon parallelogram and the bimedians

Third proof. The area of a convex quadrilateral is twice the area of its Varignon parallelogram [3, p.53]. The diagonals in that parallelogram are the bimedians m and n in the quadrilateral, that is, the line segments connecting the midpoints of opposite sides (see Figure 3). Using that the area K of a convex quadrilateral is given by one half the product of its diagonals and sine for the angle between the diagonals (this was proved in [5]), we have that

$$K = mn\sin\phi \tag{4}$$

where ϕ is the angle between the bimedians. In [7, p.19] we proved that the diagonals of a convex quadrilateral are congruent if and only if the bimedians are perpendicular. Hence the area of a convex quadrilateral is

$$K = mn \tag{5}$$

if and only if the diagonals are congruent (it is an equidiagonal quadrilateral). The length of the bimedians in a convex quadrilateral can be expressed in terms of two opposite sides and the distance v between the midpoints of the diagonals as

$$\begin{aligned} m &= \tfrac{1}{2}\sqrt{2(b^2+d^2)-4v^2}, \\ n &= \tfrac{1}{2}\sqrt{2(a^2+c^2)-4v^2} \end{aligned} \tag{6}$$

(see [6, p.162]). Using these expressions in (5), we have that the area of a convex quadrilateral is given by

$$K = \tfrac{1}{4}\sqrt{(2(a^2+c^2)-4v^2)(2(b^2+d^2)-4v^2)}$$

if and only if the diagonals are congruent. Now solving the equation

$$\tfrac{1}{2}\sqrt{(a^2+c^2)(b^2+d^2)} = \tfrac{1}{4}\sqrt{(2(a^2+c^2)-4v^2)(2(b^2+d^2)-4v^2)}$$

yields $8v^2 = 0$ or $a^2+b^2+c^2+d^2 = 2v^2$. The second equality is not satisfied in any quadrilateral, since according to Euler's extension of the parallelogram law, in all convex quadrilaterals

$$a^2+b^2+c^2+d^2 = p^2+q^2+4v^2 > 2v^2$$

where p and q are the lengths of the diagonals [1, p.126]. Thus we conclude that $v = 0$ is the only valid solution. Hence a convex quadrilateral has the area given by (1) if and only if the diagonals are congruent and bisect each other. A parallelogram, the quadrilateral characterized by bisecting diagonals ($v = 0$), has congruent diagonals if and only if it is a rectangle.

Fourth proof. Combining equations (4) and (6) yields that the area of a convex quadrilateral with consecutive sides a, b, c, d is given by

$$K = \tfrac{1}{4}\sqrt{(2(a^2+c^2)-4v^2)(2(b^2+d^2)-4v^2)}\sin\phi$$

where v is the the distance between the midpoints of the diagonals and ϕ is the angle between the bimedians. Since parallelograms are characterized by $v = 0$, we have that the area is

$$K = \tfrac{1}{2}\sqrt{(a^2+c^2)(b^2+d^2)}\sin\phi$$

if and only if the quadrilateral is a parallelogram. In a parallelogram, ϕ is equal to one of the vertex angles since each bimedian is parallel to two opposite sides (see Figure 4). A parallelogram is a rectangle if and only if one of the vertex angles is a right angle. The equation $\sin\phi = 1$ only has one possible solution $\phi = \frac{\pi}{2}$; hence we have that the area of a convex quadrilateral is given by (1) if and only if it is a rectangle.

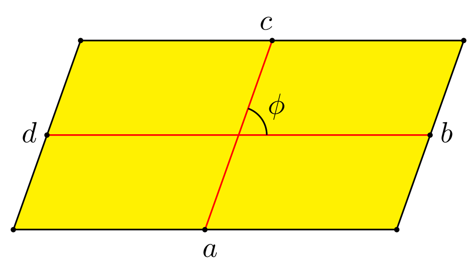

Figure 4. The angle between the bimedians in a parallelogram

Fifth proof. A convex quadrilateral with consecutive sides a, b, c, d and diagonals p, q has the area [4, p.27]

$$K = \tfrac{1}{4}\sqrt{4p^2q^2 - (a^2 - b^2 + c^2 - d^2)^2}.$$

Now solving the equation

$$\tfrac{1}{2}\sqrt{(a^2 + c^2)(b^2 + d^2)} = \tfrac{1}{4}\sqrt{4p^2q^2 - (a^2 - b^2 + c^2 - d^2)^2}$$

we get

$$(2pq)^2 - (a^2 + b^2 + c^2 + d^2)^2 = 0$$

with only one positive solution $a^2 + b^2 + c^2 + d^2 = 2pq$. Using again Euler's extension of the parallelogram law

$$a^2 + b^2 + c^2 + d^2 = p^2 + q^2 + 4v^2,$$

where v is the distance between the midpoints of the diagonals p and q, yields

$$p^2 + q^2 + 4v^2 = 2pq \quad \Leftrightarrow \quad (2v)^2 = -(p - q)^2.$$

Here the left hand side is never negative, whereas the right hand side is never positive. Thus for equality to hold, both sides must be zero. Hence $v = 0$ and $p = q$. This is equivalent to that the quadrilateral is a parallelogram with congruent diagonals, *i.e.*, a rectangle.

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