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Characterizations of Trapezoids

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Abstract. We review eight and prove an additional 13 necessary and sufficient conditions for a convex quadrilateral to be a trapezoid. One aim for this paper is to show that many of the known properties of trapezoids are in fact characterizations.

1. Introduction

A trapezoid (in British English it is called a trapezium) is a quadrilateral with a pair of opposite parallel sides. But there is some disagreement if the definition shall state *exactly* one pair or *at least* one pair. The former is called an exclusive definition and the latter an inclusive definition. The exclusive seems to be common in textbooks at lower levels of education, whereas the inclusive is common among mathematicians and at higher levels of education (beyond high school) [10, p. xiii]. What is the reason for and significance of the two possible definitions?

One likely explanation for the exclusive definition is that when students first encounter shapes like a trapezoid or a rhombus, they could get confused if a rhombus also can be called a trapezoid. When proving properties of a trapezoid it is important to actually draw it with only one pair of opposite parallel sides, so the proof covers the general case. Here the exclusive definition has its merits. But when students are to progress in their mathematical education, the exclusive definition has some drawbacks.

First of all, the main strength of the inclusive definition is the fact that a property that is proved to hold for a trapezoid automatically also holds for all quadrilaterals with two pairs of opposite parallel sides, that is, for parallelograms, rhombi, rectangles, and squares. This is a major advantage, since then we do not have to repeat arguments for those classes. Other benefits are that the taxonomy for quadrilaterals is more perspicuous within the inclusive definition, and features like symmetry and duality becomes more prominent. Also, there is the trapezoid rule for calculating integrals. But these trapezoids do not always just have one pair of opposite parallel sides; sometimes they are in fact rectangles. That would make the name of the rule confusing if a rectangle was not considered to be a special case of a trapezoid. These are some of the reasons why mathematicians nowadays prefer the inclusive definition, that is, *a trapezoid is a quadrilateral with* at least *one pair of opposite parallel sides*.

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We claim that many geometry textbooks do not put much effort into summarizing even the most basic characterizations of trapezoids. The trapezoid is one of the six simplest types of quadrilaterals, so it is usually covered in books at lower levels of education. In those texts the authors often covers quite extensively methods for proving that a quadrilateral is one of the other five types: parallelograms, rhombi, rectangles, squares, and isosceles trapezoids.[1] But not the general trapezoid. Why is that? One reason could be that authors consider the topic already covered in connection with the treatment of parallel lines. But if so, then why not instead take this opportunity to connect that theory with quadrilaterals to show how it all fits together?

Anyway, we will now summarize a handful of the simplest characterizations of trapezoids. These are the ones that rely only upon the theory of parallel lines or similarity. Then we shall prove a dozen of other characterizations, and in doing so we will demonstrate that most of the well known properties of trapezoids are in fact necessary and sufficient conditions for a quadrilateral to be a trapezoid.

First a comment on notations. The consecutive sides of a convex quadrilateral $ABCD$ will be denoted $a = AB$, $b = BC$, $c = CD$, and $d = DA$. In most of the characterizations we only consider the case when $a \parallel c$ and $a \geq c$. We trust the reader can then reformulate the characterizations in the other main case $b \parallel d$ using symmetry.

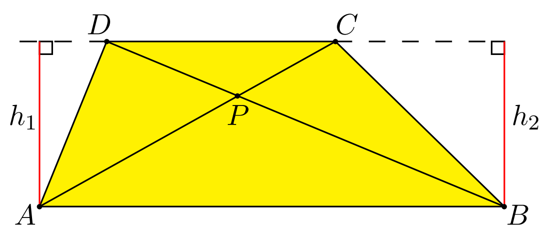

Figure 1. Two altitudes to the side CD

If the extensions of opposite sides AB and CD in a convex quadrilateral intersect at an angle ξ, then the quadrilateral is a trapezoid if and only if $\xi = 0$. A second characterization is that *the quadrilateral $ABCD$ is a trapezoid with parallel sides AB and CD if and only if $\angle ABD = \angle CDB$*, see Figure 1. An equivalent necessary and sufficient condition is that *a convex quadrilateral is a trapezoid if and only if two pairs of adjacent angles are supplementary, that is*

$$A + D = \pi = B + C. \tag{1}$$

[1]The kite must also be considered to be one of the basic quadrilaterals. Perhaps since there are only a few known characterizations of these, it usually don't get that much attention. If we are to include the possibility of a tangential quadrilateral as well (i.e. one that has an incircle), then there are a further dozen of (less well known) characterizations of kites, see [6]. We note that Theorem 2 (ix) in that paper contained a misprint. It should state: The incenter lies on the diagonal that is a line of symmetry (which is not necessarily the longest one).

From the theory of parallel lines we also have that *the line segments AB and CD are the bases of a trapezoid $ABCD$ if and only if the triangles ACD and BCD have equal altitudes to the common side CD* ($h_1 = h_2$ in Figure 1).

Two characterizations concerning similarity are the following. *A convex quadrilateral $ABCD$ is a trapezoid if and only if the diagonals divide each other in the same ratio, that is*

$$\frac{AP}{CP} = \frac{BP}{DP}, \tag{2}$$

where P is the intersection of the diagonals. A closely related necessary and sufficient condition states that *the diagonals divide a convex quadrilateral into four non-overlapping triangles, of which two opposite are similar if and only if the quadrilateral is a trapezoid* ($ABP \sim CDP$ in Figure 1).

2. Trigonometric characterizations

As part of the proof of Theorem 2 in [7] we have already proved two trigonometric characterizations of trapezoids, so we just restate them here. A convex quadrilateral $ABCD$ is a trapezoid if and only if

$$\sin A \sin C = \sin B \sin D.$$

An equivalent necessary and sufficient condition is

$$\cos(A - C) = \cos(B - D).$$

In fact, both of these conditions incorporate the possibility for either pair of opposite sides to be parallel, not just $a \parallel c$.

The first theorem and the subsequent proposition are trigonometric versions of the adjacent angle characterization (1).

Theorem 1. *A convex quadrilateral $ABCD$ is a trapezoid with parallel sides AB and CD if and only if*

$$\cos A + \cos D = \cos B + \cos C = 0.$$

Proof. ($\Rightarrow$) If the quadrilateral is a trapezoid, then $A + D = \pi$. Hence

$$\cos A + \cos D = \cos A + \cos(\pi - A) = \cos A - \cos A = 0.$$

The second equality is proved in the same way.

($\Leftarrow$) We do an indirect proof of the converse. Assume the quadrilateral is not a trapezoid and without loss of generality that $A > \pi - D$. Since $0 < A < \pi$ and the cosine function is decreasing on that interval, we get $\cos A < \cos(\pi - D)$. Hence

$$\cos A + \cos D < \cos(\pi - D) + \cos D = 0.$$

From the sum of angles in a quadrilateral we also have that

$$A > \pi - D \quad \Rightarrow \quad B < \pi - C \quad \Rightarrow \quad \cos B + \cos C > 0.$$

So if the quadrilateral is not a trapezoid, then $\cos A + \cos D \neq \cos B + \cos C$, and neither side is equal to 0. This completes the indirect proof. $\square$

Proposition 2. *A convex quadrilateral $ABCD$ is a trapezoid with parallel sides AB and CD if and only if*

$$\cot A + \cot D = \cot B + \cot C = 0.$$

Proof. Since the cotangent function is decreasing on the interval $0 < x < \pi$ and $\cot(\pi - x) = -\cot x$, the proof is identical to that of Theorem 1. □

So far we have characterizations with sine, cosine and cotangent. Next we prove one for the tangents of the half angles.

Theorem 3. *A convex quadrilateral $ABCD$ is a trapezoid with parallel sides AB and CD if and only if*

$$\tan\frac{A}{2}\tan\frac{D}{2} = \tan\frac{B}{2}\tan\frac{C}{2} = 1.$$

Proof. ($\Rightarrow$) If the quadrilateral is a trapezoid, then $A + D = \pi = B + C$. Using these, the equalities in the theorem directly follows since $\tan\frac{D}{2} = \cot\frac{A}{2}$ and $\tan\frac{C}{2} = \cot\frac{B}{2}$.

($\Leftarrow$) Assume the quadrilateral is not a trapezoid and without loss of generality that $A + D > \pi$ and $B + C < \pi$. From the addition formula for tangent, we get

$$0 > \tan\left(\frac{A}{2} + \frac{D}{2}\right) = \frac{\tan\frac{A}{2} + \tan\frac{D}{2}}{1 - \tan\frac{A}{2}\tan\frac{D}{2}}.$$

The angles $\frac{A}{2}$ and $\frac{C}{2}$ are acute, so the numerator is positive. Then the denominator must be negative, so $\tan\frac{A}{2}\tan\frac{D}{2} > 1$. In the same way $\tan\frac{B}{2}\tan\frac{C}{2} < 1$. Hence

$$\tan\frac{A}{2}\tan\frac{D}{2} \neq \tan\frac{B}{2}\tan\frac{C}{2}$$

and neither side is equal to 1. □

3. Characterizations concerning areas

The first proposition about areas concerns a bimedian, that is, a line segment that connects the midpoints of two opposite sides.

Proposition 4. *A convex quadrilateral is a trapezoid if and only if one bimedian divide it into two quadrilaterals with equal areas.*

Proof. ($\Rightarrow$) In a trapezoid, the bimedian between the bases (see the left half of Figure 2) divide it into two quadrilaterals with equal altitudes and two pairs of equal bases. Hence these two quadrilaterals, which are also trapezoids, have equal areas according to the well known formula for the area of a trapezoid.[2]

($\Leftarrow$) If $T_1 + T_2 = T_3 + T_4$ in a convex quadrilateral with notations as in the right half of Figure 2, then we have $T_1 = T_4$ since T_2 and T_3 are equal due to equal bases and equal altitudes. But T_1 and T_4 also have equal bases, so then their altitudes must be equal as well. This means that the quadrilateral is a trapezoid. □

[2]The area of a trapezoid is the arithmetic mean of the bases times the altitude.

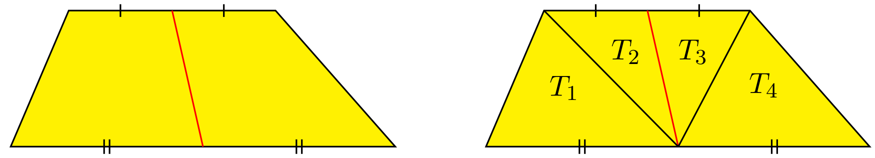

Figure 2. A trapezoid (left) and $T_1 + T_2 = T_3 + T_4$ (right)

We will need the next proposition in the proofs of the following two characterizations. A different proof was given as the solution to Problem 4.14 in [9, pp.80, 89].

Proposition 5. *If the diagonals in a convex quadrilateral $ABCD$ intersect at P, then it is a trapezoid with parallel sides AB and CD if and only if the areas of the triangles APD and BPC are equal.*

Proof. We have that the sides AB and CD are parallel if and only if (see Figure 1)

$$h_{ACD} = h_{BCD} \quad \Leftrightarrow \quad T_{ACD} = T_{BCD} \quad \Leftrightarrow \quad T_{APD} = T_{BPC}$$

where h_{XYZ} and T_{XYZ} stands for the altitude and area of triangle XYZ respectively. $\square$

The following theorem was proved by us using trigonometry as Theorem 2 in [7]. Here we give a different proof using the previous characterization.

Theorem 6. *A convex quadrilateral is a trapezoid if and only if the product of the areas of the triangles formed by one diagonal is equal to the product of the areas of the triangles formed by the other diagonal.*

Proof. We use notations on the subtriangle areas as in Figure 3. Then we have

$$\begin{aligned} & (S + U_1)(T + U_2) = (S + U_2)(T + U_1) \\ \Leftrightarrow \quad & SU_2 + TU_1 = SU_1 + TU_2 \\ \Leftrightarrow \quad & S(U_2 - U_1) = T(U_2 - U_1) \\ \Leftrightarrow \quad & (S - T)(U_2 - U_1) = 0. \end{aligned}$$

The last equality is equivalent to $S = T$ or $U_2 = U_1$, where either of these equalities is equivalent to that the quadrilateral is a trapezoid according to Proposition 5. $\square$

In the proof of the next theorem we will use the following lemma about a property that all convex quadrilaterals have. Observe that the triangles in this lemma are not the same as the ones in Theorem 6.

Lemma 7. *The diagonals of a convex quadrilateral divide it into four non-overlapping triangles. The product of the areas of two opposite triangles is equal to the product of the areas of the other two triangles.*

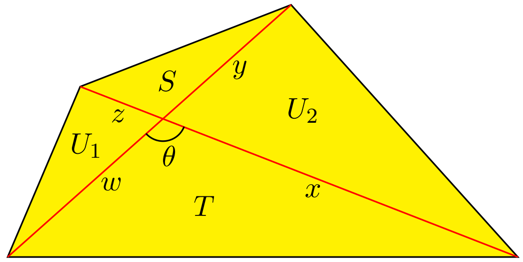

Figure 3. The diagonal parts and subtriangle areas

Proof. We denote the diagonal parts by w, x, y, z and the consecutive subtriangle areas by S, U_1, T, U_2, see Figure 3. These areas satisfy

$$ST = \tfrac{1}{4}wxyz\sin^2\theta = U_1U_2$$

where θ is the angle between the diagonals.[3] □

Our last characterization concerning areas is a beautiful formula for the area of a trapezoid. It can be proved using similarity as in [1, p.50]. We give a short proof establishing it to be both a necessary and sufficient condition.

Theorem 8. *The diagonals of a convex quadrilateral divide it into four non-overlapping triangles. If two opposite of these have areas S and T, then the quadrilateral has the area*

$$K = \left(\sqrt{S} + \sqrt{T}\right)^2$$

if and only if it is a trapezoid whose parallel sides are the two sides in the triangles in question that are not parts of the diagonals.

Proof. A convex quadrilateral has the area (see Figure 3)

$$\begin{aligned} K &= S + T + U_1 + U_2 \\ &= S + T + 2\sqrt{ST} - 2\sqrt{U_1U_2} + U_1 + U_2 \\ &= \left(\sqrt{S} + \sqrt{T}\right)^2 + \left(\sqrt{U_1} - \sqrt{U_2}\right)^2 \end{aligned}$$

where we in the second equality used that $ST = U_1U_2$ according to the Lemma. We have that the quadrilateral is a trapezoid if and only if $U_1 = U_2$ (by Proposition 5), so it is a trapezoid if and only if it has the area $K = \left(\sqrt{S} + \sqrt{T}\right)^2$. □

As a corollary we note that the area K of a convex quadrilateral satisfies the inequality [4]

$$\sqrt{K} \geq \sqrt{S} + \sqrt{T},$$

[3] This equality can also be proved without trigonometry. If the altitudes in the two triangles on respective side of the diagonal $w + y$ are h_1 and h_2, then we have that $ST = \frac{1}{4}wyh_1h_2 = U_1U_2$.

[4] We have seen this inequality before, but we cannot recall a reference.

where there is equality if and only if the quadrilateral is a trapezoid.

Theorem 8 was formulated as if there could be only one pair of opposite parallel sides (a general trapezoid). If there are two pairs of opposite parallel sides, then the two triangles with areas S and T could be be any one of the two pairs of opposite triangles formed by the diagonals.

4. Characterizations concerning sides and distances

The following simple characterization concerns the ratio of two opposite sides and the ratio of the sine of two adjacent angles.

Proposition 9. *The convex quadrilateral $ABCD$ is a trapezoid with parallel sides AB and CD if and only if*

$$\frac{DA}{BC} = \frac{\sin C}{\sin D}.$$

Proof. The quadrilateral is a trapezoid if and only if the triangles ACD and BCD have equal altitudes to the side CD, which is equivalent to that the areas of these two triangles are equal. This in turn is equivalent to

$$\tfrac{1}{2}CD \cdot DA \sin D = \tfrac{1}{2}CD \cdot BC \sin C,$$

which is equivalent to the equality in the theorem. □

The parallelogram law states that in a parallelogram, the sum of the squares of the four sides equals the sum of the squares of the two diagonals. Euler generalized this to a convex quadrilateral with sides a, b, c, d and diagonals p, q as

$$a^2 + b^2 + c^2 + d^2 = p^2 + q^2 + 4v^2 \tag{3}$$

where v is the distance between the midpoints of the diagonals. A proof can be found in [2, p.126]. We shall now derive another generalization of the parallelogram law, that will give us a characterization of trapezoids as a special case. This equality was stated in [4, p.249], but Dostor's derivation was very scarce.

Theorem 10. *If a convex quadrilateral has consecutive sides a, b, c, d and diagonals p, q, then*

$$p^2 + q^2 = b^2 + d^2 + 2ac\cos\xi$$

where ξ is the angle between the extensions of the sides a and c.

Proof. In a convex quadrilateral $ABCD$, let the extensions of AB and CD intersect at J. Other notations are as in Figure 4, where $AC = p$, $BD = q$, $AB = a$, and $AE = x$. We construct GC parallel to AB. Then $\angle DCG = \angle BJC$. We also have $EF = GC = c\cos\xi$, $DG = c\sin\xi$, $ED = h - c\sin\xi$, and $FB = a - c\cos\xi - x$.

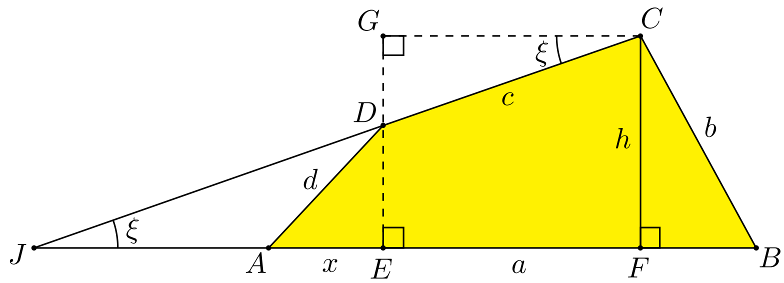

Figure 4. The alternate angles ξ

Applying the Pythagorean theorem in triangles ACF, BDE, BCF, AED, we get respectively

$$p^2 = h^2 + (x + c\cos\xi)^2, \tag{4}$$

$$q^2 = (a - x)^2 + (h - c\sin\xi)^2, \tag{5}$$

$$b^2 = h^2 + (a - c\cos\xi - x)^2, \tag{6}$$

$$d^2 = x^2 + (h - c\sin\xi)^2. \tag{7}$$

Expanding the parentheses and adding (4) and (5), we get

$$p^2 + q^2 = 2(h^2 + x^2) + 2x(c\cos\xi - a) + a^2 + c^2 - 2hc\sin\xi. \tag{8}$$

From (6) and (7),

$$b^2 + d^2 = 2(h^2 + x^2) + a^2 + c^2 - 2hc\sin\xi - 2ac\cos\xi + 2x(c\cos\xi - a). \tag{9}$$

Comparing (9) and (8), we see that

$$b^2 + d^2 = p^2 + q^2 - 2ac\cos\xi$$

and the equation in the theorem follows. □

Corollary 11. *A convex quadrilateral with consecutive sides a, b, c, d and diagonals p, q is a trapezoid with parallel sides a and c if and only if*

$$p^2 + q^2 = b^2 + d^2 + 2ac.$$

Proof. This characterization is a direct consequence of Theorem 10, since the quadrilateral is a trapezoid if and only if $\xi = 0$. □

The next two theorems concerns the distances between the midpoints of the diagonals and the midpoints of two opposite sides (a bimedian).

Theorem 12. *A convex quadrilateral is a trapezoid with parallel sides a and c if and only if the distance v between the midpoints of the diagonals has the length*

$$v = \frac{|a - c|}{2}.$$

Proof. Inserting the equation in Corollary 11 into (3), we get that a convex quadrilateral is a trapezoid if and only if

$$a^2+b^2+c^2+d^2=b^2+d^2+2ac+4v^2 \quad\Leftrightarrow\quad (a-c)^2=4v^2.$$

Hence we get the characterization $v=\frac{1}{2}|a-c|$. $\square$

Remark. According to the formula, the diagonals bisect each other ($v=0$) if and only if $a=c$. In this case the quadrilateral is a parallelogram, which is a special case of a trapezoid within the inclusive definition.

Theorem 13. *A convex quadrilateral with consecutive sides a, b, c, d is a trapezoid with parallel sides a and c if and only if the bimedian n that connects the midpoints of the sides b and d has the length*

$$n=\frac{a+c}{2}.$$

Proof. The length of the bimedian n that connects the midpoints of the sides b and d in a convex quadrilateral is given by

$$4n^2=p^2+q^2+a^2-b^2+c^2-d^2$$

according to [3, p.231] and post no 2 at [5] (both with other notations). Substituting p^2+q^2 from Corollary 11, we get that a convex quadrilateral is a trapezoid if and only if

$$4n^2=b^2+d^2+2ac+a^2-b^2+c^2-d^2 \quad\Leftrightarrow\quad 4n^2=(a+c)^2.$$

Hence $n=\frac{1}{2}(a+c)$. $\square$

The last characterization on sides and distances is about formulas for the length of the diagonals.

Theorem 14. *A convex quadrilateral $ABCD$ with consecutive sides a, b, c, d is a trapezoid with $a\parallel c$ and $a\neq c$ if and only if the length of the diagonals AC and BD are respectively*

$$p=\sqrt{\frac{ac(a-c)+ad^2-cb^2}{a-c}},$$

$$q=\sqrt{\frac{ac(a-c)+ab^2-cd^2}{a-c}}.$$

Proof. We prove the second formula first. Using the law of cosines in the two triangles formed by diagonal $BD=q$ in a convex quadrilateral, we have $d^2=a^2+q^2-2aq\cos u$ and $b^2=c^2+q^2-2cq\cos v$ (see Figure 5). Thus

$$\cos u=\frac{a^2+q^2-d^2}{2aq}$$

and

$$\cos v=\frac{c^2+q^2-b^2}{2cq}.$$

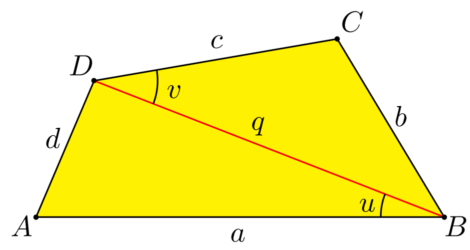

Figure 5. A diagonal and two alternate angles

The quadrilateral is a trapezoid with $a \parallel c$ if and only if $u = v$, which is equivalent to $\cos u = \cos v$. This in turn is equivalent to

$$\frac{a^2 + q^2 - d^2}{2aq} = \frac{c^2 + q^2 - b^2}{2cq}$$

which we can rewrite as

$$ac(a - c) + ab^2 - cd^2 = (a - c)q^2.$$

Now if $a \neq c$, the second formula follows.

The first formula can be proved in the same way, or we can use symmetry and need only to make the change $b \leftrightarrow d$ in the formula we just proved. □

Remark. The quadrilateral is a trapezoid with $a \parallel c$ and $a = c$ if and only if it is a parallelogram. In that case the sides alone do not uniquely determine neither the quadrilateral nor the length of the diagonals.

5. A collinearity characterization

The following theorem has been stated as a collinearity, but another possibility is to state it as a concurrency: *a convex quadrilateral is a trapezoid if and only if the two diagonals and one bimedian are concurrent, in which case the two sides that the bimedian connects are parallel.* The proof of the converse is cited from [11].

Theorem 15. *Two opposite sides in a convex quadrilateral are parallel if and only if the midpoints of those sides and the intersection of the diagonals are three collinear points.*

Proof. $(\Rightarrow)$ In a trapezoid, let E and G be the midpoints of the sides AB and CD respectively, and P the intersection of the diagonals. Triangles CDP and ABP are similar due to two pairs of equal angles (see Figure 6). Note that PG and PE are medians in those triangles, but we do not yet know that $\angle DPG = \angle BPE$. This is what we shall prove. From the similarity, we get

$$\frac{PD}{PB} = \frac{CD}{AB} = \frac{2GD}{2EB} = \frac{GD}{EB}.$$

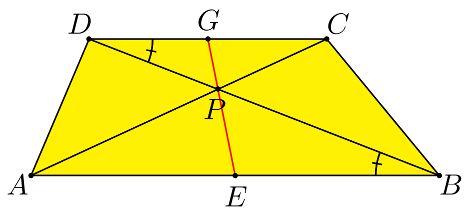

Figure 6. Are E, P and G collinear?

Also $\angle PDG = \angle PBE$, so triangles PDG and PBE are similar. Hence $\angle DPG = \angle BPE$, and since BPD is a straight line, then so is EPG.

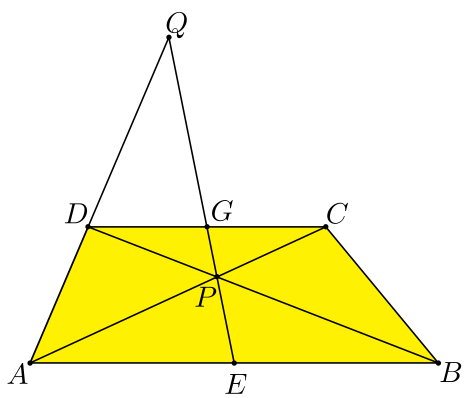

Figure 7. Is $ABCD$ a trapezoid?

($\Leftarrow$) In a convex quadrilateral $ABCD$ where E and G are the midpoints of AB and CD and P is the intersection of the diagonals, we know that E, P and G are collinear. We shall prove that AB and CD are parallel. Extend AD and EG to intersect at Q (see Figure 7). We apply Menelaus' theorem to triangles ABD and ACD using the transversal $EPGQ$. Then

$$\frac{AE}{EB}\cdot\frac{BP}{PD}\cdot\frac{DQ}{QA}=1 \tag{10}$$

and

$$\frac{AP}{PC}\cdot\frac{CG}{GD}\cdot\frac{DQ}{QA}=1. \tag{11}$$

Since $AE = EB$ and $CG = GD$, equations (10) and (11) yields

$$\frac{BP}{PD}=\frac{AP}{PC}.$$

This equality states that the diagonals divide each other in the same ratio, which is the well known sufficient condition (2) for the sides AB and CD to be parallel. $\square$

6. Can all convex quadrilaterals be folded into a trapezoid?

A convex quadrilateral is not uniquely determined by its sides alone. This means that there can be different types of quadrilaterals having the same consecutive sides.[5] Let us make a model of a convex quadrilateral as four very thin rods connected by hinges at their endpoints. We assume that the length of any rod is shorter that the sum of the other three, which ensures that the rods can be the sides of a convex quadrilateral. What we shall explore is if it's always possible to fold the model into a trapezoid?

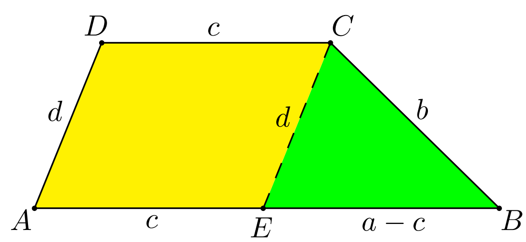

Figure 8. $AECD$ is a parallelogram

In a general trapezoid where $a \parallel c$ and $a \neq c$, we construct the triangle BCE in Figure 8 such that $CE \parallel DA$. This triangle exists whenever its sides satisfy the three triangle inequalities $a-c < b+d$, $d < a-c+b$, and $b < a-c+d$. The first of these is always satisfied if the quadrilateral exists. The second and third can be merged into

$$|a-c| > |b-d|,$$

which is a necessary condition for $a \parallel c$ when $a \neq c$. But it is also a sufficient condition, since if it is satisfied, it is possible to construct the triangle BCE and then the trapezoid. In the same way, we have that

$$|a-c| < |b-d|$$

is a necessary and sufficient condition for $b \parallel d$ when $b \neq d$. Thus the only case when we can't fold a convex quadrilateral into a trapezoid is when

$$|a-c| = |b-d|.$$

This is the characterization for when the quadrilateral has an excircle (so it is an extangential quadrilateral) according to [8, p.64].

Let us examine why we can't get a trapezoid in this case. We use the "semi factored" version of Heron's formula for the area T of a triangle to get a formula for the altitude in a trapezoid. In a triangle with sides x, y, z, the altitude h to the side x has the length

$$h = \frac{2T}{x} = \frac{\sqrt{((y+z)^2 - x^2)\,(x^2 - (y-z)^2)}}{2x}.$$

[5]For instance, a quadrilateral having the sides a, b, a, b can be either a (general) parallelogram or a rectangle.

The triangle BCE has the same altitude as the trapezoid. Inserting $x = a - c$, $y = b$, and $z = d$ yields the trapezoid altitude

$$h = \frac{\sqrt{((b+d)^2 - (a-c)^2)\,((a-c)^2 - (b-d)^2)}}{2|a-c|}$$

which is valid when $a \neq c$. Here we see that when $|a - c| = |b - d|$, the trapezoid altitude is zero. This means that the trapezoid has collapsed into a line segment. If we don't consider that degenerate case to be a trapezoid, this is the reason why an extangential quadrilateral (with a finite exradius) can never be folded into a trapezoid.

Finally we have the case $|a - c| = |b - d| = 0$. Then both pairs of opposite sides have equal length, so the quadrilateral is a parallelogram.[6] This is already a trapezoid (within the inclusive definition), so no folding is needed.

We conclude by stating the conclusions above in the following theorem.

Theorem 16. *If four line segments a, b, c, d have the property that any one of them is shorter than the sum of the other three, then they can always constitute the consecutive sides of a non-degenerate trapezoid except when $|a-c| = |b-d| \neq 0$. In that case they will be the sides of an extangential quadrilateral.*

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[6]Note that parallelograms can be considered to be extangential quadrilateral with infinite exradius, see [8, p.76].