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The Most Inaccessible Point of a Convex Domain

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Abstract. The inaccessibility of a point p in a bounded domain $D \subset \mathbb{R}^n$ is the minimum of the lengths of segments through p with boundary at ∂D. The points of maximum inaccessibility I_D are those where the inaccessibility achieves its maximum. We prove that for strictly convex domains, I_D is either a point or a segment, and that for a planar polygon I_D is in general a point. We study the case of a triangle, showing that this point is not any of the classical notable points.

1. Introduction

The story of this paper starts when the second author was staring at some workers spreading cement over the floor of a square place to construct a new floor over the existing one. The procedure was the following: first they divided the area into triangular areas (actually quite irregular triangles, of around 50 square meters of area). They put bricks all along the sides of the triangles and then poured the liquid cement in the interior. To make the floor flat, they took a big rod of metal, and putting it over the bricks on two of the sides, they moved the rod to flatten the cement. Of course, they had to be careful as they were reaching the most inner part of the triangle.

The question that arose in this situation is: *What is the minimum size for the rod? Even more, which is the most inaccessible point, i.e. the one that requires the full length of the rod? Is it a notable point of the triangle?*

The purpose of this paper is to introduce the concept of maximum inaccessibility for a domain. This is done in full generality for a bounded domain in $\mathbb{R}^n$. The inaccessibility function $\mathbf{r}$ assigns to a point of the domain D the minimum length of a segment through it with boundary in ∂D. We introduce the sets $D_r = \{x \mid \mathbf{r}(x) > r\}$ and the most inaccessible set I_D given by the points where the inaccessibility function achieves its maximum value (the notion has to be suitable modified for the case where $\mathbf{r}$ only has suppremum).

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Then we restrict to convex domains to prove convexity properties of the sets D_r and I_D. For strictly convex domains, I_D is either a point or a segment. For planar convex domains not containing pairs of regular points with parallel tangent lines (e.g. polygons without parallel sides), I_D is a point. In some sense, domains for which I_D is not a point are of very special nature. When $I_D = \{p_D\}$ is a point, we call p_D *the point of maximum inaccessibility* of D.

In the final section, we shall study in detail the case of a polygonal domain in the plane, and more specifically the case of a triangle, going back to the original problem. One of the results is that the point p_T, for a triangle T, is not a notable point of T. It would be nice to determine explicitly this point in terms of the coordinates of the vertices. We do it in the case of an isosceles triangle.

2. Accessibility for domains

Let $D \subset \mathbb{R}^n$ be a bounded domain, that is an open subset such that $\overline{D}$ is compact. Clearly also ∂D is compact. For a point $p \in D$, we consider the function:

$$f_p : S^{n-1} \to \mathbb{R}_+ \,,$$

which assigns to every unit vector v the length $l(\gamma)$ of the segment γ given as the connected component of $(p + \mathbb{R}v) \cap D$ containing p.

Lemma 1. *f_p is lower-semicontinuous, hence it achieves its minimum.*

Proof. Let us introduce some notation: for $p \in D$ and $v \in S^{n-1}$, we denote $\gamma_{p,v}$ the connected component of $(p + \mathbb{R}v) \cap D$ containing p. (So $\overline{\gamma_{p,v}} = [P, Q]$ for some $P, Q \in \partial D$.) Now define the function

$$H : D \times S^{n-1} \to \mathbb{R}_+ \,,$$

by $H(p, v) = f_p(v)$. Let us see that H is lower-semicontinuous (see [3] for general definitions of continuity). Suppose that $(p_n, v_n) \to (p, v)$. Let $\overline{\gamma_{p_n,v_n}} = [P_n, Q_n]$, where $P_n, Q_n \in \partial D$. As ∂D is compact, then there are convergent subsequences (which we denote as the original sequence), $P_n \to P$, $Q_n \to Q$. Clearly $P, Q \in \partial D$. Let γ be the open segment with $\overline{\gamma} = [P, Q]$. Then $p \in \gamma \subset (p + \mathbb{R}v)$. So $\gamma_{p,v} \subset \gamma$ and

$$H(p_n, v_n) = l(\gamma_{p_n,v_n}) = ||P_n - Q_n|| \to ||P - Q|| = l(\gamma) \geq l(\gamma_{p,v}) = H(p, v) \,.$$

Clearly, $f_p(v) = H(p, v)$, obtained by freezing p, is also lower-semicontinuous. □

Remark. In Lemma 1, if D is moreover convex, then H is continuous. This follows from the observation that a closed segment $\sigma = [P, Q]$ with endpoints $P, Q \in \partial D$ either is fully contained in ∂D or $\sigma \cap \partial D = \{P, Q\}$. The segment γ in the proof of Lemma 1 has endpoints in ∂D and goes through p, therefore it coincides with $\gamma_{p,v}$. So $H(p_n, v_n) \to H(p, v)$, proving the continuity of H.

We say that a point $p \in D$ is *r-accessible* if there is a segment of length at most r with boundary at ∂D and containing p. Equivalently, let

$$\mathbf{r}(p) = \min_{v \in S^{n-1}} f_p(v) \,,$$

which is called *accessibility* of p. Then p is r-accessible if $\mathbf{r}(p) \leq r$. Extend $\mathbf{r}$ to $\overline{D}$ by setting $\mathbf{r}(p) = 0$ for $p \in \partial D$.

Proposition 2. *The function* $\mathbf{r} : \overline{D} \to \mathbb{R}_{\geq 0}$ *is lower-semicontinuous.*

Proof. We first study the function $\mathbf{r} : D \to \mathbb{R}_+$. As $\mathbf{r}(p) = \min_v H(p, v)$, the lower-semicontinuity of H gives the lower-semicontinuity of $\mathbf{r}$: If $p_n \to p$, take v_n such that $\mathbf{r}(p_n) = H(p_n, v_n)$. After taking a subsequence, we can assume that $(p_n, v_n) \to (p, v)$. So

$$\underline{\lim}\, \mathbf{r}(p_n) = \underline{\lim}\, H(p_n, v_n) \geq H(p, v) \geq \mathbf{r}(p)\,,$$

as required.

Finally, as we define $\mathbf{r}(p) = 0$ if $p \in \partial D$, those points give no problem to lower-semicontinuity. □

We have some easy examples where f_p or $\mathbf{r}$ are not continuous. For instance, if we consider the domain

$$D = \{(x, y) | x^2 + y^2 < 1, x \leq 0\} \cup \{(x, y) | x^2 + y^2 < 4, x > 0\}\,,$$

and let $p = (0, 0)$. Then $f_p : S^1 \to \mathbb{R}^+$ has constant value 3 except at the horizontal vectors where it has value 2. Also $\mathbf{r}$ is not continuous, since $\mathbf{r}(p) = 2$, but $\mathbf{r}((\epsilon, 0)) \approx 3$, for $\epsilon > 0$ small.

Remark. If D is convex, then $\mathbf{r} : D \to \mathbb{R}_+$ is continuous. Let $p_n \to p$. Take w so that $H(p, w) = \mathbf{r}(p)$. Then $\mathbf{r}(p) = H(p, w) = \lim H(p_n, w) \geq \overline{\lim}\, \mathbf{r}(p_n)$, using the continuity of H and $H(p_n, w) \geq \mathbf{r}(p_n)$. So $\mathbf{r}$ is upper-semicontinuous, and hence continuous.

The function $\mathbf{r} : \overline{D} \to \mathbb{R}_{\geq 0}$ may not be continuous, even for convex domains. Take a semicircle $\{(x, y) | x^2 + y^2 < 1, x > 0\}$. Then $\mathbf{r}((\epsilon, 0)) = 1$, for $x > 0$ small, but $\mathbf{r}((0, 0)) = 0$.

We introduce the sets:

$$\begin{aligned} D_r &= \{p \in D \,|\, \mathbf{r}(p) > r\}, \\ E_r &= \overline{\{p \in D \,|\, \mathbf{r}(p) \geq r\}}. \end{aligned}$$

D_r is open by Proposition 2, and E_r is compact. The function $\mathbf{r}$ is clearly bounded, so it has a suppremum.

Definition. We call $R = \sup \mathbf{r}$ the *inaccessibility* of D. We call

$$I_D := \bigcap_{r < R} E_r$$

the set of points of *maximum inaccessibility* of D.

The set I_D may intersect the boundary of D. For instance, $D = \{(x, y) | x^2 + y^2 < 1, x > 0\}$. Then $R = 1$. It can be seen that $I_D = E_R = \{(x, 0) | 0 \leq x \leq \frac{\sqrt{3}}{2}\}$.

Moreover, I_D can be a point of the boundary. Take $D = \{(x,y)|\frac{x^2}{4} + y^2 < 1\} - \{(x,0)|x \leq 0\}$. Then $R = 2$, and $I_D = \{(0,0)\}$. The sets D_r, for $1 < r < 2$ are petals with vertex at the origin.

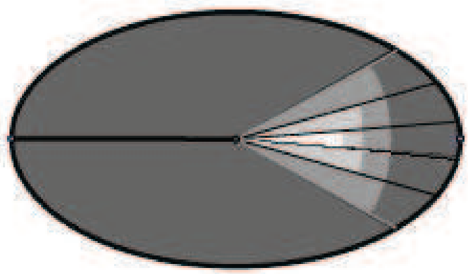

Figure 1. The sets D_r for the ellipse with a long axis removed. The set I_D is in the boundary

Note that **r** does not achieve the maximum is equivalent to $I_D \subset \partial D$. This does not happen for convex D, as will be seen in the next section.

3. Convex domains

From now on, we shall suppose that D is a convex bounded domain (see [4] for general results on convex sets). This means that if $x, y \in D$, then the segment $[x, y]$ is completely included in D. There are several easy facts: $\overline{D}$ is a compact convex set, the interior of $\overline{D}$ is D, and $\overline{D}$ is the convex hull of ∂D.

There is an alternative characterization for convex sets. Let v be a unit vector in $\mathbb{R}^n$. Then the function $f(x) = \langle x, v\rangle$ achieves its maximum in ∂D, say c. Then $f(x) \leq c$ for $x \in D$. Consider the half-space

$$H_v^- = \{x \in \mathbb{R}^n \mid f(x) < c\}\,.$$

Then $D \subset H_v^-$. We call

$$H_v = \{x \in \mathbb{R}^n \mid f(x) = c\}$$

a *supporting hyperplane* for D (see [2, p. 129]). Note that $\partial D \cap H_v \neq \varnothing$. Let also $H_v^+ = \{x \in \mathbb{R}^n \mid f(x) > c\}$.

Lemma 3. *The convex set D is the intersection*

$$\bigcap_{|v|=1} H_v^- \,,$$

and conversely, any such intersection is a convex set. Moreover,

$$\overline{D} = \bigcap_{|v|=1} \overline{H}_v^- \,.$$

Proof. The second assertion is clear, since the intersection of convex sets is convex.

For the first assertion, we have the trivial inclusion $D \subset \bigcap_{|v|=1} H_v$. Now suppose $p \notin D$. We have two cases:

- $p \notin \overline{D}$. Then take $q \in \overline{D}$ such that $d(p,q)$ achieves its minimum, say $s > 0$. Let v be the unit vector from p to q. Let H_v be the hyperplane through q determined by v. It is enough to see that the half-space H_v^+ is disjoint from D, since $p \in H_v^+$. Suppose that $x \in D \cap H_v^+$. Then the segment from q to x should be entirely included in $\overline{D}$, but it intersects the interior of the ball of centre p and radius s. This contradicts the choice of q.
- $p \in \partial D$. Consider $p_n \to p$, $p_n \notin \overline{D}$. By the above, there are $q_n \in \partial D$ and vectors v_n such that $D \subset H_{v_n}^- = \{\langle x - q_n, v_n\rangle < 0\}$. We take subsequences so that $q_n \to q \in \partial D$ and $v_n \to v$. So $D \subset \overline{H_v^-} = \{\langle x - q, v\rangle \leq 0\}$. But D is open, so $D \subset H_v^-$. Moreover, as $d(p_n, D) \to 0$, then $d(p_n, q_n) \to 0$, so $p = q$, and the hyperplane determining H_v goes through p, so $p \notin H_v^-$ (actually $p \in H_v$).

□

Remark. The proof of Lemma 3 shows that if $p \in \partial D$, then there is a supporting hyperplane H_v through p. We call it a *supporting hyperplane at* p, and we call v a *supporting vector at* p. When a point p has several supporting hyperplanes, it is called a *corner point*. The set

$$\mathbb{R}_+ \cdot \{v | v \text{ is supporting vector at } p\} \subset \mathbb{R}^n$$

is convex. Note that if ∂D is piecewise smooth, and $p \in \partial D$ is a smooth point, then p is non-corner and the tangent space to ∂D is the supporting hyperplane.

Now we want to study the sets D_r and E_r. First note that $\mathbf{r}$ is continuous on D. Therefore

$$E_r \cap D = \{x | \mathbf{r}(x) \geq r\}$$

is closed on D.

Proposition 4. *If D is convex, then $\mathbf{r}$ achieves its supremum R at D. Moreover, $I_D \cap D = E_R \cap D = \{p \,|\, \mathbf{r}(p) = R\}$ and $I_D = E_R$ (which is the closure of $E_R \cap D$).*

Proof. Let $p \in I_D \cap \partial D$, and take a supporting hyperplane H_v at p. We claim that the open semiball $B_R(p) \cap H_v^- \subset D$. If not, then there is a point $q \in \partial D$, $d(p,q) < R$, $q \in H_v^-$. Then all the segments $[q, x]$, with $x \in B_\epsilon(p) \cap \partial D$, have length $\leq r_0 <$ (for suitable small ϵ). Therefore, there is neighbourhood U of p such that $\mathbf{r}(x) \leq r_0$, $\forall x \in U \cap D$. Contradiction.

Now all points in the ray $p + tv$, $t \in (0, \epsilon)$ are not r-accessible for any $r < R$. Therefore they belong to $\mathbf{r}^{-1}(R)$. So p is in the closure of $\mathbf{r}^{-1}(R)$, which is E_R.

Therefore $I_D \cap \partial D \subset E_R$. Also, by continuity of $\mathbf{r}$ on D, we have that $E_r \cap D = \mathbf{r}^{-1}[r, \infty)$. Thus $I_D \cap D = \bigcap_{r<R} \mathbf{r}^{-1}[r, \infty) = \mathbf{r}^{-1}(R) = E_R \cap D$. All together, $I_D \subset E_R$. Obviously, $E_R \cap D = \mathbf{r}^{-1}(R) \subset I_D$, and taking closures,

$E_R = \overline{E_R \cap D} \subset I_D$. So $I_D = E_R$. Finally, as I_D is always non-empty, we have that R is achieved by $\mathbf{r}$. □

Now we prove a useful result. Given two points P, Q, we denote $\overrightarrow{PQ} = Q - P$ the vector from P to Q.

Lemma 5. *Let $p \in D$ and $r = \mathbf{r}(p)$. Let $[P, Q]$ be a segment of length r with $P, Q \in \partial D$ and $p \in [P, Q]$. Let v_P, v_Q be supporting vectors at P, Q respectively. Then*

(1) *If v_P, v_Q are parallel, then: $v_P = -v_Q$, P, Q are non-corner points, $\overrightarrow{PQ} \parallel v_P$, and $r = R$.*
(2) *If v_P, v_Q are not parallel, then: $\overrightarrow{PQ}$ is in the plane π spanned by them, there is a unit vector $v \perp \overrightarrow{PQ}$, $v \in \pi$, such that for $H_v = \{\langle x - p, v\rangle = 0\}$, it is $E_r \subset \overline{H}_v^-$; and $\mathbf{r}(x) < r$ for $x \in [P, Q] - \{p\}$ close to p.*

Proof. (1) Suppose first that v_P, v_Q are parallel. So D is inside the region between the parallel hyperplanes H_{v_P} and H_{v_Q}. Clearly $v_P = -v_Q$. Let $x \in D$, and draw the segment parallel to $[P, Q]$ through x with endpoints in the hyperplanes. It has length r. The intersection of this segment with D is of length $\leq r$. Therefore $\mathbf{r}(x) \leq r$, for all $x \in D$, so $R = r$.

If $\overrightarrow{PQ}$ is not parallel to v_P, take a small vector w such that $\langle w, v_P\rangle = 0$, $\langle w, \overrightarrow{PQ}\rangle > 0$. Let $t \in (0, 1)$ so that $p = (1-t)P + tQ$. Then $P' = P + tw \in H_{v_P}$ and $Q' = Q - (1-t)w \in H_{v_Q}$, and $p \in [P', Q']$. First, $||\overrightarrow{P'Q'}|| = ||\overrightarrow{PQ} - w|| < ||\overrightarrow{PQ}|| = r$. Also $P', Q' \notin D$, so the segment $[P', Q'] \cap D$ is of length at most $||\overrightarrow{P'Q'}||$. Therefore $\mathbf{r}(p) < r$, a contradiction.

The assertion that P, Q are non-corner points is proved below.

(2) Suppose now that v_P, v_Q are not parallel. Again D is inside the region between the hyperplanes H_{v_P} and H_{v_Q}. Let π be the plane spanned by v_P, v_Q. Let w be the projection of $\overrightarrow{PQ}$ on the orthogonal complement to π, and suppose $w \neq 0$. Clearly $\langle w, \overrightarrow{PQ}\rangle > 0$. Let $t \in (0, 1)$ so that $p = (1-t)P + tQ$. Then $P' = P + tw \in H_{v_P}$ and $Q' = Q - (1-t)w \in H_{v_Q}$, and $p \in [P', Q']$. So $l([P', Q'] \cap D) \leq ||\overrightarrow{P'Q'}|| < ||\overrightarrow{PQ}|| = r$, which is a contradiction. Therefore $\overrightarrow{PQ} \in \pi$.

Let $v \in \pi$ be a unit vector such that $v \perp \overrightarrow{PQ}$. Now consider unit vectors e_1, e_2 in π so that $e_1 \perp v_P$, $e_2 \perp v_Q$, The vector

$$u = \frac{1}{\langle e_1, v\rangle \langle e_2, v\rangle}(\langle e_1, v\rangle e_2 - \langle e_2, v\rangle e_1)$$

is perpendicular to v, hence parallel to $\overrightarrow{PQ}$. We arrange that $\langle u, \overrightarrow{PQ}\rangle < 0$ by changing the sign of v if necessary. Denote $H_v = \{\langle x - p, v\rangle = 0\}$. Let us see that this satisfies the statement. Consider w so that $\langle w, v\rangle > 0$. Let $w_1 = \frac{\langle w, v\rangle}{\langle e_1, v\rangle} e_1 \in$

H_{v_P} and $w_2 = \frac{\langle w,v\rangle}{\langle e_2,v\rangle} e_2 \in H_{v_Q}$. Then

$$w_2 - w_1 = \frac{\langle w, v\rangle}{\langle e_1, v\rangle \, \langle e_2, v\rangle}(\langle e_1, v\rangle e_2 - \langle e_2, v\rangle e_1) = \langle w, v\rangle u\,,$$

so $\langle w_2 - w_1, \overrightarrow{PQ}\rangle = \langle w, v\rangle\langle u, \overrightarrow{PQ}\rangle < 0$. Set $P' = P + w_1$, $Q' = Q + w_2$. So $[P', Q']$ is parallel to $[P, Q]$, it goes through $p + w$, and it is shorter than $[P, Q]$. So $H_v^+ \cap E_r = \varnothing$.

For the last assertion, we write $\overrightarrow{PQ} = a_1 e_1 + a_2 e_2$, where $a_1, a_2 \neq 0$. Let $P' = P + x e_1 \in H_{v_P}$, $Q' = Q + y e_2 \in H_{v_Q}$. The condition $p \in [P', Q']$ is equivalent to p, P', Q' being aligned, which is rewritten as

$$xy + (1 - t)a_2 x - t a_1 y = 0\,. \tag{1}$$

Now, the condition $||\overrightarrow{P'Q'}|| = ||\overrightarrow{PQ} + y e_2 - x e_1|| < ||\overrightarrow{PQ}|| = r$ for small r is achieved if $\langle \overrightarrow{PQ}, y e_2 - x e_1\rangle < 0$. This is a linear equation of the form $\alpha_1 x + \alpha_2 y < 0$. The intersection of such half-plane with the hyperbola (1) is non-empty except if $\alpha_1 x + \alpha_2 y = 0$ is tangent to the hyperbola at the origin. So (α_1, α_2) is a multiple of $((1 - t)a_2, -t a_1)$. This determines t uniquely. So for $s \neq t$ (and close to t), we have that $p_s = (1 - s)P + sQ$ satisfies $\mathbf{r}(p_s) < r$. (Note incidentally, that it cannot be $\overrightarrow{PQ} \parallel v_P$. If so, then $\alpha_1 = 0$, and then $(1 - t)a_2 = 0$, so $t = 1$, which is not possible.)

Now we finish the proof of (1). Suppose that Q is a corner point. Then we can choose another supporting vector v'_Q. On the one hand $\overrightarrow{PQ} \parallel v_P = -v_Q$. On the other, as $v_P \not\parallel v'_Q$, we must have $\overrightarrow{PQ} \not\parallel v_P$, by the discussion above. Contradiction. □

Theorem 6. *The sets D_r, E_r are convex sets, for $r \in [0, R]$, $R = \max \mathbf{r}$. Moreover, $\partial D_r \cap D$ is $\mathbf{r}^{-1}(r)$, for $r \in (0, R)$.*

Proof. The assertion for E_r follows from that of D_r: knowing that D_r is convex, then

$$E_r = \overline{\bigcap_{\epsilon > 0} D_{r-\epsilon}}$$

is convex since the intersection of convex sets is convex, and the closure of a convex set is convex.

Let $0 < r < R$, and let us see that D_r is convex. Let $p \notin D_r$. Then $\mathbf{r}(p) \leq r$. By Lemma 5, there is a segment $[P, Q]$ of length r, with $P, Q \in \partial D$, $v_P \not\parallel v_Q$, and a vector $v \perp \overrightarrow{PQ}$ such that $E_r \subset \overline{H}_v^-$. Then $D_r \subset H_v^-$, and $p \notin H_v^-$. So D_r is the intersection of half-spaces, hence convex.

For the last assertion, note that the continuity of $\mathbf{r}$ implies that $D \cap \partial D_r \subset \mathbf{r}^{-1}(r)$. For the reversed inclusion, suppose that $\mathbf{r}(p) = r$, but $p \notin \partial D_r$. Then there is some $\epsilon > 0$ so that $B_\epsilon(p) \subset \mathbf{r}^{-1}(0, r]$. Now $\mathbf{r}^{-1}[r, \infty)$ is convex, so it is the closure of its interior, call it V. Therefore $V \cap B_\epsilon(p)$ is open, convex, and contains p in its adherence. Moreover $V \cap B_\epsilon(p) \subset \mathbf{r}^{-1}(r)$. But this is impossible, since an easy consequence of Lemma 5 is that $\mathbf{r}^{-1}(r)$ has no interior for any $r \in (0, R)$. □

Proposition 7. *Suppose D is a convex* planar *set. Let $r \in (0, R)$. Then ∂D_r is the envelope of the segments of length r with endpoints at ∂D.*

Proof. As we proved before, the boundary of D_r is $\mathbf{r}^{-1}(r)$, so the points of ∂D_r are r-accessible, but not r'-accessible for $r' < r$. Let $p \in \partial D_r$ be a smooth point. Then there is a segment of length r and D_r is at one side of it. Therefore the segment is tangent to ∂D_r at p. □

4. Strictly convex domains

Recall that D is *strictly convex* if there is no segment included in its boundary. We assume that D is strictly convex in this section. Therefore, for each unit vector v, there is a unique point of contact $H_v \cap \partial D$. We define the function

$$g : S^{n-1} \to \partial D\,.$$

Lemma 8. *If D is strictly convex, then g is continuous.*

Proof. Let $v_n \in S^{n-1}$, $v_n \to v$. Consider $p_n = g(v_n) \in \partial D$, and the supporting hyperplane $\langle x - p_n, v_n\rangle \leq 0$. Let $p = g(v)$, with supporting hyperplane $\langle x - p, v\rangle \leq 0$. After taking a subsequence, we can suppose $p_n \to q \in \partial D$. Now $p \in \overline{D} \implies \langle p - p_n, v_n\rangle \leq 0$, and taking limits, $\langle p - q, v\rangle \leq 0$. On the other hand, $p_n \in \overline{D} \implies \langle p_n - p, v\rangle \leq 0$, and taking limits, $\langle q - p, v\rangle \leq 0$. So $\langle q - p, v\rangle = 0$. By strict convexity, $q = p$, so $g(v_n) \to g(v)$, and g is continuous. □

Now suppose that ∂D is C^1. Then for each point $p \in \partial D$, there is a normal vector $\mathbf{n}(p)$. We have a well defined function

$$\phi : \partial D \to S^{n-1}, \quad \phi(p) = \mathbf{n}(p)\,.$$

Note that $p \in H_{\mathbf{n}(p)} \cap \overline{D}$. Therefore if D is C^1 and strictly convex, both ϕ and g are defined and inverse to each other.

In general, for D convex, there are pseudo-functions $g : S^n \to \partial D$, $\phi : \partial D \to S^n$. A pseudo-function assigns to each point $v \in S^n$ a subset $g(v) \subset \partial D$ in such a way that the graph $\{(v, p) \,|\, p \in g(v)\}$ is closed. The inverse of a pseudo-function is well-defined, and g and ϕ are inverse to each other. The set $\phi(p)$ is the set of supporting vectors at p (see Remark 3).

Lemma 9. *Suppose D strictly convex. For all $0 < r < R$, $\partial D_r \cap \partial D = \varnothing$, so $\partial D_r = \mathbf{r}^{-1}(r)$.*

Proof. Take a point $p \in \partial D$, and let H_v be a supporting hyperplante. Consider a small ball B around p of radius $\leq r/2$. By strict convexity, $d(\partial B \cap D, H) = \epsilon_0 > 0$. Now we claim that $B_{\epsilon_0}(p) \cap D$ does not intersect D_r, so $p \notin \overline{D}_r$. Let $q \in B_{\epsilon_0}(p) \cap D$, and consider a line l parallel to H through q. The segment $l \cap B$ has endpoints $P, Q \in \partial B$. But $d(P, H) = d(Q, H) < \epsilon_0$, so $P, Q \notin D$. So the connected component $[P, Q] \cap D$ has length $< ||\overrightarrow{PQ}|| < r$, and q is r'-accessible for some $r' < r$. □

Corollary 10. *For D strictly convex, $\mathbf{r} : \overline{D} \to \mathbb{R}_{\geq 0}$ is continuous.*

Proof. By Remark 2, $\mathbf{r}$ is continuous on D. The continuity at ∂D follows from the proof of Lemma 9. □

Therefore, if D is strictly convex, then

$$I_D = E_R = \mathbf{r}^{-1}(R)\,.$$

As $I_D \subset D$, we have that I_D does not touch ∂D.

Theorem 11. *Let D be strictly convex. For all $0 < r < R$, D_r is strictly convex.*

Proof. Suppose that ∂D_r contains a segment l. Let p be a point in the interior of l. As it is r-accessible, there is a segment $[P, Q]$ of length r through p, where $P, Q \in \partial D$. By Lemma 5, v_P, v_Q are not parallel, and all points in $[P, Q]$ different from p are r'-accessible for some $r' < r$. Therefore l is transversal to $[P, Q]$. Let H_v be the hyperplane produced by Lemma 5 (2). Then all points at one side of H_v are r'-accessible for some $r' < r$, hence l cannot be transversal to H_v, so $l \subset H_v$.

Now let $x \in l$, $x \neq p$. Consider the segment parallel to $[P, Q]$ through x, call it σ. It has length r and endpoints at H_{v_P}, H_{v_Q}. But D is strictly convex, so it only touches the supporting hyperplanes at one point. Hence $\sigma \cap D$ is strictly contained in σ. Therefore $\mathbf{r}(x) < r$. Contradiction. □

5. Set of maximum inaccessibility

In this section we suppose that D is convex. Then $\mathbf{r}$ is continuous on D and it achieves its maximum R on D. Then $I_D = E_R$ and $I_D \cap D = E_R \cap D = \mathbf{r}^{-1}(R)$, by Proposition 4.

We want to characterize the case where I_D contains interior. Let us see an example where this situation happens. Let D be a rectangle. In this case R is the length of the shortest edge of the rectangle, and we have an open set with $\mathbf{r}(p) = R$ (see Figure 5). Note that it might happen that ∂E_R intersects ∂D.

Proposition 12. *If I_D has non-empty interior, then ∂D contains two open subsets which are included in parallel hyperplanes, which are at distance R.*

Proof. Consider an interior point $p \in I_D$, so $\mathbf{r}(p) = R$. Take a segment $l = [P, Q]$ of length R with endpoints $P, Q \in \partial D$. Let v_P, v_Q be vectors orthogonal to the supporting hyperplanes at P, Q. By Lemma 5, if they are not parallel, then there is a hyperplane through p such that E_R is contained in one (closed) half-space. This is not possible, as p is an interior point of E_R. So v_P, v_Q are parallel, and $\overrightarrow{PQ} \parallel v_P$. Now take any point x close to p, and consider the segment $[P', Q']$ through x parallel to $[P, Q]$, which has endpoints in H_{v_P}, H_{v_Q}. If $[P', Q'] \cap D$ is properly contained in $[P', Q']$, then $\mathbf{r}(x) < R$, which contradicts that $x \in E_R$. So $P' \in H_{v_P}$, $Q' \in H_{v_Q}$, and ∂D contains two open subsets in H_{v_P}, H_{v_Q} around P, Q, respectively. □

Theorem 13. *Let D be a strictly convex bounded domain, $R = \max \mathbf{r}$. Then I_D is a point or a segment.*

Proof. Suppose that I_D is not a point. As it is convex by Theorem 6, it contains a maximal segment σ. Let us see that it cannot contain two different (intersecting) segments. Let $p \in \sigma$ be an interior point of the segment. By Lemma 5, if we draw the segment $[P, Q]$ of length R through p, we have the following possibilities:

- v_P, v_Q are parallel. Then $\overrightarrow{PQ} \parallel v_P$. Then any point $x \notin [P, Q]$ lies in a segment $[P', Q']$ parallel to $[P, Q]$, with $P' \in H_{v_P}$ and $Q' \in H_{v_Q}$. By strict convexity, $l([P', Q'] \cap D) < R$, so $\mathbf{r}(x) < R$. That is, $E_R \subset [P, Q]$.
- v_P, v_Q are non-parallel. Then there is a hyperplane H_v through p such that $E_R \subset \overline{H}_v^-$. As p is an interior point of σ, σ does not cross H_v, so $\sigma \subset H_v$. Now let $x \in \sigma$, and consider the segment $[P', Q']$ parallel to $[P, Q]$ through x, with length R, $P' \in H_{v_P}$ and $Q' \in H_{v_Q}$. If $x \notin [P, Q]$ then strict convexity gives $l([P', Q'] \cap D) < R$, so $\mathbf{r}(x) < R$. That is, $E_R \subset [P, Q]$. Note that Lemma 5 (2) gives in this case that $E_R = \{p\}$.

□

Let us see an example where E_R is a segment. Let D be the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1$, where $2a > 2b$. Then $R = 2b$ and E_R is a segment contained in the short axis, delimited by the intersection of the axis with the perpendicular segments of length R with endpoints in the ellipse.

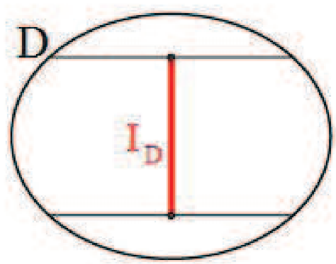

Figure 2. Ellipse

All points $(x, y) \in D$ with $x \neq 0$ can be reached by vertical segments of length $< R = 2b$. Now let $x_0 = b$, $y_0 = b\sqrt{a^2 - b^2}/a$. If $y \in (-b, -y_0) \cup (y_0, b)$ then the point $(0, y)$ is r-accessible (with a horizontal segment) with $r < R$. Now let $y \in [-y_0, y_0]$, and consider a line through $(0, y)$. Let us parametrize it as

$$r(s) = (s\, a\, \cos\theta, y + s\, b\, \sin\theta)\,,$$

with θ fixed. The intersection with the ellipse are given by $s = -\frac{y}{b}\sin\theta \pm \sqrt{1 - \frac{y^2}{b^2}\cos^2\theta}$. So the square of the distance between the two points is

$$\begin{aligned} l(\theta)^2 &= 4(1 - \frac{y^2}{b^2}\cos^2\theta)(a^2\cos^2\theta + b^2\sin^2\theta) \\ &= 4(1 - \frac{y^2}{b^2}T)((a^2 - b^2)T + b^2)\,, \end{aligned}$$

where $T = \cos^2\theta \in [0, 1]$. The minimum of this degree 2 expression on T happens for a negative value of T, therefore, we only need to check the values $T = 0, 1$. For $T = 0$, we get $4b^2$; for $T = 1$, we get $4(1 - \frac{y^2}{b^2})a^2 \geq 4(1 - \frac{y_0^2}{b^2})a^2 = 4b^2$. So $l(\theta)^2 \geq 4b^2$.

A consequence of Theorem 13 is the following: for a strictly convex bounded domain D, if I_D is not a point then there are two non-corner points $P, Q \in \partial D$ with parallel tangent hyperplanes which are moreover perpendicular to $\overrightarrow{PQ}$.

Corollary 14. *Suppose D is a* planar *convex bounded domain (not necessarily strictly convex). If I_D is not a point then there are two non-corner points $P, Q \in \partial D$ with parallel tangent hyperplanes which are moreover perpendicular to $\overrightarrow{PQ}$.*

Proof. Following the proof of Theorem 13, we only have to rule out case (2). As the hyperplane H_v is now of dimension 1, we have $\sigma \subset [P, Q] = H_v \cap \overline{D}$. But Lemma 5 says also that $E_R \cap [P, Q] = \{p\}$. So E_R does not contain a segment, i.e. it is a point. □

So, for a convex polygon D, if it does not have parallel sides, then I_D is a point.

Corollary 14 is not true in dimension ≥ 3. Take a triangle $T \subset \mathbb{R}^2$ and consider $D = T \times [0, L]$ for large L. For T, denote $I_T = \{p\}$. Then D has $I_D = \{p\} \times [a, b]$, for some $0 < a < b < L$. Certainly, there are two parallel faces (base and top), but we slightly move one of them to make them non-parallel, and I_D is still a segment.

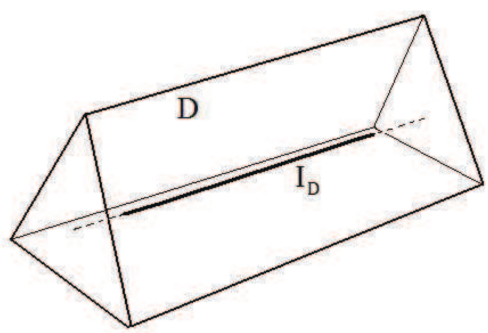

Figure 3. I_D can be positive dimensional

One can make this construction to have I_D of higher dimension (not just a segment), e.g. by considering $T \times [0, L]^N$, $N > 1$.

6. Polygons

In this section we want to study in detail the case of convex polygons in the plane, and to give some answers in the case of triangles. The starting point is the case of a sector.

Lemma 15. *Fix $\lambda \in \mathbb{R}$. Let D be the domain with boundary the half-lines $(x, 0)$, $x \geq 0$ and $(\lambda y, y)$, $y \geq 0$. Let $r > 0$. Then the boundary of D_r is the curve:*

$$\begin{cases} x = r(\cos^3\theta + \lambda(\sin^3\theta + 2\sin\theta\cos^2\theta)) \\ y = r(\sin^3\theta - \lambda\sin^2\theta\cos\theta) \end{cases} \tag{2}$$

Proof. D is not a bounded domain, but the theory works as well in this case. To find the boundary of D_r, we need to take the envelope of the segments of length r with endpoints laying on the half-rays, according to Proposition 7. Two points at $(a, 0)$ and $(\lambda b, b)$ are at distance r if

$$(\lambda b - a)^2 + b^2 = r^2 .$$

So $\lambda b - a = -r\cos\theta$, $b = r\sin\theta$, i.e. $a = \lambda r\sin\theta + r\cos\theta$. The line which passes through $(\lambda b, b)$ and $(a, 0)$ is

$$r\sin\theta\, x + r\cos\theta\, y = r^2\sin\theta\cos\theta + r^2\lambda\sin^2\theta .$$

We are going to calculate the envelope of these lines (see pp. 75-80 in [5]). Take the derivative and solve the system:

$$\begin{cases} r\sin\theta\, x + r\cos\theta\, y = r^2\sin\theta\cos\theta + r^2\lambda\sin^2\theta \\ r\cos\theta\, x - r\sin\theta\, y = -r^2\sin^2\theta + r^2\cos^2\theta + 2r^2\lambda\sin\theta\cos\theta . \end{cases}$$

We easily get the expression in the statement. The region D_r is the unbounded region with boundary the curve (2) and the two half-rays. □

We call the curve in Lemma 15 a λ-*bow* (or just a bow). Let $\lambda = \cot\alpha$, $\alpha \in (0, \pi)$. If $\lambda < 0$, we are dealing with an obtuse angle, and $\theta \in [0, \pi - \alpha]$. If $\lambda = 0$, we have a right angle, and $\theta \in [0, \frac{\pi}{2}]$. Finally, an acute angle happens for $\lambda > 0$. In this case, $\theta \in [\frac{\pi}{2} - \alpha, \frac{\pi}{2}]$. (Note that θ is the angle between the segment and the negative horizontal axis, in the proof of Lemma 15.)

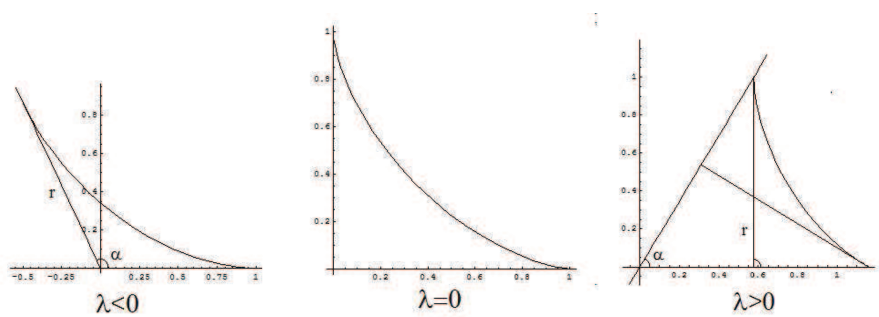

Figure 4. λ-bows with $r = 1$

As an application, we prove the following:

Corollary 16. *Let D be a planar convex polygon. Then the sets D_r, $0 < r < R$, and I_D if it is not a point, have boundaries which are piecewise C^1, and whose pieces are λ-bows and (possibly) segments in the sides of ∂D. In particular, these domains are strictly convex when ∂D_r does not intersect ∂D.*

Proof. Let $l_1, \dots, l_k$ be the lines determined by prolonging the sides of the polygon. Consider l_i, l_j. If they intersect, consider the sector that they determine in which D is contained. Lemma 15 provides us with a (convex) region D_r^{ij}. If l_i, l_j are parallel and $r < d(l_i, l_j)$ then set $D_r^{ij} = D$, and if l_i, l_j are parallel and $r \geq d(l_i, l_j)$ then set $D_r^{ij} = \varnothing$. It is fairly clear that

$$D_r = \bigcap_{i \neq j} D_r^{ij} .$$

To see the last assertion, note that at any smooth point $p \in \partial D_r$, we have strict convexity because of the shape of the bows given in Lemma 15. If $p \in \partial D_r$ is a non-smooth point, then it is in the intersection of two such curves. This means that there are segments σ_1, σ_2 of length r where σ_1 has endpoints at lines l_{i_1}, l_{j_1} and σ_2 has endpoints at lines l_{i_2}, l_{j_2}. Moreover, the endpoints should be actually in the sides of D (otherwise p would be r'-accessible for some $r' < r$). In particular, this means that σ_1, σ_2 cannot be parallel. As such segments are tangent to the bows, the curves intersect transversely at p, and p is a corner point.

A similar statement holds for $I_D = E_R$, when it is not a point, by doing the above reasoning for $r = R$. □

In particular, we see that I_D cannot be a segment for polygons.

For instance, when D is a rectangle of sides $a \geq b$, then $R = b$. We draw the bows at the vertices, to draw the set $I_D = E_R$.

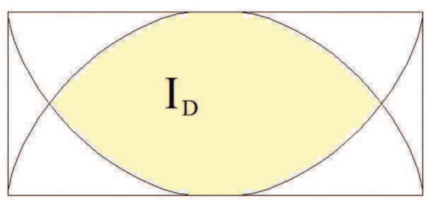

Figure 5. For a rectangle, I_D has interior

Note that I_D intersects ∂D if and only if $a \geq 2b$.

It would be nice to have a function

$$I_D = (I_1, I_2) = I_D((x_1, y_1), (x_2, y_2), \dots, (x_k, y_k)) \in \mathbb{R}^2 ,$$

which assigns the value of I_D given the vertices (x_i, y_i) of a k-polygon. Such function is only defined for polygons with non-parallel sides.

We shall produce the formula for I_D for the case of an *isosceles triangle*. Consider an isosceles triangle of height 1, and base $2\lambda > 0$. Put the vertices at the points $A = (0, 0)$, $B = (2\lambda, 0)$ and $C = (\lambda, 1)$. By symmetry, the point I_D must lie in the vertical axis $x = \lambda$. Moreover, the segment of length R through

I_D tangent to the bow corresponding to C must be horizontal. This means that $I_D = (\lambda, I_2)$ where $\frac{R}{2} = \lambda(1 - I_2)$. So

$$I_D = (I_1, I_2) = (\lambda, I_2(\lambda)) = \left(\lambda, 1 - \frac{R}{2\lambda}\right) .$$

The sector corresponding to A is that of Lemma 15, and the point I_D should lie in its λ-bow, which is the curve given in Lemma 15 for the value $r = R$. Hence

$$\begin{aligned} \lambda &= R(\cos^3\theta + \lambda(\sin^3\theta + 2\sin\theta\cos^2\theta)), \\ 1 - \frac{R}{2\lambda} &= R(\sin^3\theta - \lambda\sin^2\theta\cos\theta) . \end{aligned}$$

Eliminating R, we get

$$\lambda^2 \sin^2\theta\cos\theta + 2\lambda\sin\theta\cos^2\theta + \cos^3\theta - \frac{1}{2} = 0$$

i.e.

$$\lambda = \frac{-2\cos^2\theta + \sqrt{2\cos\theta}}{2\sin\theta\cos\theta} \tag{3}$$

(the sign should be plus, since $\lambda > 0$). Note that for an equilateral triangle, $\lambda = \frac{1}{\sqrt{3}}$, $I_2 = \frac{1}{3}$, $\theta = \frac{\pi}{3}$ and $R = \frac{4}{3\sqrt{3}}$.

Also

$$R = \frac{\lambda}{\cos^3\theta + \lambda(\sin^3\theta + 2\sin\theta\cos^2\theta)} . \tag{4}$$

One can check the following formula:

$$I_2 = 1 - \frac{R}{2\lambda} = \lambda\frac{\sin^3\theta - \lambda\sin^2\theta\cos\theta}{\cos^3\theta + \lambda(\sin^3\theta + 2\sin\theta\cos^2\theta)} . \tag{5}$$

This locates the point $I_D = (\lambda(\theta), I_2(\lambda(\theta)))$.

Remark. Do the change of variables $\cos\theta = \frac{1-u^2}{1+u^2}$, $\sin\theta = \frac{2u}{1+u^2}$, to get algebraic expressions for I_D. It is to be expected that this algebraicity property holds for a general triangle.

Recall the position of the ortocentre, incentre, baricentre and circumcentre [1]

$$\begin{aligned} H &= \left(\lambda, \lambda^2\right) . \\ I &= \left(\lambda, \frac{\lambda}{\lambda + \sqrt{\lambda^2+1}}\right) . \\ G &= \left(\lambda, \frac{1}{3}\right) , \\ O &= \left(\lambda, \frac{1-\lambda^2}{2}\right) . \end{aligned}$$

We draw the height of the point H, I, G, O, I_D as a function of λ :

A simple consequence is that these 5 points are distinct for an isosceles triangle which is not equilateral. We conjecture that this is true for a non-isosceles triangle.

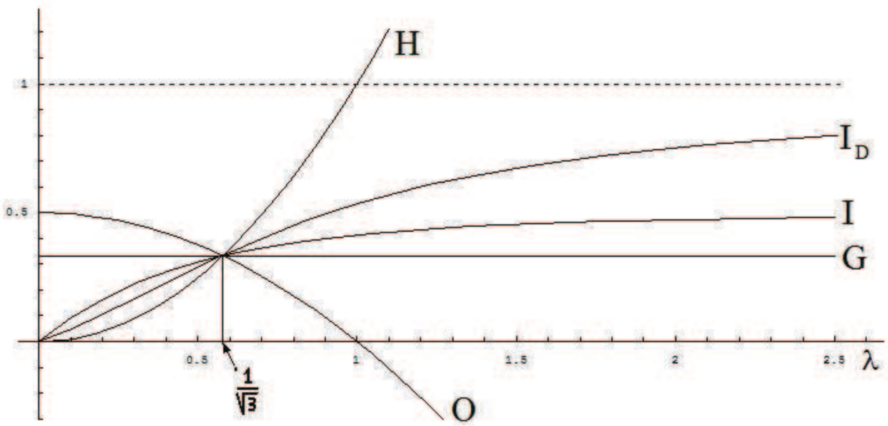

Figure 6. Notable points of a triangle of height 1 and base 2λ

Note the asymptotic for an isosceles triangle. For $\lambda \sim 0$, we have that (3) implies $\cos^3\theta \sim \frac{1}{2}$. Now (4) and (5) give that $R \sim 2\lambda$ and

$$I_2(\lambda) \sim \frac{\sin^3\theta}{\cos^3\theta}\lambda \sim (2^{2/3}-1)^{3/2}\lambda\,.$$

Rescale the triangle to have base $b = 2$ and height $h = \frac{1}{\lambda}$. Then when h is large, the point I_D approaches to be at distance $(2^{2/3}-1)^{3/2} = 0.4502$ to the base, and $R \sim 2$. Also, for $\lambda \to \infty$, we have $I_2(\lambda) \to 1$.

Remark. Consider a rectangle D with vertices $(\pm a, \pm 1)$, with $a \gg 1$. Then I_D has interior (see Figure 5). Moving slightly the vertices at the left, we get an isosceles trapezoid Z_ϵ, with vertices $(-a, \pm(1-\epsilon)), (a, \pm 1)$, for $\epsilon > 0$. Consider the triangle T_ϵ obtained by prolonging the long sides of Z_ϵ, i.e. with vertices $(a - 2a/\epsilon, 0), (a, \pm 1)$. By the above, the point $I_{T_\epsilon} \sim (a - 0.4502, 0)$. As $R \sim 2$, we have that $I_{Z_\epsilon} = I_{T_\epsilon}$.

By symmetry, if we consider the isosceles trapezoid Z'_ϵ with vertices $(-a, \pm 1)$, $(a, \pm(1-\epsilon))$, then $I_{Z'_\epsilon} \sim (-a + 0.4502, 0)$.

The polygons Z_ϵ and Z'_ϵ are nearby, but their points of maximum inaccessibility are quite far apart. So the map $D \mapsto I_D$ cannot be extended continuously (in any reasonable topology) to all polygons with 4 sides.

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