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Perpendicular Bisectors of Triangle Sides

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Abstract. Formulas, in terms of the sidelengths and area, are given for the lengths of the segments of the perpendicular bisectors of the sides of a triangle in its interior. The ratios of perpendicular bisector segments to each other are given, and the ratios of the segments into which the perpendicular bisectors are divided by the circumcenter are considered. Then we ask whether a set of three perpendicular bisector lengths uniquely determines a triangle. The answer is no in general: depending on the set of bisectors, anywhere from zero to four (but no more than four) triangles can share the same perpendicular bisector segments.

1. Introduction

It is well-known that the perpendicular bisectors of the sides of a triangle meet at a single point, which is the center of the circumcircle. Bui [1] gives results for similar triangles associated with the perpendicular bisectors. In this paper we first find formulas for the lengths of the segments of the perpendicular bisectors in the *interior* of a given triangle. Then we study the question of existence of triangles with prescribed lengths of perpendicular bisector segments.

Lemma 1. *The perpendicular bisector segment through the midpoint of one side terminates at a point on the longer of the remaining two sides* (or at their intersection if these sides are equal).

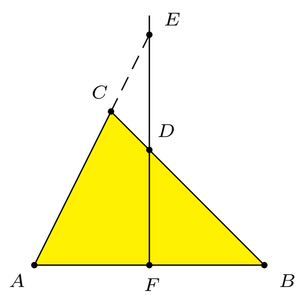

Figure 1

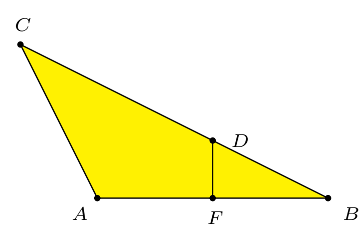

Figure 2

Proof. Let the perpendicular bisector pass through the midpoint F of the side AB of triangle ABC. Consider the case when both angles A and B are acute. In Figure 1, $AF = FB$ and $FE \geq FD$. We show that $BC \geq AC$. Now,

$$\tan A = \frac{EF}{AF} \geq \frac{DF}{AF} = \frac{DF}{BF} = \tan B.$$

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Since $\tan A \geq \tan B$ and angles A and B are both acute, $\angle A \geq \angle B$. It follows that $BC \geq AC$.

The other cases (when one of angles A and B is obtuse or a right angle) are clear (see Figure 2 in the case of an obtuse angle A). $\square$

Lemma 1 will be used in constructing the perpendicular bisectors in Figures 3-5. Henceforth we will adopt the notational convention that the sides a, b, and c opposite to angles A, B, and C are such that $a \geq b \geq c$ (and hence $A \geq B \geq C$). We denote by p_a , p_b, p_c the lengths of the perpendicular bisector segments on the sides BC, CA, AB respectively. Also, Δ denotes the area of the triangle.

Theorem 2. *Let $a \geq b \geq c$.*

$$p_a = \frac{2a\Delta}{a^2 + b^2 - c^2}, \qquad p_b = \frac{2b\Delta}{a^2 + b^2 - c^2}, \qquad p_c = \frac{2c\Delta}{a^2 - b^2 + c^2}.$$

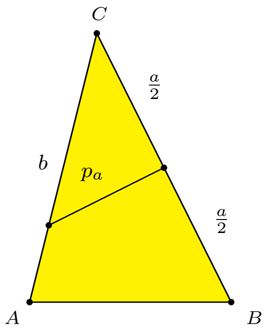


Figure 3

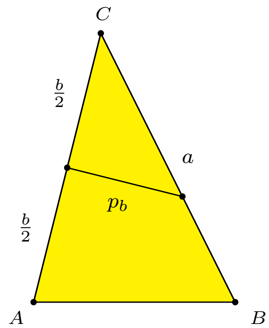


Figure 4

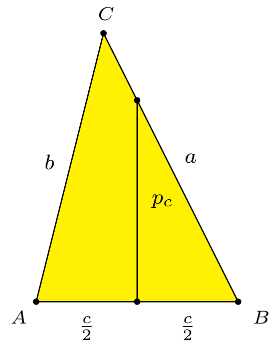


Figure 5

Proof. (i) Figure 3 illustrates the case where $\angle A$ is acute; the proof is identical when $\angle A$ is right or obtuse. We have $\tan C = \frac{p_a}{\frac{a}{2}}$ since by Lemma 1 bisector p_a meets side b because $b \geq c$. We know from [2] (which applies since $\angle C$ is oblique since $\angle A \geq \angle C$) that $\Delta = \frac{\tan C}{4}(a^2 + b^2 - c^2)$. Combining these gives $p_a = \frac{2a\Delta}{a^2+b^2-c^2}$.

(ii) Figure 4 illustrates the case where $\angle A$ is acute; the proof is again identical when $\angle A$ is right or obtuse. We have $\tan C = \frac{p_b}{\frac{b}{2}}$ (since by Lemma 1 bisector p_b intersets side a because $a \geq c$). Again $\Delta = \frac{\tan C}{4}(a^2 + b^2 - c^2)$. Combining these gives $p_b = \frac{2b\Delta}{a^2+b^2-c^2}$.

(iii) Figure 5 illustrates the case where $\angle A$ is acute; the proof is again identical when $\angle A$ is right or obtuse. We have $\tan B = \frac{p_c}{\frac{c}{2}}$ since by Lemma 1 bisector p_c intersects side a because $a \geq b$. By [2] (which applies since $\angle B$ is oblique since $\angle A \geq \angle B$) we have $\Delta = \frac{\tan B}{4}(a^2 - b^2 + c^2)$. Combining these gives $p_c = \frac{2c\Delta}{a^2-b^2+c^2}$. $\square$

Since Heron's well-known formula gives Δ in terms of a, b, and c, Theorem 2 gives each of the perpendicular bisectors in terms of the three sides. Moreover, we can apply the law of cosines for each angle to obtain these symmetric area formulas in terms of one side, another perpendicular bisector and a third angle.

Corollary 3. $\Delta = p_a b \cos C = p_b a \cos C = p_c a \cos B$.

Theorem 4. *Let* $a \geq b \geq c$.
(i) $p_a \geq p_b$*;*
(ii) $p_c \geq p_b$*;*
(iii) $p_a \geq p_c$*,* $p_a = p_c$ *and* $p_a < p_c$ *are all possible.*

Proof. (i) By Theorem 2, $\frac{p_a}{p_b} = \frac{a}{b} = \frac{\sin A}{\sin B}$. Since $A \geq B$ and $A + B < \pi$, $\sin A \geq \sin B$. It follows that $p_a \geq p_b$.

(ii) From Figures 4 and 5,

$$\frac{p_b}{p_c} = \frac{\frac{b}{2}\tan C}{\frac{c}{2}\tan B} = \frac{b\sin C}{\cos C} \cdot \frac{\cos B}{c\sin B} = \frac{\cos B}{\cos C} \leq 1$$

since $B \geq C$ are acute angles.

(iii) We show that all scenarios are possible by examples. Let $a = 6$ and $c = 4$, so that $4 \leq b \leq 6$.

b	Δ	p_a	p_c	
4	$3\sqrt{7}$	$\frac{\Delta}{3}$	$\frac{2\Delta}{9}$	$p_a > p_c$
$2\sqrt{\frac{29}{5}}$	$\frac{48}{5}$	$\frac{5\Delta}{18}$	$\frac{5\Delta}{18}$	$p_a = p_c$
5	$\frac{15\sqrt{7}}{4}$	$\frac{4\Delta}{15}$	$\frac{8\Delta}{27}$	$p_a < p_c$

□

Since we also know [3] that the distances from the circumcenter - which is the intersection of the perpendicular bisectors - to the sides AC and AB are in the ratios $\frac{\cos B}{\cos C}$, we have in Figure 6 that $\frac{OE}{OF} = \frac{\cos B}{\cos C} = \frac{ED'}{FD}$ (by Theorem 4(ii)) so $\frac{OE}{ED'} = \frac{OF}{FD}$. Hence:

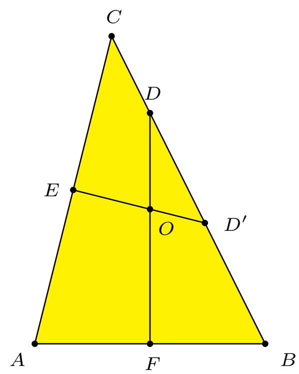


Figure 6

Corollary 5. *In an acute triangle the circumcenter divides the perpendicular bisectors* p_b *and* p_c *in equal proportions.*

A similar result applies when the triangle is obtuse (in which case the circumcenter lies outside the triangle):

Corollary 6. *In an obtuse triangle, the perpendicular bisectors p_b and p_c extended to the circumcenter are divided by their respective intersecting triangle sides in equal proportions.*

2. Do the perpendicular bisectors uniquely determine a triangle?

Theorem 7. *Given positive p_a, p_b, p_c satisfying $p_a \geq p_b$ and $p_c \geq p_b$, there are no more than two non-congruent triangles with $a \geq b \geq c$ and perpendicular bisector segments of lengths p_a, p_b, p_c.*

Proof. By Theorem 2, $\frac{p_a}{p_b} = \frac{a}{b}$, and

$$\frac{p_c}{p_b} = \frac{\frac{c}{b}\cdot(a^2+b^2-c^2)}{a^2-b^2+c^2} = \frac{\frac{c}{b}\left(\left(\frac{a}{b}\right)^2+1-\left(\frac{c}{b}\right)^2\right)}{\left(\frac{a}{b}\right)^2-1+\left(\frac{c}{b}\right)^2}$$

Putting $\alpha := \frac{p_a}{p_b} \geq 1$, $\gamma := \frac{p_c}{p_b} \geq 1$, and $x := \frac{c}{b}$, we rearrange this as

$$f(x) = x^3 + \gamma x^2 - (\alpha^2+1)x + \gamma(\alpha^2-1) = 0. \tag{1}$$

Since x is a ratio of sides, and $a - b < c \leq b$, we must have $x \in (\alpha - 1,\ 1]$.

If $\alpha - 1 = 0$ (the isosceles case of $\frac{a}{b} = \frac{p_a}{p_b} = 1$), (1) becomes $x(x^2+\gamma x-2) = 0$, and has exactly one solution in the interval $(\alpha - 1,\ 1]$.

If $\alpha - 1 > 0$, (1) exhibits two switches in the signs of parameters, so by Descartes' rule the cubic has either 2 or 0 positive roots. Thus the number of non-similar triangles cannot be more than 2. Since similar but non-congruent triangles cannot share the same absolute sizes of perpendicular bisectors, the number of non-congruent triangles sharing the same (p_a, p_b, p_c) can be no more than 2. □

Remark. Since $c > a - b$, we must have $x > \alpha - 1$. If $\gamma = 1$, (1) becomes

$$(x + \alpha + 1)(x - 1)(x - (\alpha - 1)) = 0.$$

It follows that $x = 1$ is the only admissible root. This results in an isosceles triangle.

Having already given an exact number of triangles above for the cases of $p_a = p_b$ and $p_c = p_b$, we next find specific parameter conditions that are necessary and sufficient for the number of triangles with specified (p_a, p_b, p_c), *i.e.*, the number of admissible solutions of (1) - to be 0, 1, or 2 when α and γ both exceed 1, *i.e.*, $p_a > p_b$ and $p_c > p_b$.

It is easy to see that the cubic $f(x)$ in (1) has a local minimum at

$$x_0 = \frac{-\gamma + \sqrt{\gamma^2 + 3(\alpha^2+1)}}{3} > 0,$$

and a local maximum at $x_1 = \frac{-\gamma - \sqrt{\gamma^2+3(\alpha^2+1)}}{3} < 0$.

It is routine to verify that the local minimum $x_0 \in (\alpha - 1, 1)$ if and only if $\gamma \in \left(\frac{\alpha^2-2}{2},\ \frac{-\alpha^2+3\alpha-1}{\alpha-1}\right)$.

Theorem 8. *Given $p_a > p_b$ and $p_c > p_b$, let $\alpha = \frac{p_a}{p_b}$ and $\gamma = \frac{p_c}{p_b}$. The number of triangles with perpendicular bisector segments of lengths p_a, p_b, p_c is*
2 *if and only if* $\gamma \in \left(\frac{\alpha^2-2}{2},\ \frac{-\alpha^2+3\alpha-1}{\alpha-1}\right)$ *and* $f(x_0) < 0$,
1 *if and only if* $\gamma \in \left(\frac{\alpha^2-2}{2},\ \frac{-\alpha^2+3\alpha-1}{\alpha-1}\right)$ *and* $f(x_0) = 0$,
0 *otherwise.*

Proof. Note that $f(\alpha-1) = 2\alpha(\alpha-1)(\gamma-1) > 0$ and $f(1) = \alpha^2(\gamma-1) > 0$.

If the local minimum $x_0 \in (\alpha-1,\ 1)$, then the number of roots of $f(x)$ in the interval 2, 1, or 0 according as $f(x_0) < 0, = 0$, or > 0.

If $x_0 \notin (\alpha-1,\ 1)$, then $f(x)$ is monotonic and has no root in the interval. $\square$

While no more than two distinct triangles can share the same p_a, p_b, p_c, one must also consider the possibility that up to two more triangles could share the same three perpendicular bisectors segments $\{p_1, p_2, p_3\}$ with different assignments of the bisectors to the long, medium, and short sides of the triangle. This is because, by Theorem 4, the medium-length side b has the shortest perpendicular bisector but either the longest side a or the shortest side c can have the longest bisector. Therefore, given segments of lengths $p_1 \geq p_2 \geq p_3$, we seek triangles (a, b, c) with $a \geq b \geq c$ and $(p_a, p_b, p_c) = (p_1, p_3, p_2)$ or (p_2, p_3, p_1). Thus:

Corollary 9. *There are a maximum of four triangles with the three segments of given lengths as perpendicular bisector segments.*

We conclude this paper by giving explicit examples showing that the number of triangles in Corollary 9 can be any of $0, 1, 2, 3, 4$.

(i) $n = 4$: Consider $(p_1, p_2, p_3) = (20,\ 18,\ 15)$.

If $(p_a, p_b, p_c) = (20, 15, 18)$, $\alpha = \frac{4}{3}$, $\gamma = \frac{6}{5}$, and

$$f_1(x) = x^3 + \frac{6}{5}x^2 - \frac{25}{9}x + \frac{14}{15} = \left(x - \frac{7}{15}\right)\left(x - \frac{\sqrt{97}-5}{6}\right)\left(x + \frac{\sqrt{97}+5}{6}\right).$$

This gives two triangles similar to $(20, 15, 7)$ and $(8, 6, \sqrt{97}-5)$.

On the other hand, if $(p_a, p_b, p_c) = (18, 15, 20)$, $\alpha = \frac{6}{5}$, $\gamma = \frac{4}{3}$, and

$$f_2(x) = x^3 + \frac{4}{3}x^2 - \frac{61}{25}x + \frac{44}{75} = \left(x - \frac{4}{5}\right)\left(x - \frac{\sqrt{421}-16}{15}\right)\left(x + \frac{\sqrt{421}+16}{15}\right).$$

This gives two triangles similar to $(6, 5, 4)$ and $(18, 15, \sqrt{421}-16)$.

(ii) For $n = 3$, let $(p_1, p_2, p_3) = (3,\ 2\sqrt{2},\ \sqrt{5})$.

If $(p_a, p_b, p_c) = (3, \sqrt{5},\ 2\sqrt{2})$, $\alpha = \frac{\sqrt{5}}{3}$, $\gamma = \frac{2\sqrt{2}}{\sqrt{5}}$, and

$$f_3(x) = x^3 + \frac{2\sqrt{2}}{\sqrt{5}}x^2 - \frac{14}{5}x + \frac{8\sqrt{2}}{5\sqrt{5}} = \left(x + \frac{4\sqrt{2}}{\sqrt{5}}\right)\left(x - \frac{\sqrt{2}}{\sqrt{5}}\right)^2.$$

This gives a triangle similar to $(3, \sqrt{5},\ \sqrt{2})$.

If $(p_a, p_b, p_c) = (2\sqrt{2}, \sqrt{5},\ 3)$, then

$$f_4(x) = x^3 + \frac{3}{\sqrt{5}}x^2 - \frac{13}{5}x + \frac{9}{5\sqrt{5}} = \left(x - \frac{1}{\sqrt{5}}\right)\left(x - \frac{\sqrt{13}-2}{\sqrt{5}}\right)\left(x + \frac{\sqrt{13}+2}{\sqrt{5}}\right).$$

The two roots in the interval $(0, 1)$ give the triangles similar to $(2\sqrt{2},\ \sqrt{5},\ 1)$ and $(2\sqrt{2},\ \sqrt{5},\ \sqrt{13} - 2)$.

(iii) For $n = 2$, let $(p_1, p_2, p_3) = (39,\ 30,\ 25)$. If $(p_a, p_b, p_c) = (30, 25, 39)$, then $\alpha = \frac{6}{5}$, $\gamma = \frac{39}{25}$, and

$$f_5(x) = x^3 + \frac{39}{25}x^2 - \frac{61}{25}x + \frac{429}{625} = \left(x + \frac{13}{5}\right)\left(x - \frac{11}{25}\right)\left(x - \frac{3}{5}\right).$$

The two positive roots are in $(\alpha - 1, 1) = \left(\frac{1}{5}, 1\right)$. These give two triangles similar to $(30, 25, 11)$ and $(6, 5, 3)$.

On the other hand, if $(p_a, p_b, p_c) = (39, 25, 30)$, the cubic

$$f_6(x) = x^3 + \frac{6}{5}x^2 - \frac{2146}{625}x + \frac{5376}{3125}$$

has only one real root which is negative (see Figure 7). There is no triangle with $(p_a, p_b, p_c) = (39, 25, 30)$.

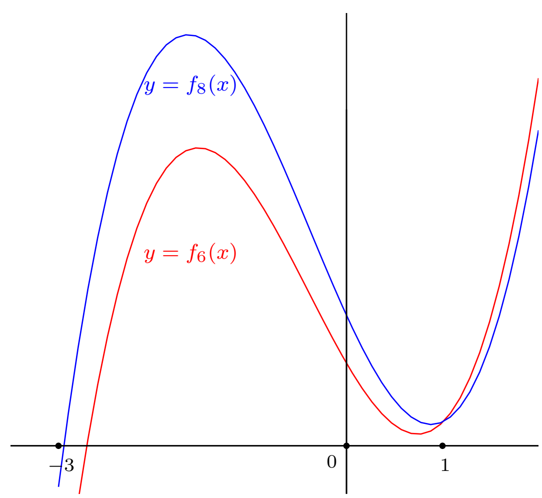


Figure 7

(iv) For $n = 1$, consider $(p_1, p_2, p_3) = (8,\ 5,\ \sqrt{19})$. If $(p_a, p_b, p_c) = (5,\ \sqrt{19},\ 8)$, then $(\alpha,\ \gamma) = \left(\frac{5}{\sqrt{19}},\ \frac{8}{\sqrt{19}}\right)$, and

$$f_7(x) = x^3 + \frac{8}{\sqrt{19}}x^2 - \frac{44}{19}x + \frac{48}{19\sqrt{19}} = \left(x + \frac{12}{\sqrt{19}}\right)\left(x - \frac{2}{\sqrt{19}}\right)^2.$$

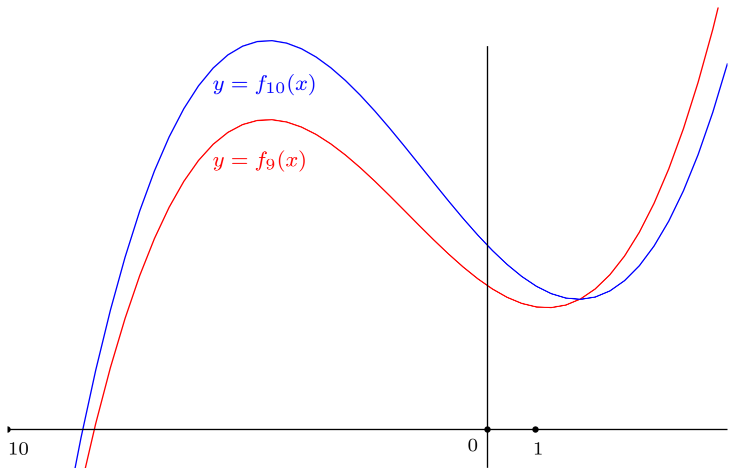
This give a single triangle similar to $(5,\ \sqrt{19},\ 2)$.

On the other hand, with $(p_a, p_b, p_c) = (8, \sqrt{19}, 5)$, we have $(\alpha, \gamma) = \left(\frac{8}{\sqrt{19}}, \frac{5}{\sqrt{19}}\right)$, and

$$f_8(x) = x^3 + \frac{5}{\sqrt{19}}x^2 - \frac{83}{19}x + \frac{225}{19\sqrt{19}}$$

has only one real root which is negative (see Figure 7). There is no such triangle.

(v) Finally, for $n = 0$, we take $(p_1, p_2, p_3) = (5, 4, 1)$. The two cubic polynomials are $f_9(x) = x^3 + 5x^2 - 17x + 75$ and $f_{10}(x) = x^3 + 4x^2 - 26x + 96$. Each of these has exactly one real root which is negative (see Figure 8). Therefore, there is no triangle with perpendicular bisector segments $(5, 4, 1)$.



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