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Why Are the Side Lengths of the Squares Inscribed in a Triangle so Close to Each Other?

Victor Oxman and Moshe Stupel

Abstract. We compare the side lengths of the squares inscribed in a non-obtuse triangle.

Given non-obtuse triangle ABC we consider an inscribed square. The construction is well known (see, for instance, [1] and [2, Problem 9, pp.16,64]). One can easily note that the side lengths of the squares based on the various sides of the triangle are almost equal to each other (Figure 1). Why does it happen? In this note we will give the answer to this question.

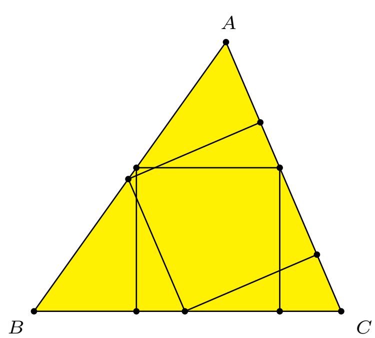

Figure 1

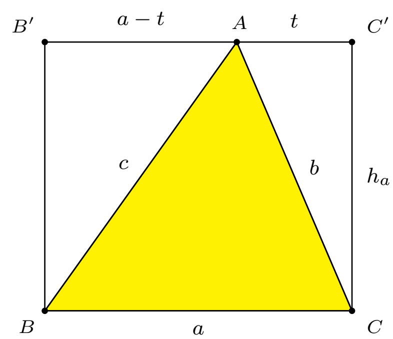

Figure 2

Let $BC = a$, $AC = b$, $AB = c$, and h_a, h_b, h_c the corresponding altitudes of triangle ABC. Denote the side lengths of the squares based on BC and AC by x_a and x_b respectively. One can easily verify that $x_a = \frac{ah_a}{a+h_a}$. Given a and $h_a > 0$, consider a non-obtuse triangle ABC with $BC = a$ and altitude h_a on the side BC. The vertex A lies on the side $B'C'$ of the rectangle $BB'C'C$ with $BB' = h_a$ (see Figure 2).

Let $AC' = t$. We have $b^2 = t^2 + h_a^2$ and $c^2 = (a-t)^2 + h_a^2$. Note that

$$x_b = \frac{bh_b}{b+h_b} = \frac{ah_a}{\sqrt{t^2+h_a^2} + \frac{ah_a}{\sqrt{t^2+h_a^2}}} = \frac{ah_a\sqrt{t^2+h_a^2}}{t^2+h_a(a+h_a)}.$$

We shall assume $a \geq b$. This requires $h_a \leq a$ and $t \in [0,\ t_0]$, where $t_0 = \sqrt{a^2-h_a^2}$. Since we require triangle ABC to be non-obtuse, $b^2+c^2 \geq a^2$. This implies $t^2 - at + h_a^2 \geq 0$. This is always the case when $h_a \geq \frac{a}{2}$. When $h_a < \frac{a}{2}$,

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we must further restrict $t \in [0, t_1] \cup [t_2, t_0]$, where

$$t_1 = \frac{a - \sqrt{a^2 - 4h_a^2}}{2}, \qquad t_2 = \frac{a + \sqrt{a^2 - 4h_a^2}}{2}.$$

Note that $t_0^2 - t_2^2 = \frac{a(a-\sqrt{a^2-4h_a^2})}{2} \geq 0$.

Now consider the function

$$f(t) := \frac{x_a}{x_b} = \frac{t^2 + h_a(a + h_a)}{(a + h_a)\sqrt{t^2 + h_a^2}}$$

defined on

$$\mathcal{D} := \begin{cases} [0, t_0], & \text{if } \frac{a}{2} \leq h_a \leq a, \\ [0, t_1] \cup [t_2, t_0] & \text{if } 0 < h_a < \frac{a}{2} \end{cases}.$$

This is a restriction of the function

$$F(t) = \frac{t^2 + h_a(a + h_a)}{(a + h_a)\sqrt{t^2 + h_a^2}}$$

defined on $[0, a]$. It has derivative

$$F'(t) = \frac{1}{a + h_a} \cdot \frac{t(t^2 - h_a(a - h_a))}{(t^2 + h_a^2)^{\frac{3}{2}}}.$$

From this it is clear that the only interior critical point is $t^* = \sqrt{h_a(a - h_a)}$, and that $F(t)$ is decreasing on $[0, t^*]$ and increasing on $[t^*, a]$. Therefore, $F(t) \geq F(t^*) = \frac{2\sqrt{ah_a}}{a+h_a}$ for every $t \in [0, a]$.

Note that $F(0) = F(t_0) = 1$. This means $f(t) \leq 1$ for $t \in \mathcal{D}$, and $x_a \leq x_b$.

(1) For $h_a \geq \frac{a}{2}$, comparing $f(t^*) = \frac{2\sqrt{ah_a}}{a+h_a}$ with the boundary values $f(0) = 1$ and $f(t_0) = 1$, we conclude that

$$\min\{f(t) : t \in \mathcal{D}\} = \frac{2\sqrt{ah_a}}{a + h_a}.$$

As a function of $h_a \in \left[\frac{a}{2},\ a\right]$, $\frac{2\sqrt{ah_a}}{a+h_a}$ is increasing. Therefore, for $t \in \mathcal{D} = [0,\ t_0]$,

$$\frac{x_a}{x_b} = f(t) \geq \frac{2\sqrt{ah_a}}{a + h_a} \geq \frac{2\sqrt{a \cdot \frac{a}{2}}}{a + \frac{a}{2}} = \frac{2\sqrt{2}}{3}.$$

(2) If $h_a < \frac{a}{2}$, then

$$t^{*2} - t_1^2 = \frac{a(\sqrt{a^2 - 4h_a^2} - (a - 2h_a))}{2} = \frac{a\sqrt{a - 2h_a}(\sqrt{a + 2h_a} - \sqrt{a - 2h_a})}{2} > 0,$$

$$t_2^2 - t^{*2} = \frac{a(\sqrt{a^2 - 4h_a^2} + (a - 2h_a))}{2} > 0.$$

It follows that the critical point t^* is not in the interior of $\mathcal{D} = [0, t_1] \cup [t_2, t_0]$. Comparing boundary values, we have

$$\min\{f(t) : t \in \mathcal{D}\} = \min\{f(t_1), f(t_2)\}.$$

Note that $f(t_1) = f(t_2)$ because when $t = t_1$ or t_2, $\angle BAC = 90^\circ$. In fact, for $j = 1, 2$,

$$f(t_j) = \frac{a(t_j + h_a)}{(a + h_a)\sqrt{at_j}} = \frac{\sqrt{a}}{a + h_a} \cdot \sqrt{\frac{(t_j + h_a)^2}{t_j}} = \frac{\sqrt{a(a + 2h_a)}}{a + h_a}.$$

As a function of $h_a \in \left(0, \frac{a}{2}\right]$, $\frac{\sqrt{a(a+2h_a)}}{a+h_a}$ is decreasing. Therefore, for $t \in \mathcal{D} = [0, t_1] \cup [t_2, t_0]$,

$$\frac{x_a}{x_b} = f(t) \geq \frac{\sqrt{a(a + 2h_a)}}{a + h_a} \geq \frac{\sqrt{a(a + 2 \cdot \frac{a}{2})}}{a + \frac{a}{2}} = \frac{2\sqrt{2}}{3}.$$

We conclude that in all cases,

$$1 \geq \frac{x_a}{x_b} \geq \frac{2\sqrt{2}}{3} = 0.94\ldots.$$

The difference between x_a and x_b is less than 6% of x_b. This explains why the lengths of the sides of the inscribed squares are very close to each other.

Note that from the above reasoning the smallest inscribed square in a non-obtuse triangle is based on the longest side.

References

[1] F. M. van Lamoen, Inscribed squares, *Forum Geom.*, 207–214.

[2] I. M. Yaglom, *Geometric Transformations*, II, Random Hon muse, New York, 1968.

Victor Oxman: Western Galilee College, P.O.B. 2125, Acre 24121 Israel
E-mail address: victor.oxman@gmail.com

Moshe Stupel: Shaanan College, P.O.B. 906, Kiryat Shmuel, Haifa 26109 Israel
E-mail address: stupel@bezeqint.net