Forum Geometricorum
Volume 13 (2013) 133–134.

FORUM GEOM
ISSN 1534-1178

Derivation of the Law of Cosines via the Incircle

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Abstract. The law of cosines can be derived without using special cases and without using the Pythagorean theorem.

Most trigonometry textbooks treat the derivation of the law of cosines as three separate cases depending on whether the triangle is acute, obtuse, or right. In this note we derive a proof using an arbitrary general triangle and we do not use the Pythagorean theorem.

We let ABC be any triangle whose incircle is tangent at D, E, and F as shown in Figure 1. Let $a = BC$, $b = CA$, $c = AB$, and r be the radius of the incircle with incenter I. We derive our formula by equating independent expressions for r.

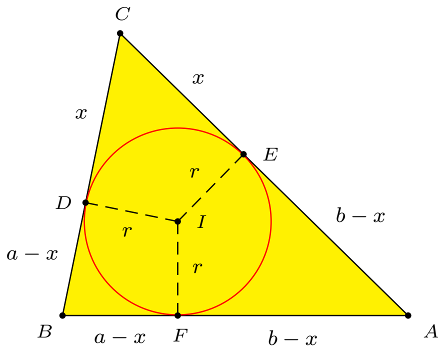

Figure 1.

If $x = CD = CE$, then $c = (a - x) + (b - x)$ so that $x = \frac{1}{2}(a + b - c)$. In the right triangle CEI, $x = r \cot \frac{C}{2}$. Solving $\frac{1}{2}(a + b - c) = r \cot \frac{C}{2}$ for r yields

$$r = \frac{a + b - c}{2 \cot \frac{C}{2}}. \tag{1}$$

By computing the area of triangle ABC in two ways we obtain

$$\frac{1}{2}ab \sin C = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr,$$

and

$$r = \frac{ab \sin C}{a + b + c}. \tag{2}$$

Publication Date: August 27, 2013. Communicating Editor: Paul Yiu.

From (1) and (2), we have $\frac{a+b-c}{2\cot\frac{C}{2}} = \frac{ab\sin C}{a+b+c}$. Cross multiplying and making use of some well known trigonometric identities we obtain the following.

$$\begin{aligned}(a+b)^2 - c^2 &= 2ab\sin C\cot\frac{C}{2}\\ &= 2ab\left(2\sin\frac{C}{2}\cos\frac{C}{2}\right)\cdot\frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}\\ &= 4ab\cos^2\frac{C}{2}.\end{aligned}$$

Therefore,

$$\begin{aligned}c^2 &= (a+b)^2 - 4ab\cos^2\frac{C}{2}\\ &= a^2+b^2-2ab\left(2\cos^2\frac{C}{2}-1\right)\\ &= a^2+b^2-2ab\cos C.\end{aligned}$$

Since the identity $\sin^2 x + \cos^2 x = 1$ (and hence $\cos 2x = 2\cos^2 x - 1$) can be derived independently [2] without the Pythagorean theorem, we have derived the law of cosines without the use of the Pythagorean theorem. This also answers a question in a footnote in [1, p. 135] whether this is possible.

References

[1] O. Byer, F. Lazebnik, and D. L. Smeltzer, *Methods for Euclidean Geometry*, MAA, 2010.
[2] J. Zimba, On the possibility of trigonometric proofs of the Pythagorean theorem, *Forum Geom.*, 9 (2009) 275–278.

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