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Jigsawing a Quadrangle from a Triangle

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Abstract. For an acute-angled triangle ABC, we construct two isotomic points P and Q on BC and P' and Q' on AB and AC respectively, such that BPP' and CQQ' are right triangles which, when rotated about P' and Q' respectively through appropriate angles fit with $AP'Q'$ to a quadrangle with a new fourth vertex A'. We show that AA' passes trough the circumcenter O.

Given an acute-angled triangle ABC consider the construction of a pair of isotomic points P and Q on the side BC (with $BP = QC < \frac{1}{2} \cdot BC$) such that the perpendiculars to BC at P and Q intersect AB and AC at P' and Q' satisfying $P'Q' = PP' + QQ'$.

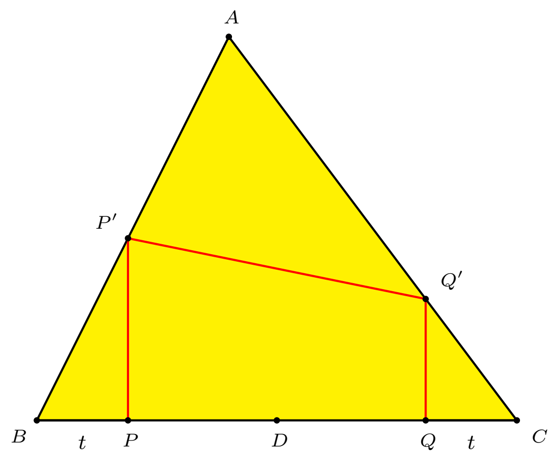

Figure 1

If we put $BP = QC = t$, then $PP' = t \cdot \tan B$, $QQ' = t \cdot \tan C$, and $P'Q' = t(\tan B + \tan C) = \frac{t \sin A}{\cos B \cos C}$. Applying the law of cosines to triangle $AP'Q'$, we have

$$\left(c - \frac{t}{\cos B}\right)^2 + \left(b - \frac{t}{\cos C}\right)^2 - 2\left(c - \frac{t}{\cos B}\right)\left(b - \frac{t}{\cos C}\right)\cos A = t^2 \cdot \frac{1 - \cos^2 A}{\cos^2 B \cos^2 C}.$$

Clearing denominators and rearranging terms, we have

$$\begin{aligned}&(1 - \cos^2 A - \cos^2 B - \cos^2 C + 2\cos A \cos B \cos C)t^2 \\ &+ 2\cos B \cos C(b \cos B + c \cos C - a \cos A)t - a^2 \cos^2 B \cos^2 C = 0. \qquad (1)\end{aligned}$$

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Making use of the following identities

$$b\cos B + c\cos C - a\cos A = 2a\cos B\cos C,$$
$$1-\cos^2 A - \cos^2 B - \cos^2 C = 2\cos A\cos B\cos C,$$

we rewrite (1) as

$$4t^2\cos A + 4at\cos B\cos C - a^2\cos B\cos C = 0 \qquad (2)$$

after cancelling a common factor $\cos B\cos C$. Writing $x = \frac{a}{2} - t$, or $2t = a - 2x$, we have

$$(a-2x)^2\cos A + 2a(a-2x)\cos B\cos C - a^2\cos B\cos C = 0$$
$$4x^2\cos A - 4ax(\cos A + \cos B\cos C) + a^2(\cos A + \cos B\cos C) = 0$$
$$4x^2\cos A - 4ax\sin B\sin C + a^2\sin B\sin C = 0. \qquad (3)$$

From this,

$$x = \frac{a\sin B\sin C - a\sqrt{\sin B\sin C(\sin B\sin C - \cos A)}}{2\cos A}$$
$$= \frac{a\sin B\sin C - a\sqrt{\sin B\cos B\cdot\sin C\cos C}}{2\cos A} \qquad (4)$$

Applying the law of sines to triangle ABC, with $b = 2R\sin B$ and $c = 2R\sin C$ for the circumradius R, we have

$$\frac{2x}{a} = \frac{2R\sin B\sin C - 2R\sqrt{\sin B\sin C\cdot\cos B\cos C}}{2R\cos A}$$
$$= \frac{b\sin C - \sqrt{b\cos B\cdot c\cos C}}{2R\cos A}$$
$$= \frac{AX - \sqrt{BX\cdot XC}}{2R\cos A} = \frac{AX - X_1X}{AH} = \frac{AX_1}{AH},$$

where H is the orthocenter of triangle ABC, and the altitude AX intersects the semicircle with diameter BC at X_1 (see Figure 2). This leads to the following construction of the trapezoid $PQQ'P'$.

Let O and G be the circumcenter and centroid of triangle ABC, and D the midpoint of BC.

(1) Construct the semicircle with diameter BC (on the same side of A) to intersect the A-altitude at X_1.

(2) Construct the line X_1G to intersect OD at X_2.

(3) Construct the parallels to OB and OC through X_2 to meet AC in P and Q respectively.

(4) Construct the perpendiculars to BC at P and Q to intersect AB and AC at P' and Q' respectively.

These points satisfy $PP' + QQ' = P'Q'$.

Proof. In Figure 2,

$$\frac{X_2D}{OD} = \frac{PD}{BD} = \frac{2x}{a} = \frac{AX_1}{AH}.$$

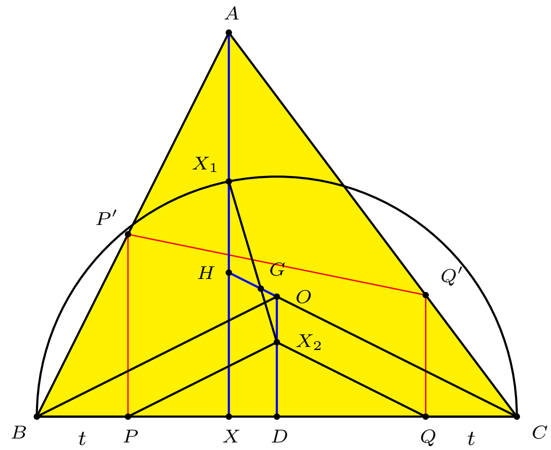

Figure 2.

Since $AH = 2 \cdot OD$, we have $AX_1 = 2 \cdot X_2D$, and $HX_1 = AH - AX_1 = 2 \cdot OD - 2 \cdot X_2D = 2 \cdot OX_2$. From this it is clear that X_1X_2 passes through the centroid G which divides HO in the ratio $HG : GO = 2 : 1$. □

We rotate triangle BPP' about P' and triangle CQQ' about C so that the images of P and Q coincide at a point on $P'Q'$. Then the images of B and C coincide at a point A'.

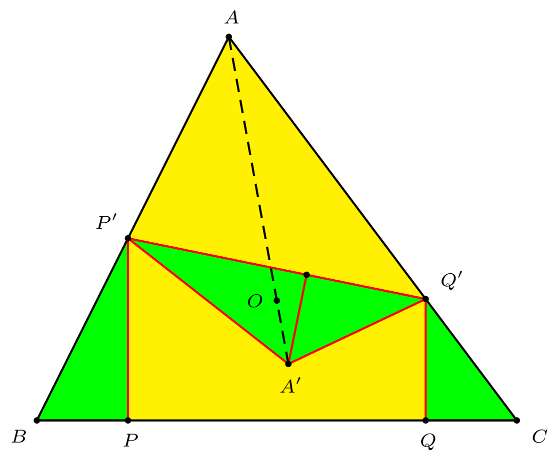

Figure 3.

Consider the quadrilateral $AP'A'Q'$. Note that $\angle Q'A'P' = \angle B + \angle C$, so that the quadrilateral is cyclic. Hence,

$$\angle Q'AA' = \angle Q'P'A' = \angle PP'B = \frac{\pi}{2} - B.$$

This means that AA' passes through the circumcenter O of triangle ABC.

By symmetry, if we perform similar constructions on the sides CA and AB, and obtain points B', C' corresponding to A', the lines BB' and CC' also pass through the circumcenter O, as does AA'.

References

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