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Pedal Polygons

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Abstract. We study the pedal polygon $H_1H_2\cdots H_n$ of a point P with respect to a polygon $\mathbf{P}$, where the points H_i are the feet of the perpendiculars drawn from P to the sides of $\mathbf{P}$. In particular we prove that if $\mathbf{P}$ is a quadrilateral which is not a parallelogram, there exists one and only one point P for which the points H_i are collinear.

1. Introduction

Consider a polygon $A_1A_2\cdots A_n$ and call it $\mathbf{P}$. Let P be a point and let H_i be the foot of the perpendicular from P to the line A_iA_{i+1}, $i = 1, 2, \ldots, n$ (with indices i taken modulo n). The points H_i usually form a polygon $H_1H_2\cdots H_n$, which we call the *pedal polygon* of P with respect to $\mathbf{P}$, and denote by $\mathbf{H}$ (see Figure 1). We call P the *pedal point*. See ([2, p.22]) for the notion of pedal triangle.

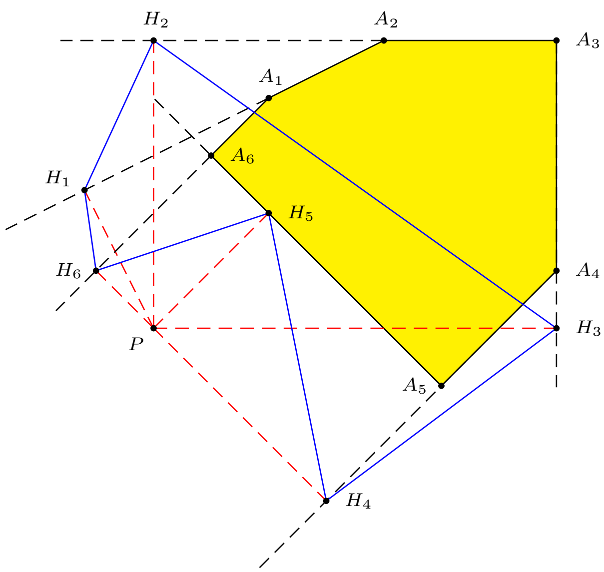

Figure 1

In this article we find some properties of the pedal polygon $\mathbf{H}$ of a point P with respect to $\mathbf{P}$. In particular, when $\mathbf{P}$ is a triangle we find the points P such that the pedal triangle $\mathbf{H}$ is a right, obtuse or acute triangle. When $\mathbf{P}$ is a quadrilateral which is not a parallelogram, we prove that there exists one and only one point P for which the points H_i are collinear. Moreover, we find the points P for which

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the pedal quadrilateral $\mathbf{H}$ of P has at least one pair of parallel sides. We also prove that, in general, there exists one and only one pedal point with respect to which $\mathbf{H}$ is a parallelogram. In the last part of the paper, we find some properties of the pedal polygon $\mathbf{H}$ in the general case of a polygon $\mathbf{P}$ with n sides.

2. Properties of the pedal triangle

Let $\mathbf{P}$ be a triangle. The pedal triangle of the circumcenter of $\mathbf{P}$ is the medial triangle of $\mathbf{P}$; the one of the orthocenter is the orthic triangle of $\mathbf{P}$; the one of the incenter is the Gergonne triangle of $\mathbf{P}$ (*i.e.*, the triangle whose vertices are the points in which the incircle of $\mathbf{P}$ touches the sides of $\mathbf{P}$).

Theorem 1. [2, p.41] *If $\mathbf{P}$ is a triangle, the points H_i are collinear if and only if P lies on the circumcircle of $\mathbf{P}$.*

The line containing the points H_i is called *Simson line* of the point P with respect to $\mathbf{P}$ (see Figure 2).

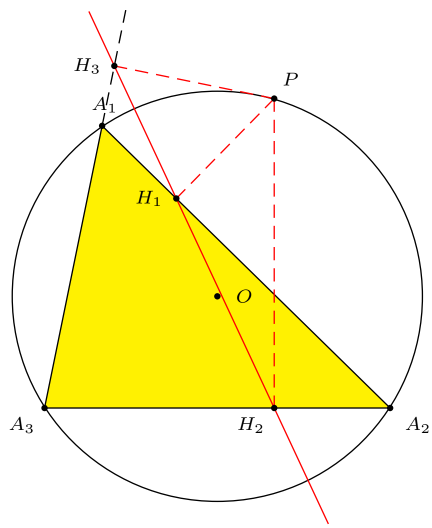

Figure 2.

Theorem 2. [1, p.108] *The points $\mathbf{P}$ for which the pedal triangle $\mathbf{H}$ is isosceles are all and only the points that lie on at least one of the Apollonius circles associated to the vertices of $\mathbf{P}$.*

The Apollonius circle associated to the vertex A_i is the locus of points P such that $PA_{i+1} : PA_{i+2} = A_iA_{i+2} : A_iA_{i+1}$. The three Apollonius circles are coaxial and they intersect in the two isodynamic points of the triangle $\mathbf{P}$, I_1 and I_2. Therefore, *the isodynamic points of $\mathbf{P}$ are the only points whose pedal triangles with respect to $\mathbf{P}$ are equilateral* (see Figure 3).

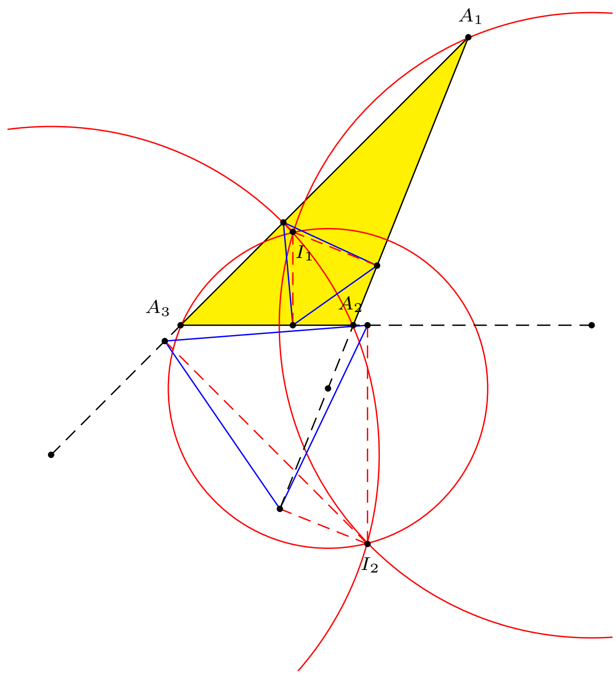

Figure 3

We will find now the points P whose pedal triangle is a right, acute, obtuse triangle. Let P be a point (see Figure 4) and let $A_iA_{i+1} = a_{i+2}$, $PA_i = x_i$, $H_iH_{i+1} = h_{i+2}$. Since the quadrilateral $A_iH_iPH_{i+2}$ is cyclic, $h_i = x_i \sin A_i$ ([2, p.2]).

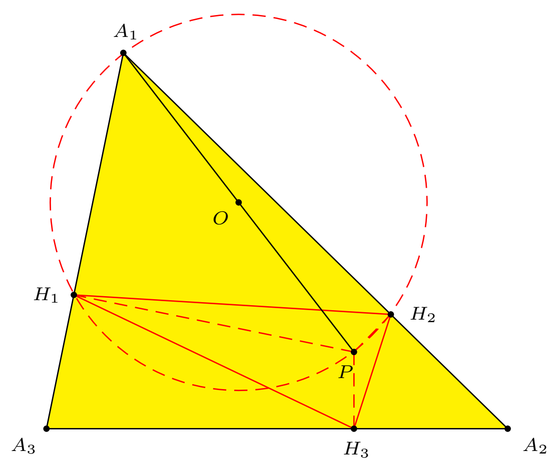

Figure 4

By the Pythagorean theorem and its converse, the pedal triangle of P is right in H_i if and only if

$$x_i^2 \sin^2 A_i = x_{i+1}^2 \sin^2 A_{i+1} + x_{i+2}^2 \sin^2 A_{i+2}.$$

By the law of sines, this is equivalent to

$$a_i^2 x_i^2 = a_{i+1}^2 x_{i+1}^2 + a_{i+2}^2 x_{i+2}^2. \qquad (1)$$

This relation represents the locus γ_i of points P for which the triangle $\mathbf{H}$ is right in H_i. Therefore, the locus of points P whose pedal triangle is a right triangle is $\gamma_1 \cup \gamma_2 \cup \gamma_3$. Observe that γ_i contains the points A_{i+1} and A_{i+2}; moreover, γ_i and γ_{i+1} intersect only in the point A_{i+2}.

We verify now that γ_1 is a circle. Set up an orthogonal coordinate system such that $A_2 \equiv (1, 0)$ and $A_3 \equiv (-1, 0)$; let $A_1 \equiv (a, b)$ and $P \equiv (x, y)$. The relation (1) becomes:

$$4((x-a)^2+(y-b)^2) = ((a+1)^2+b^2)((x-1)^2+y^2)+((a-1)^2+b^2)((x+1)^2+y^2).$$

Simplifying, we obtain the equation of a circle:

$$(a^2 + b^2 - 1)(x^2 + y^2) + 4by - (a^2 + b^2 - 1) = 0.$$

Moreover, it is not hard to verify that the tangents to the circumcircle of $\mathbf{P}$ in the points A_2 and A_3 pass through the center of γ_1.

Analogously the same holds for γ_2 and γ_3. We can then state that γ_i is a circle passing through the points A_{i+1} and A_{i+2}; the tangents to the circumcircle of $\mathbf{P}$ in the points A_{i+1} and A_{i+2} pass through the center C_i of γ_i; moreover, γ_i and γ_{i+1} are tangent in A_{i+2}. Then, if $C_1C_2C_3$ is the tangential triangle of $A_1A_2A_3$, γ_i is the circle with center C_i passing through A_{i-1} and A_{i+1} (see Figure 5).

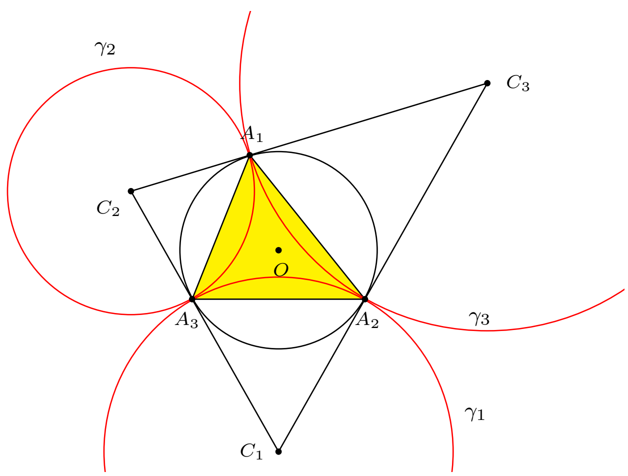

Figure 5

Observe that, by the law of cosines, the angle in H_i of the pedal triangle of P is obtuse if and only if:

$$a_i^2 x_i^2 > a_{i+1}^2 x_{i+1}^2 + a_{i+2}^2 x_{i+2}^2,$$

i.e., the point P lies inside the circle γ_i. Thus, we have established the following theorem.

Theorem 3. *The pedal triangle of a point* P *is*
(a) *a right triangle if and only if* P *lies on one of the circles* γ_i,
(b) *an obtuse triangle if and only if* P *is inside one of the circles* γ_i,
(c) *an acute triangle if and only if* P *is external to all the circles* γ_i.

3. Properties of the pedal quadrilateral

Let $\mathbf{P}$ be a cyclic quadrilateral. The pedal quadrilateral of the circumcenter of $\mathbf{P}$ is the Varignon parallelogram of $\mathbf{P}$, and the one of the anticenter ([6, p.152]) is the principal orthic quadrilateral of $\mathbf{P}$ ([5, p.80]).

Let $\mathbf{P}$ be a tangential quadrilateral. The pedal quadrilateral of the incenter of $\mathbf{P}$ is the contact quadrilateral of $\mathbf{P}$, *i.e.*, the quadrilateral whose vertices are the points in which the incircle of $\mathbf{P}$ touches the sides of $\mathbf{P}$.

For a generic quadrilateral, we consider the problem of finding the pedal points for which the points H_i are collinear.

It is easy to verify that if $\mathbf{P}$ has only one pair of parallel sides, there is only one pedal point P for which the points H_i are collinear. P is the common point to the lines containing opposite and non parallel sides of $\mathbf{P}$, and the points H_i lie on the perpendicular from P to the parallel sides of $\mathbf{P}$. On the other hand, if $\mathbf{P}$ is a parallelogram, there is no point with respect to which the points H_i are collinear.

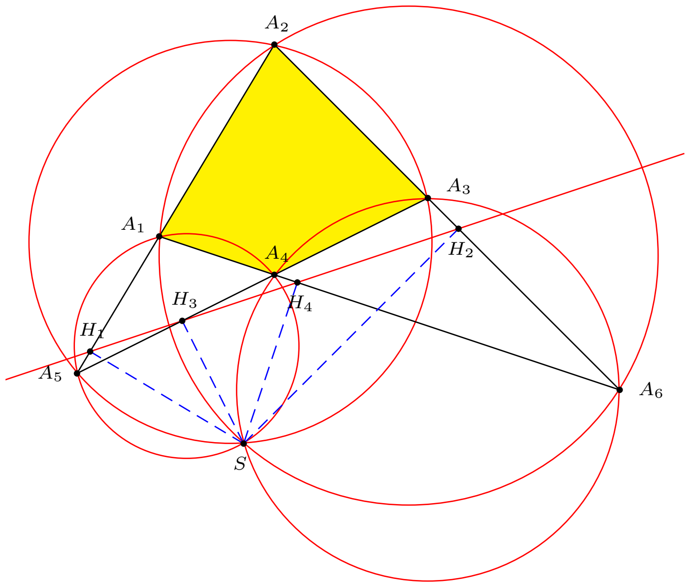

Figure 6

Suppose now that $\mathbf{P}$ is a quadrilateral without parallel sides (see Figure 6). Let A_5 be the common point to the lines A_1A_2 and A_3A_4, and A_6 the common point to A_2A_3 and A_1A_4. Consider the four triangles $A_1A_2A_6$, $A_2A_3A_5$, $A_3A_4A_6$, $A_1A_4A_5$, and let $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$ be their circumcircles, respectively.

If the pedal point P lies on one of the circles $\mathcal{C}_i$, then, by Theorem 1, at least three of the points H_i are collinear. It follows that the four points H_i are collinear if and only if P lies in every $\mathcal{C}_i$. The four circles are concurrent in the Miquel point of the quartet of lines containing the sides of $\mathbf{P}$ ([3, p.82]). Thus, we have established the following theorem.

Theorem 4. *If $\mathbf{P}$ is a quadrilateral, that is not a parallelogram, there exists one and only one pedal point with respect to which the points H_i are collinear.*

We call this point the *Simson point* of the quadrilateral $\mathbf{P}$, and denote it by S. We call the *Simson line* of $\mathbf{P}$ the line containing the points H_i. Observe that the points H_i determine a quadrilateral if and only if $P \notin \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$.

Theorem 5. *If $\mathbf{P}$ is a quadrilateral which is not a parallelogram, the reflections of the Simson point with respect to the lines containing the sides of $\mathbf{P}$ are collinear and the line ℓ containing them is parallel to the Simson line.*

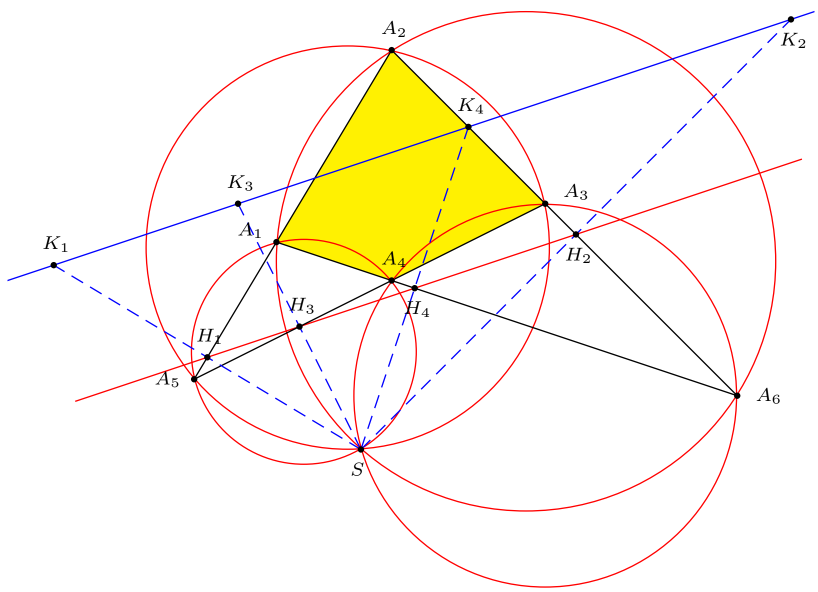

Figure 7

Proof. The theorem is trivially true if $\mathbf{P}$ has one pair of parallel sides. Suppose that $\mathbf{P}$ is without parallel sides (see Figure 7). Let K_i, $i = 1, 2, 3, 4$, be the reflection of S with respect to the line A_iA_{i+1}. The points S, H_i and K_i are collinear and $SH_i = H_iK_i$, then K_i is the image of H_i under the homothety $\mathsf{h}(S, 2)$. Then, since the points H_i are collinear (Theorem 4), the points K_i are also collinear. Moreover, the line ℓ containing the points K_i is parallel to the Simson line of $\mathbf{P}$. □

Conjecture. If $\mathbf{P}$ is a cyclic quadrilateral without parallel sides, the line ℓ passes through the anticenter H of $\mathbf{P}$, and the Simson line bisects the segment SH.

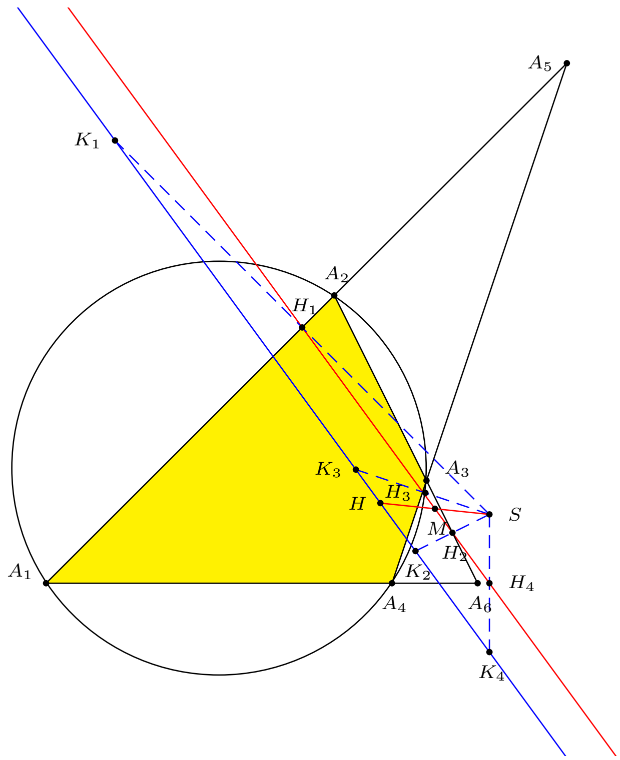

Figure 8

The conjecture was suggested by using a dynamic geometry software (see Figure 8). However, we have been unable to prove it.

If $\mathbf{P}$ is a cyclic quadrilateral with a pair of parallel sides, then $\mathbf{P}$ is an isosceles trapezoid. The line ℓ coincides with the Simson line, *i.e.*, the line joining the midpoints of the bases of $\mathbf{P}$, and passes through the anticenter of $\mathbf{P}$. In this case the Simson line contains the segment SH.

We now find the points P whose pedal quadrilaterals have at least on pair of parallel sides.

If $\mathbf{P}$ is a parallelogram, then the points P whose pedal quadrilaterals have at least one pair of parallel sides are all and only the points of the diagonals of $\mathbf{P}$.

Suppose now that $\mathbf{P}$ is not a parallelogram. We prove that the locus of *the point P whose pedal quadrilateral has the sides H_1H_4 and H_2H_3 parallel is the circle* A_1A_3S (see Figure 9).

First observe that S is a point with respect to whom H_1H_4 and H_2H_3 are parallel because the points H_i are collinear. Set up now an orthogonal coordinate system such that $A_1 \equiv (-1, 0)$ and $A_3 \equiv (1, 0)$; let $A_2 \equiv (a, b)$, $A_4 \equiv (c, d)$ and

$P \equiv (x, y)$. If H_1H_4 and H_2H_3 are parallel, then P lies on the circle γ of equation:

$$(hd + kb)x^2 + (hd + kb)y^2 - (hk - 4bd)y = hd + kb,$$

where $h = a^2 + b^2 - 1$ and $k = c^2 + d^2 - 1$.

Note that the points A_1 and A_3 are on γ, and γ is the circle A_1A_3S.

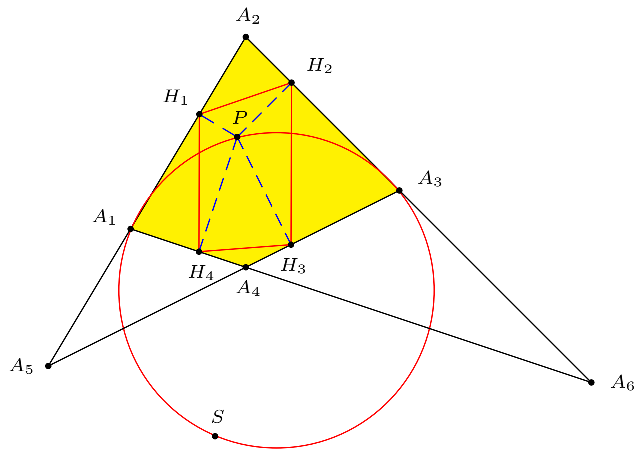

Figure 9

Analogously we can prove that *the points P whose pedal quadrilateral has the sides H_1H_2 and H_3H_4 parallel is the circle A_2A_4S.* Therefore we have established the following theorem.

Theorem 6. *The points P whose pedal quadrilaterals have at least one pair of parallel sides are precisely those on the circles A_1A_3S and A_2A_4S.*

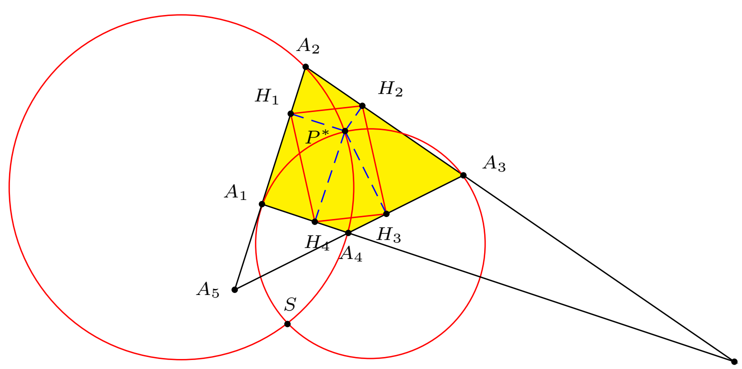

Figure 10

In general, the circles A_1A_3S and A_2A_4S intersect at two points, the Simson point S and one other point P^* (see Figure 10). The pedal quadrilateral of P^* is a parallelogram. We call P^* the *parallelogram point* of $\mathbf{P}$. Observe that if $\mathbf{P}$ is a parallelogram the parallelogram point is the intersection of the diagonals of $\mathbf{P}$.

If $\mathbf{P}$ is cyclic, the pedal quadrilateral of the circumcenter O of $\mathbf{P}$ is the Varignon parallelogram of $\mathbf{P}$. Therefore, the parallelogram point of $\mathbf{P}$ is O. It follows that *if $\mathbf{P}$ is cyclic, the Simson point is the intersection point of the circles A_1A_3O and A_2A_4O, other than O.*

4. Some properties of the pedal polygon

Let $\mathbf{P}$ be a polygon with n sides. Consider the pedal polygon $\mathbf{H}$ of a point P with respect to $\mathbf{P}$. We denote by $\mathbf{Q}_i$ the quadrilateral $PH_iA_{i+1}H_{i+1}$, for $i = 1, 2, \dots, n$. Since the angles in H_i and in H_{i+1} are right, $\mathbf{Q}_i$ cannot be concave.

Lemma 7. *If $ABCD$ is a convex or a crossed quadrilateral such that ABC and CDA are right angles, then it is cyclic. Moreover, its circumcenter is the midpoint of AC and its anticenter is the midpoint of BD.*

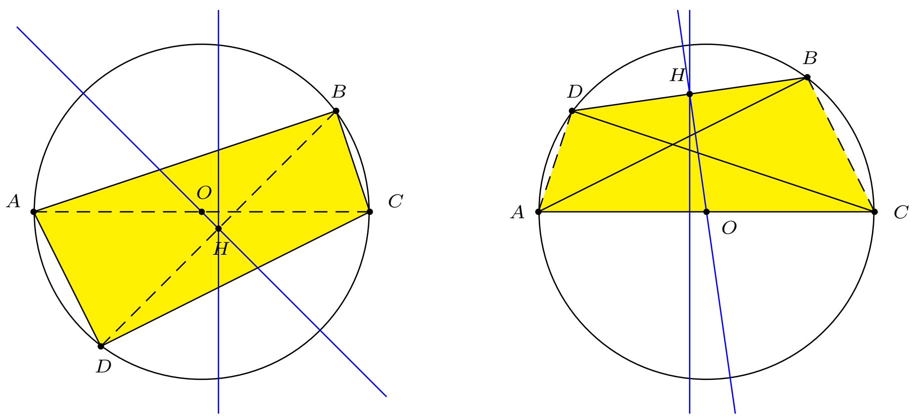

Figure 11

Proof. Let $ABCD$ be a convex or a crossed quadrilateral with ABC and CDA right angles (see Figure 11). Then, it is cyclic with the diagonal AC as diameter. (When it is a crossed quadrilateral it is inscribed in the semicircle with diameter AC). Then its circumcenter O is the midpoint of AC.

Consider the maltitudes with respect to the diagonals AC and BD. The maltitude through O is perpendicular to the chord BD of the circumcircle, then it passes through the midpoint H of BD. But also the maltitude relative to AC passes through H. Then, the anticenter of the quadrilateral is H. Note that the maltitudes of a crossed quadrilateral $ABCD$ are concurrent because they are also the maltitudes of the cyclic convex quadrilateral $ACBD$. $\square$

By Lemma 7, the quadrilaterals $\mathbf{Q}_i$ are cyclic. Denote by O_i and A'_i the circumcenter and the anticenter of $\mathbf{Q}_i$ respectively. We call $O_1O_2\dots O_n$ the *polygon of the circumcenters* of $\mathbf{P}$ with respect to P and denote it by $\mathbf{P}_c(P)$. We call $A'_1A'_2\dots A'_n$ the *polygon of the anticenters* of $\mathbf{P}$ with respect to P and we denote it with $\mathbf{P}_a(P)$.

Theorem 8. *The polygon $\mathbf{P}_c(P)$ is the image of $\mathbf{P}$ under the homothety $\mathsf{h}\left(P, \frac{1}{2}\right)$.*

Proof. By Lemma 7, the circumcenter O_i of $\mathbf{Q}_i$ is the midpoint of A_iP (see Figure 12 for a pentagon $\mathbf{P}$). □

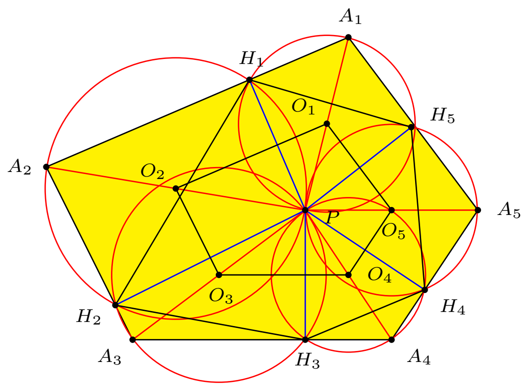

Figure 12

Note that by varying P the polygons $\mathbf{P}_c(P)$ are all congruent to each other (by translation).

Theorem 9. *The polygon $\mathbf{P}_a(P)$ is the medial polygon of $\mathbf{H}$, with vertices the midpoints of the segments H_iH_{i+1} for $i = 1, 2, \dots, n$.*

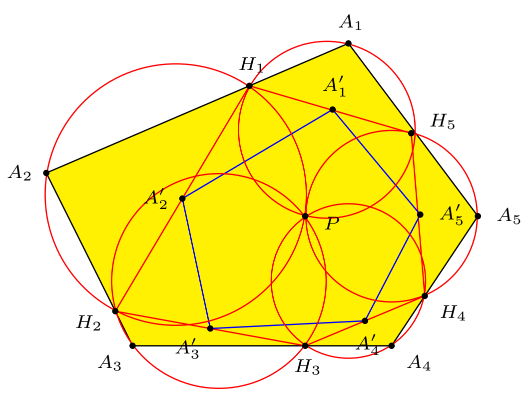

Figure 13

Proof. By Lemma 7, the anticenter A_i' of $\mathbf{Q}_i$ is the midpoint of H_iH_{i+3} (see Figure 13 for a pentagon). □

Corollary 10. (a) *If* $\mathbf{P}$ *is a triangle,* $\mathbf{P}_a(P)$ *is the medial triangle of* $\mathbf{H}$.
(b) *If* $\mathbf{P}$ *is a quadrilateral,* $\mathbf{P}_a(P)$ *is the Varignon parallelogram of* $\mathbf{H}$.

Theorem 11. *If* $\mathbf{H}$ *is cyclic, the Euler lines of the quadrilaterals* $\mathbf{Q}_i$ *are concurrent at the circumcenter of* $\mathbf{H}$.

Proof. The Euler line of the quadrilateral $\mathbf{Q}_i$ passes through the circumcenter O_i of $\mathbf{Q}_i$ and through the anticenter A_i' of $\mathbf{Q}_i$, that is the midpoint of H_iH_{i+3}, then it is the perpendicular bisector of a side of $\mathbf{H}$ (see Figure 14 for a quadrilateral). □

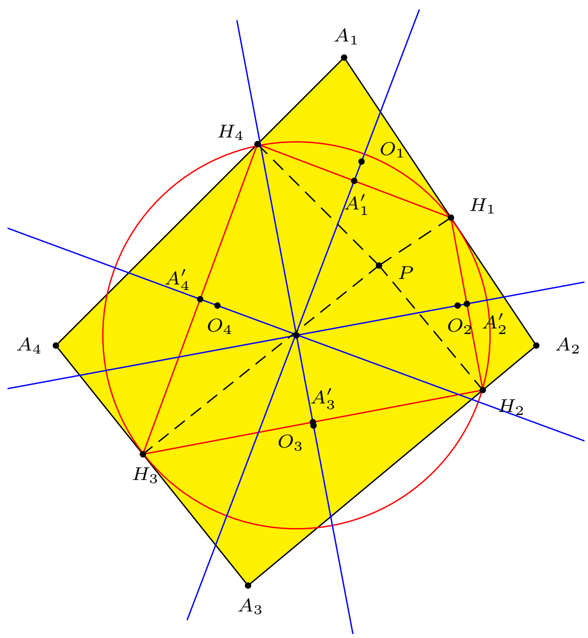

Figure 14

Corollary 12. *If* $\mathbf{P}$ *is a triangle, the Euler lines of the quadrilaterals* $\mathbf{Q}_i$ *are concurrent at the circumcenter of* $\mathbf{H}$ (see Figure 15).

Remark. If $\mathbf{P}$ is a quadrilateral and $\mathbf{H}$ is not cyclic, the Euler lines of the quadrilaterals $\mathbf{Q}_i$ bound a quadrilateral affine to $\mathbf{H}$ ([4, p.471]).

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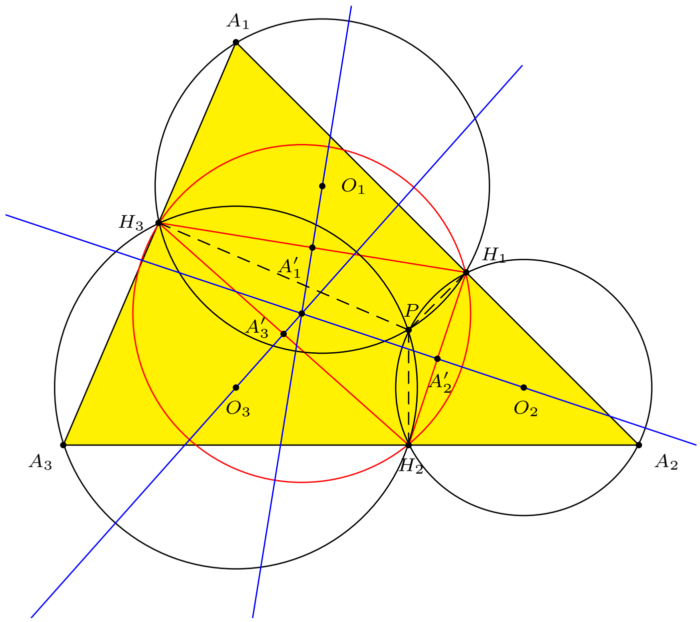

Figure 15

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