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Special Inscribed Trapezoids in a Triangle

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Abstract. We give a generalization Floor van Lamoen's recent result on jigsawing a quadrangle in a triangle.

1. Construction of an inscribed trapezoid

This note is a generalization of a recent result of Floor van Lamoen [1].

For an arbitrary point A' on the side BC of a given triangle ABC, we want to find on BC two points P, P' isotomic with respect to B and C such that the parallels from P, P' to AA' meet their closest sides AB, AC at the points Q, Q' and we have in the trapezoid $QPP'Q'$,

$$QQ' = PQ + P'Q'. \tag{1}$$

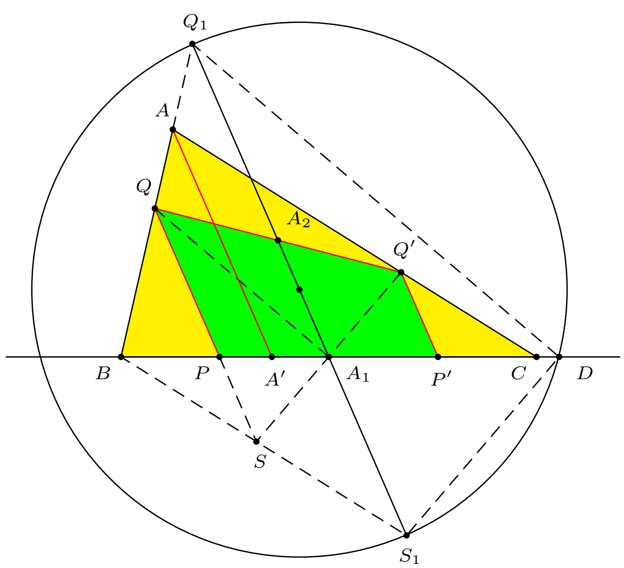

Figure 1.

Let A_1 be the midpoint of BC. The parallel from A_1 to AA' meets QQ' at its midpoint A_2, the line AB at Q_1, and the parallel from B to AC at a point S_1. The symmetric of Q' in A_1 is the point S, the intersection of BS_1 and QP. Since $A_1A_2 = \frac{PQ+P'Q'}{2} = \frac{QQ'}{2}$, the triangle A_1QQ' is right angled, and the same holds for the triangle A_1QS. The parallel from Q_1 to QA_1 meets the line BC at a point

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D and from $\frac{BS}{BS_1} = \frac{BQ}{BQ_1} = \frac{BA_1}{BD}$, we conclude that DS_1 is parallel to A_1S. Hence angle Q_1DS_1 is a right angle and the point D lies on the circle with diameter S_1Q_1. This leads to the following construction.

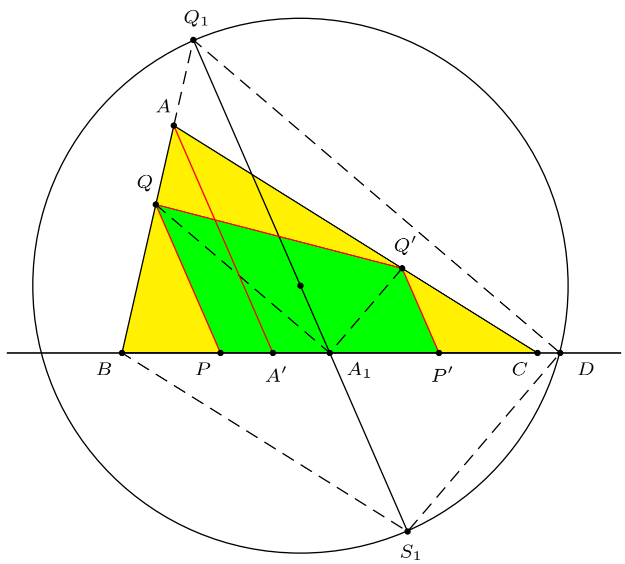

Figure 2.

Construct

(1) the parallel from A_1 to AA' to intersect the line AB at Q_1 and the parallel from B to AC at S_1,

(2) the circle with diameter Q_1S_1 to intersect BC at the point D such that A_1 is between B, D,

(3) the parallels from A_1 to DQ_1 and DS_1 to intersect AB at Q and AC at Q' respectively,

(4) the parallels to AA' from Q and Q' to intersect BC at P and P' respectively.

The points P, P' are isotomic with respect to BC, and the trapezoid $QPP'Q'$ satisfies (1) (see Figure 2).

2. An intersting property

Rotate triangle QBP about Q to $QA''P''$ with P'' on QQ'. Since $Q'P'' = Q'P'$ and $P'C = BP$, a rotation about Q' will bring triangle $Q'CP'$ to $Q'A''P''$. The lines QP'' and QA'' are the reflections of QB and QP in the bisector of angle Q of triangle AQQ'. Since QP is parallel to AA', the line QA'' contains the isogonal conjugate of the infinite point of AA' in triangle AQQ'. For the same reason, the line $Q'A''$ also contains the isogonal conjugate of the same infinite point. It follows that A'' is this isogonal conjugate, and it lies on the circumcircle of AQQ' (see Figure 3).

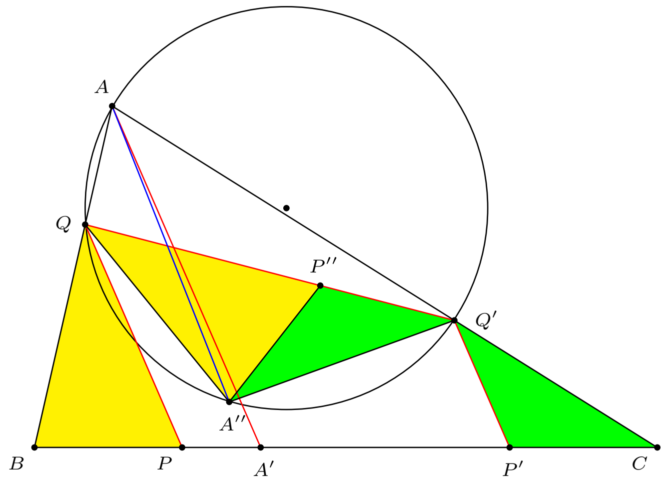

Figure 3.

In particular, if AA' is an altitude of triangle ABC, then the line AA'' passes through the circumcenter of the triangle.

Reference

[1] F. M. van Lamoen, Jigsawing a quadrangle from a triangle, *Forum Geom.*, 13 (2013) 149–152.

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