Gossard’s Perspector and Projective Consequences

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Abstract. Considering as starting point a geometric configuration studied, among others, by Gossard, we pursue the projective study of a triangle in the Euclidean plane, its Euler line and its nine-point circle, and we relate Pappus’ Theorem to the nine-point circle and Euler line.

1. Introduction

The relative position of Euler’s line with respect to the sides of a triangle has raised the geometers’ interest since the very first paper on this topic, Leonhard Euler’s classical work [10].

In 1997, problem A1 from the W. L. Putnam competition explored the case when Euler’s line is parallel to one of the sides of a triangle. Amer. Math. Monthly published Problem 10980 proposed by Ye Zhong Hao and Wu Wei Chao, whose statement is the following. Consider four distinct straight lines in the same plane, with the property that no two of them are parallel, no three are concurrent, and no three form an equilateral triangle. Prove that, if one of the lines is parallel to the Euler line of the triangle formed by the other three, then each of the four given lines is parallel to the Euler line of the triangle formed by the other three. In the Editorial Comment following the solution of problem 10980 (see vol. 111 (2004), pp.824), the editors have pointed out the meaningful contributions to the history of this problem, especially Gossard’s presentation at an A. M. S. conference in 1915. A generalization from 1999, given by Paul Yiu, is mentioned in [13].

In the Bulletin of the A. M. S. from 1916, we find O. D. Kellogg’s report on Gossard’s 1915 talk at the AMS Southwestern Section Conference (see [15]). As far as we know, Gossard’s paper has not been published, although we know from the report what he proved and what methods he used. The summary, as published by the Bulletin, is the following: "Euler proved that orthocenter, circumcenter, and centroid of a triangle are collinear, and the line through them has received the name Euler line. He also proved that the Euler line of a given triangle together with two of its sides forms a triangle whose Euler line is parallel with the third side of the given triangle. By the use of vector coordinates or ordinary projective coordinates, Professor Gossard proves the following theorem: the three Euler lines of
the triangles formed by the Euler line and the sides, taken by twos, of a given triangle, form a triangle triply perspective with the given triangle and having the same Euler line. The orthocenters, circumcenters and centroids of these two triangles are symmetrically placed as to the center of perspective.”

Our goal in the present note goes beyond providing elementary proofs for these facts, and aims to explore the deeper geometric meaning of a phenomenon seen in the above mentioned results. Application 1 is Ye and Wu’s problem. Applications 3 and 4, proved below, are just particular cases of Application 1. Proposition 1 is Gossard result from 1916, with a different proof. Furthermore, the original tools in Gossard’s work were ordinary projective coordinates. That’s why it would be natural to explore from a projective viewpoint the geometric structure inspired by Euler’s original contribution, which made the substance in Gossard’s work. In the last part of our paper, we discuss the projective viewpoint on the relative position of the Euler line and the three lines forming a given triangle. We will show how Euler’s line can be regarded as the axis of a projectivity between two sides of a triangle. This result was also proved by D. Barbilian (see [4]), and it appears in a note unpublished during Barbilian’s life. The result is presented below in our Proposition 4. We have been able to reconstruct the context of Barbilian’s work and we have obtained incidence results that complete the discussion on Ye and Wu’s problem.

Finally, with Propositions 3 and 4, which as far as we know appear for the first time here, we extend the projective analysis on this geometric structure (i.e., a triangle, its Euler line and its nine-point circle) and will relate Pappus’ Theorem to the nine-point circle and Euler line. We also study the parallelism of Euler’s line with one of the sides of the triangle from the projective viewpoint. In conclusion, one of the most important consequences of our investigation is that we are able to better understand the geometric connections between Euler’s line and the nine-point circle using projective methods. Our geometric motivation was the belief that beyond the synthetic and analytic methods, one can fathom the entire depth of a geometry problem by understanding the projective background of a certain geometric structure.

2. Synthetic and analytic viewpoint

First, we prove a Lemma which will become our main tool of investigation. This Lemma was inspired by Ye and Wu’s problem. Consider the Euclidean plane and a Cartesian frame. Let A, B, C be three arbitrary points in the Euclidean plane.

Lemma 1. Denote by $m_E$ the slope of Euler’s line in $\Delta ABC$ and by $m_1, m_2, m_3$ the slopes of the lines $BC$, $AC$, and $AB$, respectively. Then

$$m_E = \frac{-m_1m_2 + m_3m_1 + m_2m_3 + 3}{m_1 + m_2 + m_3 + 3m_1m_2m_3}$$

Proof. Measuring the slope of the angle between $BC$ and the Euler’s line of $\Delta ABC$, we have (see Figure 1):
\[
\frac{m_1 - m_E}{1 + m_1 m_E} = \tan \angle(HOS) = \frac{HS}{MN} = \frac{AM - AH - ON}{BN - BM} \\
= \frac{2R \sin B \sin C - 2R \cos A - R \cos A}{R \sin A - 2R \sin C \cos B} \\
= \frac{2 \sin B \sin C + 3 \cos(B + C)}{\sin(B + C) - 2 \sin C \cos B} \\
= \frac{3 \cos B \cos C - \sin B \sin C}{\sin B \cos C - \sin C \cos B} \\
= 3 - \tan B \tan C.
\]

Replacing in the last relation the following expressions
\[
\tan B = \frac{m_3 - m_1}{1 + m_1 m_3}, \quad \tan C = \frac{m_1 - m_2}{1 + m_2 m_1},
\]
we get the equality
\[
\frac{m_1 - m_E}{1 + m_1 m_E} = 3 \frac{m_3 - m_1}{1 + m_1 m_3} \cdot \frac{m_1 - m_2}{1 + m_2 m_1}.
\]

Cross-multiplying and collecting the like-terms, we obtain:
\[
m_1 m_2 + m_1 m_3 + m_1 m_E + m_2 m_3 + m_2 m_E + m_3 m_E + 3 m_E m_1 m_2 m_3 + 3 = 0.
\]

Solving for \(m_E\) in this relation immediately yields the relation from the statement of our lemma. \(\square\)

We should remark here that any other relative positions of the points \(A, B, C\) yield the same result. Now, we present several applications of this lemma.
Application 1. (Problem 10980, *American Mathematical Monthly*, proposed by Ye Zhong Hao and Wu Wei Chao, 109 (2002) 921, solution, 110 (2004) 823–824.) Consider four distinct straight lines in the same plane, with the property that no two of them are parallel, no three are concurrent, and no three form an equilateral triangle. Prove that, if one of the lines is parallel to the Euler line of the triangle formed by the other three, then each of the four given lines is parallel to the Euler line of the triangle formed by the other three.

Solution: Denote by \( m_1, m_2, m_3, \) and \( m_4 \) the slopes of the four lines \( d_1, d_2, d_3, \) and \( d_4, \) respectively. Suppose that Euler’s line of the triangle formed by the lines \( d_1, d_2, d_3, \) is parallel to \( d_4 \) and has slope \( m_E. \) Then \( m_E = m_4 \) and we get

\[
m_1m_2 + m_1m_3 + m_4m_1 + m_2m_4 + m_3m_4 + 3m_4m_1m_2m_3 + 3 = 0.
\]

This relation is symmetric in any one of the slopes and the conclusion follows immediately.

Application 2. Consider \( \triangle ABC \) and \( \triangle A'B'C' \) such that the measure of the oriented angles between the straight lines \( AB \) and \( A'B', \) \( AC \) and \( A'C', \) and \( BC \) and \( B'C', \) respectively, are equal to \( \alpha. \) Then the measure of the angle between Euler’s line of \( \triangle ABC \) and Euler’s line of \( \triangle A'B'C' \) is also \( \alpha. \)

Solution: We consider the following construction (see Figure 2). On the circumcircle of \( \triangle ABC, \) we consider the points \( A'', B'', \) and \( C'' \) such that \( A'B'' \parallel A'B', \) \( A''C'' \parallel A'C', \) and \( B''C'' \parallel B'C'. \) More precisely, we choose \( A'' \) such that the angle \( \hat{(AOA'')} \) is \( \alpha. \)

Let us consider now the rotation \( R_o^\alpha \) of center \( O \) (\( O \) is the circumcenter of \( \triangle ABC \)) and oriented angle \( \alpha. \) Then \( m(\triangle AB, A'B'') = m(\triangle AC, A''C'') = m(\triangle BC, B''C'') \) yields \( A'' = R_o^\alpha(A), B'' = R_o^\alpha(B), C'' = R_o^\alpha(C). \) We denote by \( e, e' \) and \( e'' \) Euler’s lines of \( \triangle ABC, \triangle A'B'C', \) and respectively \( \triangle A''B''C''. \) Then \( \triangle A''B''C'' \) is obtained by rotating \( \triangle ABC \) about \( O \) by \( \alpha. \) Thus, all the elements of \( \triangle ABC \) rotate about \( O. \) This means \( e'' = R_o^\alpha(e), \) or \( m(\triangle(e, e'')) = \alpha. \) Since the slopes satisfy the following equalities \( m_{A'B''} = m_{A'B'}, m_{A''C''} = \)
the slopes of the straight lines $AC$, $AB$, and $m$ from Lemma 1: 

Remark. Let $\triangle ABC$ and $\triangle A'B'C'$ be two triangles with the property $AB \perp A'B'$, $AC \perp A'C'$, $BC \perp B'C'$. Then the Euler lines of the two triangles are perpendicular.

We can prove this Remark directly from Lemma 1. However, we can also provide a direct argument for its proof. Denote by $m'_1, m'_2, m'_3$ the slopes of the side and by $m'_E$ the slope of Euler’s line of $\triangle A'B'C'$. Then

$$m'_E = \frac{m'_1 m'_2 + m'_3 m'_1 + m'_2 m'_3 + 3}{m'_1 + m'_2 + m'_3 + 3 m'_1 m'_2 m'_3}$$

$$= -\frac{\left(-\frac{1}{m_1}\right)\left(-\frac{1}{m_2}\right) + \left(-\frac{1}{m_3}\right)\left(-\frac{1}{m_1}\right) + \left(-\frac{1}{m_2}\right)\left(-\frac{1}{m_3}\right)}{m_1 + m_2 + m_3 + 3m_1 m_2 m_3}$$

$$= \frac{m_1 + m_2 + m_3 + 3m_1 m_2 m_3}{m_1 m_2 + m_3 m_1 + m_2 m_3 + 3}$$

$$= -\frac{1}{m_E}.$$ 

This proves that the two lines are perpendicular.

Application 3. In the acute triangle $ABC$, Euler’s line is parallel to $BC$ if and only if $\tan B \tan C = 3$.

Note: In [17], it is mentioned that this problem was proposed by Dan Brânzei. We have discussed this application in [6]. The solution uses a direct trigonometric argument. We present here the analytic argument based on Lemma 1.

Solution: Choose a coordinate system so that the $x$-axis is parallel to $BC$. If we denote by $m_1$ the slope of the straight line $BC$, then $m_1 = 0$. Denoting $m_2, m_3, m_e$ the slopes of the straight lines $AC$, $AB$, and Euler’s line $e$, respectively, we get from Lemma 1:

$$m_e = -\frac{m_2 m_3 + 3}{m_2 + m_3}.$$

Thus, Euler’s line $e$ of $\triangle ABC$ is parallel to $BC$ if and only if $m_e = 0$, which is equivalent to $m_2 m_3 = -3$. Now we take into account that $m_2 = -\tan C$ and $m_3 = \tan B$ (or, depending on the position of $\triangle ABC$, we could have $m_2 = \tan C$ and $m_3 = -\tan B$). Consequently, $\tan B \tan C = 3$. □

For an interesting connection between the formula obtained here for $m_e$ and Tzitzeica surfaces, a topic studied in depth in affine differential geometry, see [2]. For a graphical study of Tzitzeica surfaces by using Mathematica, see [3]. For the importance of Tzitzeica’s surfaces in the development of differential geometry at the beginning of the 20th century, see [1].

Application 4. (W. L. Putnam Competition, 1997) A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle $ABC$ has $H$ as the intersection of the altitudes, $O$ the center of the circumscribed circle, $M$ the midpoint of $BC$, and $F$ the foot of the altitude from $A$. What is the length of $BC$?
Solution: Since Euler’s line is parallel to $BC$, by the previous application, we have $\tan B \tan C = 3$. This is just a consequence of the previous application. We can continue our argument as in [14], pg.233, or [6]. Expressing $\tan B$ and $\tan C$ from triangles $ABF$ and $AFC$, respectively, we get

\[
\frac{h_a}{BF} \cdot \frac{h_a}{FC} = 3.
\]

Since $HG \parallel BC$, we have $h_a = AF = 3FH = 3 \cdot 5 = 15$. Therefore, $BF \cdot FC = \frac{15 \cdot 15}{2} = 75$. Namely, we express $BC^2 = (BF+FC)^2 = (FC-BF)^2 + 4BF \cdot FC$.

To compute the first term in the last expression we write $FC-BF = FM + MC - (BM - FM) = 2FM = 2OH = 22$. Therefore, $BC^2 = 22^2 + 4 \cdot 75 = 784$, thus $BC = 28$. \hfill \Box

Lemma 2. Euler’s line of $\triangle ABC$ intersects the lines $AB$ and $AC$ in $M$, respectively $N$. Then Euler’s line of $\triangle AMN$ is parallel to $BC$.

Proof. Choose a coordinate system so that the $x$-axis is parallel to $BC$, as in Application 3 (see Figure 3). If we denote by $m_1$ the slope of the straight line $BC$, then $m_1 = 0$. Denoting $m_2, m_3, m_e$ the slopes of the straight lines $AC$, $AB$, and respectively Euler’s line $e$. By Lemma 1:

\[
m_e = -\frac{m_2m_3 + 3}{m_2 + m_3},
\]

and the slope of Euler’s line of $\triangle AMN$ is

\[
m_e' = \frac{m_em_2 + m_em_3 + m_2m_3 + 3}{m_e + m_2 + m_3 + 3m_em_2m_3}.
\]
In fact, the numerator of the last expression is
\[
m_{e}m_{2} + m_{e}m_{3} + m_{2}m_{3} + 3
\]
\[
= m_{e}(m_{2} + m_{3}) + m_{2}m_{3} + 3
\]
\[
= \left(-\frac{m_{2}m_{3} + 3}{m_{2} + m_{3}}\right)(m_{2} + m_{3}) + m_{2}m_{3} + 3 = 0.
\]
In fact, we proved that \(m_{e'} = 0\), which means that \(e' \parallel BC\).

**Application 5.** Consider two triangles such that \(\Delta ABC \equiv \Delta A'B'C'\) and they have the same Euler's line. Then \(\Delta A'B'C'\) is obtained from \(\Delta ABC\) either by a translation, or by a central symmetry.

**Example 1.** Problem 244 in [19] states the following. Let \(H\) be the orthocenter of \(\Delta ABC\), and \(O_{a}, O_{b}, O_{c}\) the circumcenters of triangles \(BHC, CHA, AHB\). Then \(\Delta ABC \equiv \Delta O_{a}O_{b}O_{c}\) have the same nine-point circle and the same Euler's line. This provides us an example of two triangles that have the same Euler's line (see Figure 4).

![Figure 4.](image-url)

**Example 2.** Now we describe two triangles of interest that have the same Euler’s line. Consider \(\Delta ABC\) and its circumcircle \(C\). Consider also the incircle tangent to \(BC, AC\) and \(AB\) respectively in \(D, E,\) and \(F\). On the straight lines \(AI, BI, CI\) we consider the excenters (i.e., the centers of the excircles) \(I_{a}, I_{b},\) and \(I_{c}\). Remark that...
the circumcircle of $\triangle ABC$ is the nine-point circle of $\triangle I_a I_b I_c$, because $A, B, C$ are the feet of the altitudes (e.g. $AI_a \perp I_b I_c$).

Thus, $I$ is the orthocenter in $\triangle I_a I_b I_c$, and $O$ is the center of the nine-point circle in $\triangle I_a I_b I_c$. Therefore, $OI$ is Euler’s line in $\triangle I_a I_b I_c$. Remark that $\triangle DEF$ and $\triangle I_a I_b I_c$ have parallel sides. Therefore their Euler’s lines must be parallel (we may say that this is a consequence of Application 2). But the circumcenter of $\triangle DEF$ is the point $I$. This means that the Euler’s line of $\triangle DEF$ passes through $I$ and, being parallel to $OI$, must be $OI$. □

3. Gossard’s perspector

In this section we present an elementary proof of Gossard’s result cited in [15].

**Proposition 3** (Gossard, [15]). Denote by $e$ the Euler line of an arbitrary $\triangle ABC$ in the Euclidean plane. Suppose that $e$ intersects $BC, AB, AC$ in $M, N,$ and respectively $P$. Denote by $e_1, e_2, e_3$ Euler’s lines of $\triangle ANP, \triangle BMN,$ and $\triangle CPM, respectively$. Denote $A’, B’, C’$ the intersection of the following pair of lines: $e_2 \cap e_3, e_1 \cap e_3,$ and $e_1 \cap e_2,$ respectively. Then $\triangle A’B’C’ \equiv \triangle ABC$, and $\triangle A’B’C’$ has the same Euler line $e$, and there exists a point $I_G$ (called Gossard’s perspector) on the line $e$ such that $\triangle A’B’C’$ is the symmetric of $\triangle ABC$ by the symmetry centered in $I_G$.

The proof presented below is based on Lemma 1. Thus, we claim that it may be more elementary than Gossard’s original proof, as it is presented by Kellogg in [15]. An important rôle in the proof is played by the conditions $e_1 \parallel BC, e_2 \parallel AC, e_3 \parallel AB$. 
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**Proof.** We choose coordinate axis such that the vertices of \( \Delta ABC \) have the coordinates \( A(0, 1), B(b, 0), C(c, 0) \) (see Figure 4). Let \( G \) be the gravity center of \( \Delta ABC \); then \( G(\frac{b+c}{3}, \frac{1}{3}) \). The slope of Euler’s line in \( \Delta ABC \) is given by

\[
m_e = \frac{m_2 m_3 + 3}{m_2 + m_3} = \frac{\left(-\frac{1}{c}\right) \left(-\frac{1}{b}\right) + 3}{-\frac{1}{c} - \frac{1}{b}} = \frac{3bc + 1}{b + c}.
\]

Thus, the equation of Euler’s line is \( y = \frac{3bc + 1}{b + c} x - bc \). The coordinates of the points \( M, N, \) and \( P \) are:

\[
M\left( \frac{bc(b + c)}{3bc + 1}, 0 \right),
\]

\[
N\left( \frac{b(b + c)(bc + 1)}{3b^2c + 2b + c}, \frac{2b^2c - bc^2 + b}{3b^2c + 2b + c} \right),
\]

\[
P\left( \frac{c(b + c)(bc + 1)}{3bc^2 + 2c + b}, \frac{2bc^2 - b^2c + c}{3bc^2 + 2c + b} \right).
\]

The line \( e_1 \) passes through the center of gravity of \( \Delta ANP \) and is parallel to \( BC \), therefore it has the equation

\[
(e_1) : y = \frac{y_N + y_P + y_A}{3}.
\]

At the intersection of lines \( e \) and \( e_1 \) we have the point \( Q \) whose coordinates are

\[
Q\left( \frac{b + c}{3bc + 1}, \frac{1}{3}(E + 3bc), \frac{1}{3}E \right),
\]
where we have denoted by
\[
E = \frac{2b^2c - bc^2 + b}{3b^2c + 2b + c} + \frac{2bc^2 - b^2c + c}{3bc^2 + 2c + b} + 1.
\]
The center of gravity of \(\triangle BMN\), denoted \(R\), has the coordinates
\[
(x_R, y_R) = \left( \frac{1}{3} \left( \frac{bc(b + c)}{3bc + 1} + b + \frac{b(b + c)(bc + 1)}{3b^2c + 2b + c} \right), \quad \frac{1}{3} \left( \frac{2b^2c - bc^2 + b}{3b^2c + 2b + c} \right) \right).
\]
Euler’s line in \(\triangle BMN\) passes through \(R\) and is parallel to \(AC\), thus it has the equation
\[
(e_2): \quad y - y_R = -\frac{1}{c}(x - x_R).
\]
Denote by \(S\) the intersection of the lines \(e\) and \(e_2\). We get
\[
y_S = \frac{(3bc + 1)(x_R + cy_R) - bc(b + c)}{3bc^2 + 2c + b}.
\]
To emphasize the transformation by symmetry (as described in [15]), we claim that \(y_S + y_P = y_Q + y_M\). This is equivalent to
\[
\frac{(3bc + 1)(x_R + cy_R) - bc(b + c)}{3bc^2 + 2c + b} + \frac{2bc^2 - b^2c + c}{3bc^2 + 2c + b} = \frac{1}{3} \left( \frac{2b^2c - bc^2 + b}{3b^2c + 2b + c} + \frac{2bc^2 - b^2c + c}{3bc^2 + 2c + b} + 1 \right).
\]
By replacing \(x_R\) and \(y_R\) and simplifying the relation, we obtain the desired equality. Therefore, the segments \([PS]\) and \([QM]\) have the same midpoint. (It is not necessary to check also that \(x_P + x_S = x_Q + x_M\), since \(P, S, Q\) and \(M\) are collinear.)

Denote by \(I_G\) the common midpoint of those two segments. As above, one can prove that \(I_G\) is the midpoint of the segment \([NT]\), where \(\{T\} = e_3 \cap e\). The analogy of the computation can be further seen since the coordinates of \(I_G\) are symmetric in \(b\) and \(c\). Thus, with the above notation for \(E\), \(I_G\) has the coordinates
\[
(x_{I_G}, y_{I_G}) = \left( \frac{1}{2} \left( \frac{bc(b + c)}{3bc + 1} + \frac{b + c}{3bc + 1} \cdot \frac{1}{3}(E + 3bc) \right), \quad \frac{1}{6}E \right)
\]
We can write the coordinates in the form
\[
I_G \left( \frac{1}{6} \cdot \frac{b + c}{3bc + 1}(E + 6bc), \frac{1}{6}E \right).
\]
This is the point called the Gossard perspector. Denote \(S_{I_G}\) the symmetry of center \(I_G\) in the Euclidean plane. Since \(e_1 \parallel BC\), \(Q \in e_1 M \in BC\), and \(I_G\) is the midpoint of \([QM]\), we have \(e_1 = S_{I_G}(BC)\). Similarly \(e_2 = S_{I_G}(AC)\), \(e_3 = S_{I_G}(AB)\).

Then, we have obtained the following:
\[
\{A'\} = e_2 \cap e_3 = S_{I_G}(AC) \cap S_{I_G}(AB) = S_{I_G}(AC \cap AB) = S_{I_G}(\{A\}).
\]
Similarly, \(\{B'\} = S_{I_G}(\{B\})\), and \(\{C'\} = S_{I_G}(C)\).

Consequently, \(\triangle A'B'C' \equiv \triangle ABC\), and \(\triangle A'B'C'' = S_{I_G}(\triangle ABC)\).
Denoting $G$ and $G'$ the gravity centers of $\triangle ABC$ and $\triangle A'B'C'$, we have $\{G'\} = S_{IG}(\{G\})$. For the orthocenters we get a similar correspondence: $\{H'\} = S_{IG}(\{H\})$. Thus, $e' = S_{IG}(e)$, where $e'$ is Euler’s line of $\triangle A'B'C'$. But $I_G \in e$. Thus, Euler’s line $e$ passes through the center of symmetry. We deduce that $S_{IG}(e) = e$, or $e' = e$. Finally, we proved that $\triangle ABC$ and $\triangle A'B'C'$ have the same Euler’s line. This completes the analytic proof of Gossard’s perspector theorem, as mentioned in our introduction (see [15]).

Example 3. We have seen in Example 1 (see [19], 244) that if $H$ is the orthocenter of $\triangle ABC$, and $O_a$, $O_b$, $O_c$ are the circumcenters of triangles $BHC$, $CHA$, $AHB$, then $\triangle ABC$ and $\triangle O_aO_bO_c$ have the same Euler’s line (see Figure 4). In fact, $O_a$, $O_b$, and $O_c$ are the symmetric points of $O$ with respect to the sides $BC$, $AC$ and, respectively, $AB$. Denote by $A_1$, $B_1$, and $C_1$ the midpoints of the sides $BC$, $AC$ and, respectively, $AB$.

Then $H$ is the circumcenter of $\triangle O_aO_bO_c$. Actually, $\triangle O_aO_bO_c$ is the homothetic of $\triangle A_1B_1C_1$ by homothety of center $O$ and ratio 2. Thus, $\triangle O_aO_bO_c$ has the sides parallel and congruent to the sides of $\triangle ABC$, and, furthermore, $OO_a \perp BC$, and also $OO_a \perp O_bO_c$, (and the similar relations). This proves that $O$ is the orthocenter of $\triangle O_aO_bO_c$. Therefore $\triangle ABC$ and $\triangle O_aO_bO_c$ interchanged among them the orthocenters and the circumcenters. This is the argument to see that the Euler’s lines in the two triangles are the same and the two triangles have the same center of the nine-point circle, since $O_9$ is the midpoint of $OH$. Further, $\triangle ABC$ and $\triangle O_aO_bO_c$ are symmetric with respect to $O_9$. Therefore, Gossard’s perspector in $\triangle O_aO_bO_c$ is the symmetric of Gossard perspector in $\triangle ABC$ with respect to $O_9$, the center of the nine-point circle.

4. Projective viewpoint

Consider now a projectivity $f : d_1 \to d_2$. (See also [7, pp.39 ff], [8, pp.9-11]) The geometric locus of the points from which the the projectivity is seen as an involution of pencils of lines is called axis of the projectivity.
intersection of the cross-joints of any pairs of corresponding points (see [8, pp.36-37]). This result is known as the axis theorem. To illustrate it, if \( M_1 \rightarrow N_1 \) and \( M_2 \rightarrow N_2 \), then the point \( \{ P \} = M_1N_2 \cap M_2N_1 \) lies on the axis of the projectivity, since we have the mapping \( r_1 = PM_1 \rightarrow PN_1 = r_2 \) and \( r_2 = PM_2 \rightarrow PN_1 = r_1 \). Thus, \( r_1 \rightarrow r_2 \) and \( r_2 \rightarrow r_1 \), which means that the projectivity \( f : d_1 \rightarrow d_2 \) is seen as an involution. As a consequence, we remind here the well-known geometric structure called Pappus’ line.

\[ A' \quad B' \quad C' \]
\[ A \quad B \quad C \]

![Figure 8.](image1)

Let \( A, B, C \in d_1 \) and \( A', B', C' \in d_2 \). Then the points \( \{ M \} = AB' \cap BA' \), \( \{ N \} = AC' \cap AC' \cap CA' \), \( \{ P \} = BC' \cap CB' \), are collinear (see Figure 8). This result can be viewed as an immediate consequence of the axis theorem. Indeed, consider the projectivity \( f : d_1 \rightarrow d_2 \) uniquely determined by \( A \rightarrow A' \), \( B \rightarrow B' \), \( C \rightarrow C' \). By the axis theorem, we get immediately that the points \( \{ M \} = AB' \cap BA' \), \( \{ N \} = AC' \cap AC' \cap CA' \), \( \{ P \} = BC' \cap CB' \) are collinear. With this preparation, we are able to show that the Euler’s line of a triangle \( ABC \) can be regarded as the axis of projectivity for three suitable projectivities between the sides of \( \triangle ABC \) (see Figure 9).

\[ A \quad B' \quad C' \]
\[ A' \quad B \quad C \]

![Figure 9.](image2)
Denote by $A', B', C'$ the midpoints of the sides $BC$, $AC$, and respectively $AB$. Denote by $A_1, B_1, C_1$ the feet of altitudes from $A, B, C$. We use the standard notations for $O$, the circumcircle, $G$ the center of gravity, and $H$ the orthocenter of $\triangle ABC$. There are three projectivities, each one between two sides of $\triangle ABC$. One of them is $f_C : BC \rightarrow AC$, the projectivity determined by $B \rightarrow A$, $A_1 \rightarrow B_1$, $A' \rightarrow B'$. Since $H$ and $G$ appear as cross-joints points, they lie on the axis of projectivity of $f_C$. Specifically, \( \{ H \} = AA_1 \cap BB_1 \), \( \{ G \} = BB' \cap AA' \). Since two points determine uniquely a line, and since $G$ and $H$ determine Euler’s line, this means that the Euler’s line is identified with the axis of projectivity $f_C$. Furthermore, on the Euler’s line we get a new point: \( \{ \Omega_{AB} \} = A_1B' \cap A'B_1 \). We can also emphasize the pair of homologous points that determine $O$, the circumcenter, in this projectivity. Extend the line determined by the vertex $A$ and by $O$ and denote \( \{ X \} = AO \cap BC \). Similarly, \( \{ Y \} = BO \cap AC \). Since in our projectivity $B \rightarrow A$, then $X \rightarrow Y$. Thus, on the axis of projectivity we obtain \( \{ O \} = AX \cap BY \).

Considering similar constructions for the projectivities $f_A$ and $f_B$, we obtain the following fact.

**Proposition 4** (Barbilian [4]). In $\triangle ABC$, let $A', B', C'$ be the midpoints of the sides $BC$, $AC$, and respectively $AB$. Denote by $A_1, B_1, C_1$ the feet of altitudes from $A, B, C$. Then the points \( \{ \Omega_{AB} \} = A_1B' \cap A'B_1 \), \( \{ \Omega_{CB} \} = C_1B' \cap C'B_1 \), \( \{ \Omega_{AC} \} = A_1C' \cap A'C_1 \) are collinear and they lie on Euler’s line of $\triangle ABC$.

In the first part, we have presented Applications 3 and 4, where we give synthetic and trigonometric characterizations of the fact that Euler’s line is parallel to a side of the triangle. We study here the following question: What is projective condition that the projectivity $f_C : BC \rightarrow AC$ must satisfy such that Euler’s
line is parallel to $BC$? Denote by $(e)$ Euler’s line in $\triangle ABC$. (See Figure 10.) Let $\{T\} = AC \cap (e), \{U\} = BC \cap (e)$. We need to determine the pairs of straight lines that characterize in a projectivity the points $T$ and $U$. Recall that the projectivity $f_C$ has as homologous points $B \to A$. To get $T$, consider the pair $C \to (e) \cap AC$. Similarly, we get $U$ by the pair $(e) \cap BC \to C$. Therefore we have obtained the projective characterization of the fact that the Euler line is parallel to a side of the triangle. Thus, we are able to state the projective counterpart of Application 3, which is the trigonometric characterization of this parallelism.

**Proposition 5.** In $\triangle ABC$, let $(e)$ be the Euler’s line. The sufficient condition that $(e) \parallel BC$, is that the projectivity $f_C$ has $\infty \to C$ as pair of homologous points. Similarly, to have $(e) \parallel AC$, it is sufficient that $f_C$ has $C \to \infty$ as pairs of homologous points.

Four our next step, we need to recall here Pappus’ Theorem on the circle. Let $A, B, C$ and $A', B', C'$ six points on the circle $C$. Then the intersection points $AB' \cap A'B, AC' \cap A'C$ and $BC' \cap B'C$ are collinear. To recall the idea of the most direct proof, consider the projectivity $f : C \to C$ uniquely determined by $A \to A', B \to B', C \to C'$. Then, the intersection points mentioned in the statement lie precisely on the axis of the projectivity. With this observation, we obtain that Euler’s line is the axis of projectivity of a certain projectivity within the nine-point circle. The result is the following.

**Proposition 6.** Consider $A', B', C'$ the midpoints of the sides $BC, AC$ and respectively $AB$. Let $A_1, B_1$ and $C_1$ the feet of the altitudes. Consider the projectivity $\phi$ uniquely determined by $A_1 \to B_1$, $A' \to B', B_2 \to A_2$. Then the points $A_1A_2 \cap B_1B_2 = \{H\}$ (the orthocenter of $\triangle ABC$), $A_1B' \cap A'B_1 = \{\Omega_{AB}\}$ and $A'A_2 \cap B'B_2 = \{O_9\}$ (the center of the nine-point circle) are collinear on the axis of projectivity of $\phi$. 

![Figure 11](image-url)
The proof is just a direct application of Pappus’ Theorem on the circle, for the geometric structure described in the statement. Since $H$ and $\Omega_{AB}$ are on Euler’s line, the axis of projectivity and Euler’s line must be the same straight line. As a consequence, the third point, $O_9$, the center of the nine-point circle, must be on the axis of projectivity, thus on Euler’s line.

Proposition 4 appears in [4, pp. 40]. Actually, Dan Barbilian collected in an undated note, published in the cited collection of posthumous works, several projective properties of the nine-point circle and its connection with Euler’s line. He focused mainly on the projective properties, which represent, as we can see, an important part of the more complex phenomenon whose overall picture we tried to present here.

References

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