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Gossard's Perspector and Projective Consequences

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Abstract. Considering as starting point a geometric configuration studied, among others, by Gossard, we pursue the projective study of a triangle in the Euclidean plane, its Euler line and its nine-point circle, and we relate Pappus' Theorem to the nine-point circle and Euler line.

1. Introduction

The relative position of Euler's line with respect to the sides of a triangle has raised the geometers' interest since the very first paper on this topic, Leonhard Euler's classical work [10].

In 1997, problem A1 from the W. L. Putnam competition explored the case when Euler's line is parallel to one of the sides of a triangle. *Amer. Math. Monthly* published Problem 10980 proposed by Ye Zhong Hao and Wu Wei Chao, whose statement is the following. *Consider four distinct straight lines in the same plane, with the property that no two of them are parallel, no three are concurrent, and no three form an equilateral triangle. Prove that, if one of the lines is parallel to the Euler line of the triangle formed by the other three, then each of the four given lines is parallel to the Euler line of the triangle formed by the other three.* In the Editorial Comment following the solution of problem 10980 (see vol. 111 (2004), pp.824), the editors have pointed out the meaningful contributions to the history of this problem, especially Gossard's presentation at an A. M. S. conference in 1915. A generalization from 1999, given by Paul Yiu, is mentioned in [13].

In the *Bulletin of the A. M. S.* from 1916, we find O. D. Kellogg's report on Gossard's 1915 talk at the AMS Southwestern Section Conference (see [15]). As far as we know, Gossard's paper has not been published, although we know from the report what he proved and what methods he used. The summary, as published by the *Bulletin*, is the following: *"Euler proved that orthocenter, circumcenter, and centroid of a triangle are collinear, and the line through them has received the name Euler line. He also proved that the Euler line of a given triangle together with two of its sides forms a triangle whose Euler line is parallel with the third side of the given triangle. By the use of vector coordinates or ordinary projective coordinates, Professor Gossard proves the following theorem: the three Euler lines of*

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the triangles formed by the Euler line and the sides, taken by twos, of a given triangle, form a triangle triply perspective with the given triangle and having the same Euler line. The orthocenters, circumcenters and centroids of these two triangles are symmetrically placed as to the center of perspective."

Our goal in the present note goes beyond providing elementary proofs for these facts, and aims to explore the deeper geometric meaning of a phenomenon seen in the above mentioned results. Application 1 is Ye and Wu's problem. Applications 3 and 4, proved below, are just particular cases of Application 1. Proposition 1 is Gossard result from 1916, with a different proof. Furthermore, the original tools in Gossard's work were ordinary projective coordinates. That's why tt would be natural to explore from a projective viewpoint the geometric structure inspired by Euler's original contribution, which made the substance in Gossard's work. In the last part of our paper, we discuss the projective viewpoint on the relative position of the Euler line and the three lines forming a given triangle. We will show how Euler's line can be regarded as the axis of a projectivity between two sides of a triangle. This result was also proved by D. Barbilian (see [4]), and it appears in a note unpublished during Barbilian's life. The result is presented below in our Proposition 4. We have been able to reconstruct the context of Barbilian's work and we have obtained incidence results that complete the discussion on Ye and Wu's problem.

Finally, with Propositions 3 and 4, which as far as we know appear for the first time here, we extend the projective analysis on this geometric structure (*i.e.*, a triangle, its Euler line and its nine-point circle) and will relate Pappus' Theorem to the nine-point circle and Euler line. We also study the parallelism of Euler's line with one of the sides of the triangle from the projective viewpoint. In conclusion, one of the most important consequences of our investigation is that we are able to better understand the geometric connections between Euler's line and the nine-point circle using projective methods. Our geometric motivation was the belief that beyond the synthetic and analytic methods, one can fathom the entire depth of a geometry problem by understanding the projective background of a certain geometric structure.

2. Synthetic and analytic viewpoint

First, we prove a Lemma which will become our main tool of investigation. This Lemma was inspired by Ye and Wu's problem. Consider the Euclidean plane and a Cartesian frame. Let A, B, C be three arbitrary points in the Eulcidean plane.

Lemma 1. *Denote by m_E the slope of Euler's line in ΔABC and by m_1, m_2, m_3 the slopes of the lines BC, AC, and AB, respectively. Then*

$$m_E = -\frac{m_1m_2 + m_3m_1 + m_2m_3 + 3}{m_1 + m_2 + m_3 + 3m_1m_2m_3}.$$

Proof. Measuring the slope of the angle between BC and the Euler's line of ΔABC, we have (see Figure 1):

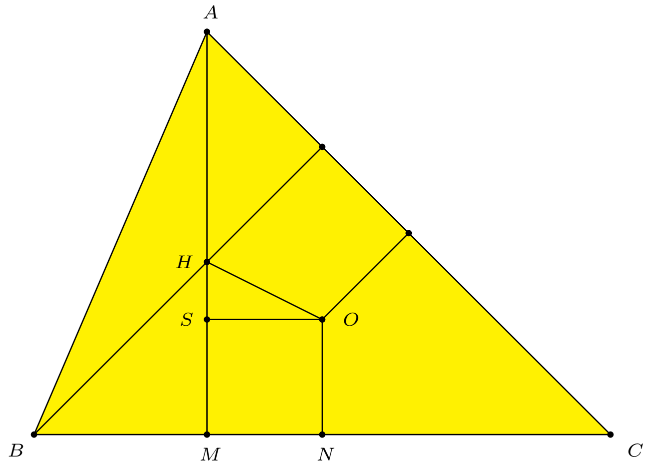

Figure 1

$$\begin{aligned}\frac{m_1 - m_E}{1 + m_1 m_E} &= \tan\angle(HOS) = \frac{HS}{MN} = \frac{AM - AH - ON}{BN - BM}\\ &= \frac{2R\sin B\sin C - 2R\cos A - R\cos A}{R\sin A - 2R\sin C\cos B}\\ &= \frac{2\sin B\sin C + 3\cos(B+C)}{\sin(B+C) - 2\sin C\cos B}\\ &= \frac{3\cos B\cos C - \sin B\sin C}{\sin B\cos C - \sin C\cos B}\\ &= \frac{3 - \tan B\tan C}{\tan B - \tan C}.\end{aligned}$$

Replacing in the last relation the following expressions

$$\tan B = \frac{m_3 - m_1}{1 + m_1 m_3}, \qquad \tan C = \frac{m_1 - m_2}{1 + m_1 m_2},$$

we get the equality

$$\frac{m_1 - m_E}{1 + m_1 m_E} = \frac{3 - \frac{m_3 - m_1}{1 + m_1 m_3} \cdot \frac{m_1 - m_2}{1 + m_1 m_2}}{\frac{m_3 - m_1}{1 + m_1 m_3} - \frac{m_1 - m_2}{1 + m_1 m_2}}.$$

Cross-multiplying and collecting the like-terms, we obtain:

$$m_1m_2 + m_1m_3 + m_1m_E + m_2m_3 + m_2m_E + m_3m_E + 3m_Em_1m_2m_3 + 3 = 0.$$

Solving for m_E in this relation immediately yields the relation from the statement of our lemma. □

We should remark here that any other relative positions of the points A, B, C yield the same result. Now, we present several applications of this lemma.

Application 1. (Problem 10980, *American Mathematical Monthly*, proposed by Ye Zhong Hao and Wu Wei Chao, 109 (2002) 921, solution, 110 (2004) 823–824.) Consider four distinct straight lines in the same plane, with the property that no two of them are parallel, no three are concurrent, and no three form an equilateral triangle. Prove that, if one of the lines is parallel to the Euler line of the triangle formed by the other three, then each of the four given lines is parallel to the Euler line of the triangle formed by the other three.

Solution: Denote by m_1, m_2, m_3, and m_4 the slopes of the four lines d_1, d_2, d_3, d_4, respectively. Suppose that Euler's line of the triangle formed by the lines d_1, d_2, d_3 is parallel to d_4 and has slope m_E. Then $m_E = m_4$ and we get

$$m_1m_2 + m_1m_3 + m_1m_4 + m_2m_3 + m_2m_4 + m_3m_4 + 3m_4m_1m_2m_3 + 3 = 0.$$

This relation is symmetric in any one of the slopes and the conclusion follows immediately. □

Application 2. Consider ΔABC and $\Delta A'B'C'$ such that the measure of the oriented angles between the straight lines AB and $A'B'$, AC and $A'C'$, and BC and $B'C'$, respectively, are equal to α. Then the measure of the angle between Euler's line of ΔABC and Euler's line of $\Delta A'B'C'$ is also α.

Solution: We consider the following construction (see Figure 2). On the circumcircle of ΔABC, we consider the points A'', B'' and C'' such that $A''B''\|A'B'$, $A''C''\|A'C'$ and $B''C''\|B'C'$. More precisely, we choose A'' such that the angle $(\widehat{AOA''})$ is α.

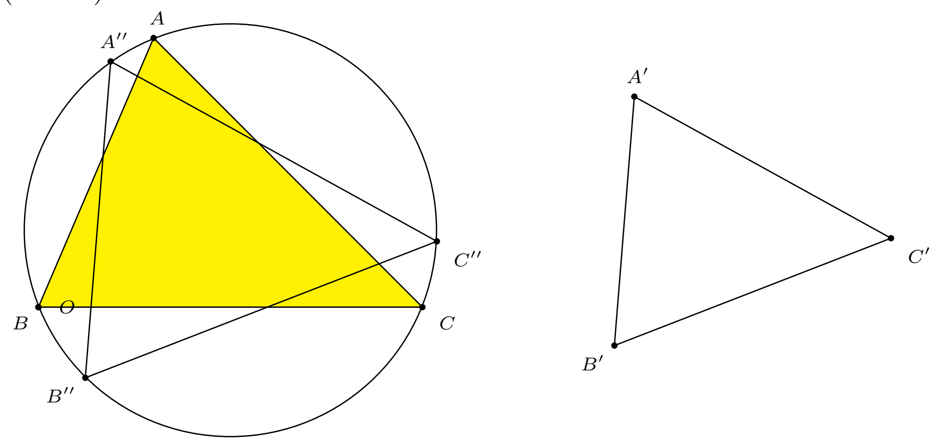

Figure 2

Let us consider now the rotation R_O^α of center O (O is the circumcenter of ΔABC) and oriented angle α. Then $m(\angle(AB, A''B'')) = m(\angle(AC, A''C'')) = m(\angle(BC, B''C''))$ yields $A'' = R_O^\alpha(A)$, $B'' = R_O^\alpha(B)$, $C'' = R_O^\alpha(C)$. We denote by e, e' and e'' Euler's lines of ΔABC, $\Delta A'B'C'$, and respectively $\Delta A''B''C''$. Then $\Delta A''B''C''$ is obtained by rotating ΔABC about O by α. Thus, all the elements of ΔABC rotate about O. This means $e'' = R_O^\alpha(e)$, or $m(\angle(e, e'')) = \alpha$. Since the slopes satisfy the following equalities $m_{A''B''} = m_{A'B'}$, $m_{A''C''} =$

$m_{A'C'}$, and $m_{B''C''} = m_{B'C'}$, then $m_{e''} = m_{e'}$, which actually means $m(\angle(e, e')) = \alpha$. $\square$

Remark. Let ΔABC and $\Delta A'B'C'$ be two triangles with the property $AB \perp A'B'$, $AC \perp A'C'$, $BC \perp B'C'$. Then the Euler lines of the two triangles are perpendicular.

We can prove this Remark directly from Lemma 1. However, we can also provide a direct argument for its proof. Denote by m_1', m_2', m_3' the slopes of the side and by m_E' the slope of Euler's line of $\Delta A'B'C'$. Then

$$\begin{aligned} m_E' &= -\frac{m_1'm_2' + m_3'm_1' + m_2'm_3' + 3}{m_1' + m_2' + m_3' + 3m_1'm_2'm_3'} \\ &= -\frac{\left(-\frac{1}{m_1}\right)\left(-\frac{1}{m_2}\right) + \left(-\frac{1}{m_3}\right)\left(-\frac{1}{m_1}\right) + \left(-\frac{1}{m_2}\right)\left(-\frac{1}{m_3}\right) + 3}{\left(-\frac{1}{m_1}\right) + \left(-\frac{1}{m_2}\right) + \left(-\frac{1}{m_3}\right) + 3\left(-\frac{1}{m_1}\right)\left(-\frac{1}{m_2}\right)\left(-\frac{1}{m_3}\right)} \\ &= \frac{m_1 + m_2 + m_3 + 3m_1m_2m_3}{m_1m_2 + m_3m_1 + m_2m_3 + 3} \\ &= -\frac{1}{m_E}. \end{aligned}$$

This proves that the two lines are perpendicular.

Application 3. In the acute triangle ABC, Euler's line is parallel to BC if and only if $\tan B \tan C = 3$.

Note: In [17], it is mentioned that this problem was proposed by Dan Brânzei. We have discussed this application in [6]. The solution uses a direct trigonometric argument. We present here the analytic argument based on Lemma 1.

Solution: Choose a coordinate system so that the x-axis is parallel to BC. If we denote by m_1 the slope of the straight line BC, then $m_1 = 0$. Denoting m_2, m_3, m_e the slopes of the straight lines AC, AB, and Euler's line e, respectively, we get from Lemma 1:

$$m_e = -\frac{m_2m_3 + 3}{m_2 + m_3}.$$

Thus, Euler's line e of ΔABC is parallel to BC if and only if $m_e = 0$, which is equivalent to $m_2m_3 = -3$. Now we take into account that $m_2 = -\tan C$ and $m_3 = \tan B$ (or, depending on the position of ΔABC, we could have $m_2 = \tan C$ and $m_3 = -\tan B$). Consequently, $\tan B \tan C = 3$. $\square$

For an interesting connection between the formula obtained here for m_e and Tzitzeica surfaces, a topic studied in depth in affine differential geometry, see [2]. For a graphical study of Tzitzeica surfaces by using Mathematica, see [3]. For the importance of Tzitzeica's surfaces in the development of differential geometry at the beginning of the 20th century, see [1].

Application 4. (W. L. Putnam Competition, 1997) A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitudes, O the center of the circumscribed circle, M the midpoint of BC, and F the foot of the altitude from A. What is the length of BC ?

Solution: Since Euler's line is parallel to BC, by the previous application, we have $\tan B \tan C = 3$. This is just a consequence of the previous application. We can continue our argument as in [14], pg.233, or [6]. Expressing $\tan B$ and $\tan C$ from triangles ABF and AFC, respectively, we get

$$\frac{h_a}{BF} \cdot \frac{h_a}{FC} = 3.$$

Since $HG \| BC$, we have $h_a = AF = 3FH = 3 \cdot 5 = 15$. Therefore, $BF \cdot FC = \frac{15 \cdot 15}{3} = 75$. Namely, we express $BC^2 = (BF+FC)^2 = (FC-BF)^2 + 4BF \cdot FC$. To compute the first term in the last expression we write $FC - BF = FM + MC - (BM - FM) = 2FM = 2OH = 22$. Therefore, $BC^2 = 22^2 + 4 \cdot 75 = 784$, thus $BC = 28$. $\square$

Lemma 2. *Euler's line of ΔABC intersects the lines AB and AC in M, respectively N. Then Euler's line of ΔAMN is parallel to BC.*

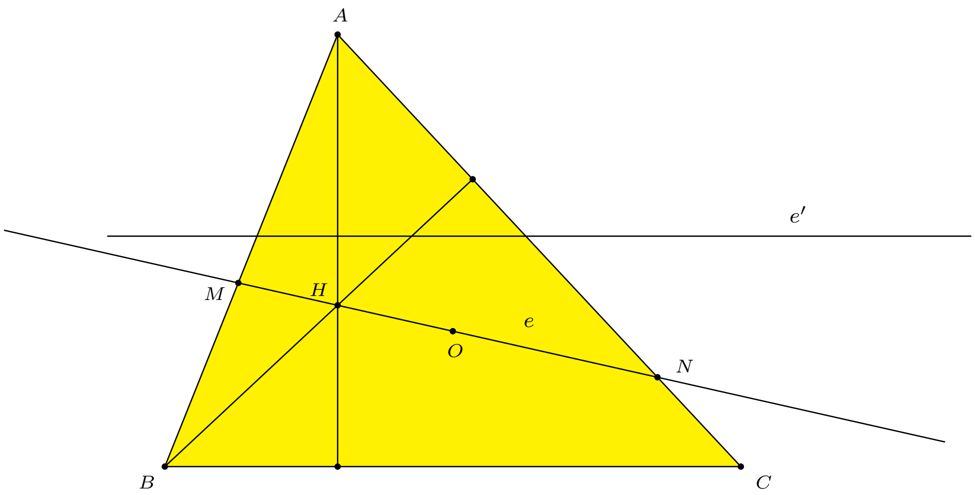

Figure 3

Proof. Choose a coordinate system so that the x-axis is parallel to BC, as in Application 3 (see Figure 3). If we denote by m_1 the slope of the straight line BC, then $m_1 = 0$. Denoting m_2, m_3, m_e the slopes of the straight lines AC, AB, and respectively Euler's line e. By Lemma 1:

$$m_e = -\frac{m_2 m_3 + 3}{m_2 + m_3},$$

and the slope of Euler's line of ΔAMN is

$$m_{e'} = -\frac{m_e m_2 + m_e m_3 + m_2 m_3 + 3}{m_e + m_2 + m_3 + 3 m_e m_2 m_3}.$$

In fact, the numerator of the last expression is

$$\begin{aligned} & m_e m_2 + m_e m_3 + m_2 m_3 + 3 \\ = \ & m_e(m_2 + m_3) + m_2 m_3 + 3 \\ = \ & \left(-\frac{m_2 m_3 + 3}{m_2 + m_3}\right)(m_2 + m_3) + m_2 m_3 + 3 = 0. \end{aligned}$$

In fact, we proved that $m_{e'} = 0$, which means that $e' \| BC$. □

Application 5. Consider two triangles such that $\Delta ABC \equiv \Delta A'B'C'$ and they have the same Euler's line. Then $\Delta A'B'C'$ is obtained from ΔABC either by a translation, or by a central symmetry. □

Example 1. Problem 244 in [19] states the following. Let H be the orthocenter of ΔABC, and O_a, O_b, O_c the circumcenters of triangles BHC, CHA, AHB. Then $\Delta ABC \equiv \Delta O_a O_b O_c$ have the same nine-point circle and the same Euler's line. This provides us an example of two triangles that have the same Euler's line (see Figure 4).

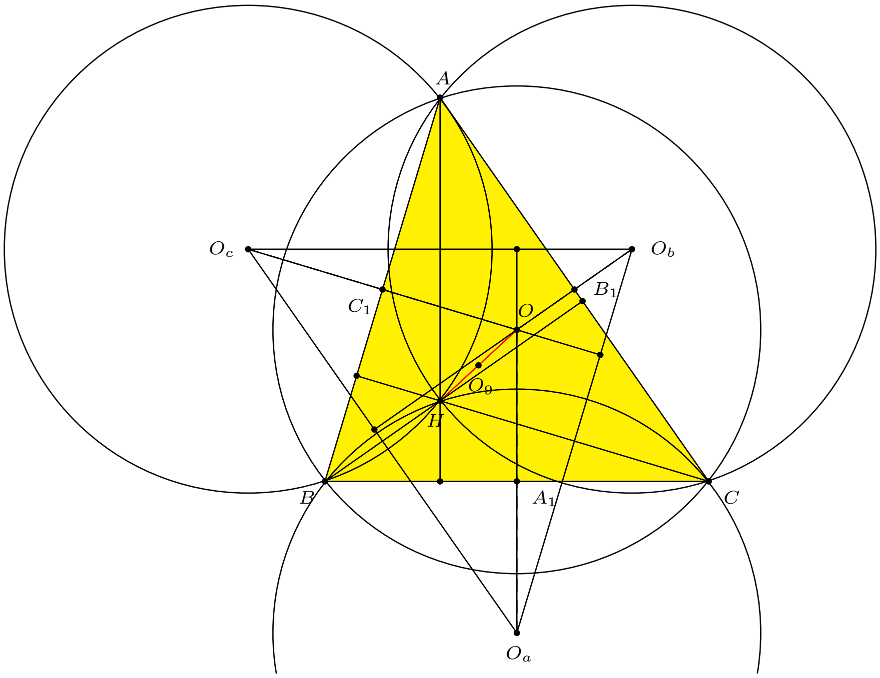

Figure 4.

Example 2. Now we describe two triangles of interest that have the same Euler's line. Consider ΔABC and its circumcircle $\mathcal{C}$. Consider also the incircle tangent to BC, AC and AB respectively in D, E, and F. On the straight lines AI, BI, CI we consider the excenters (*i.e.*, the centers of the excircles) I_a, I_b, and I_c. Remark that the circumcircle of ΔABC is the nine-point circle of $\Delta I_aI_bI_c$, because A, B, C are the feet of the altitudes (e.g. $AI_a \perp I_bI_c$).

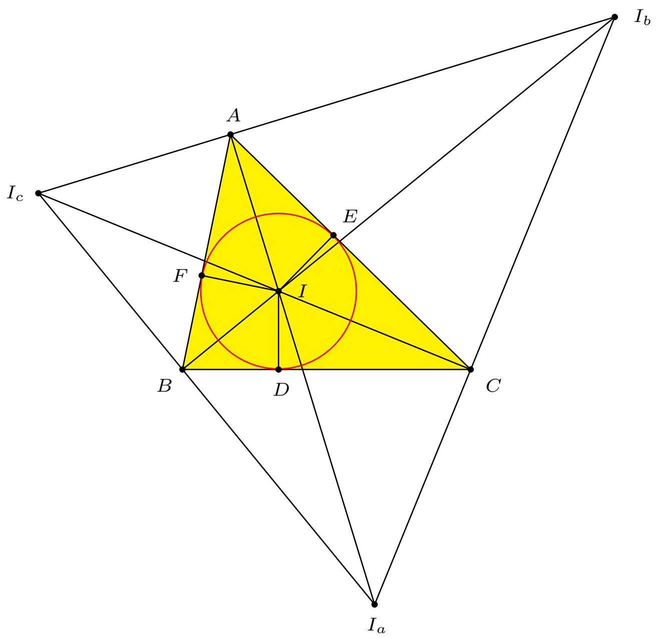

Figure 5.

Thus, I is the orthocenter in $\Delta I_aI_bI_c$, and O is the center of the nine-point circle in $\Delta I_aI_bI_c$. Therefore, OI is Euler's line in $\Delta I_aI_bI_C$. Remark that ΔDEF and $\Delta I_aI_bI_c$ have parallel sides. Therefore their Euler's lines must be parallel (we may say that this is a consequence of Application 2). But the circumcenter of ΔDEF is the point I. This means that the Euler's line of ΔDEF passes through I and, being parallel to OI, must be OI. □

3. Gossard's perspector

In this section we present an elementary proof of Gossard's result cited in [15].

Proposition 3 (Gossard, [15])**.** *Denote by e the Euler line of an arbitrary ΔABC in the Euclidean plane. Suppose that e intersects BC, AB, AC in M, N, and respectively P. Denote by e_1, e_2, e_3 Euler's lines of ΔANP, ΔBMN, and ΔCPM, respectively. Denote A', B', C' the intersection of the following pair of lines: $e_2 \cap e_3$, $e_1 \cap e_3$, and $e_1 \cap e_2$, respectively. Then $\Delta A'B'C' \equiv \Delta ABC$, and $\Delta A'B'C'$ has the same Euler line e, and there exists a point I_G (called Gossard's perspector) on the line e such that $\Delta A'B'C'$ is the symmetric of ΔABC by the symmetry centered in I_G.*

The proof presented below is based on Lemma 1. Thus, we claim that it may be more elementary than Gossard's original proof, as it is presented by Kellogg in [15]. An important rôle in the proof is played by the conditions $e_1 \| BC$, $e_2 \| AC$, $e_3 \| AB$.

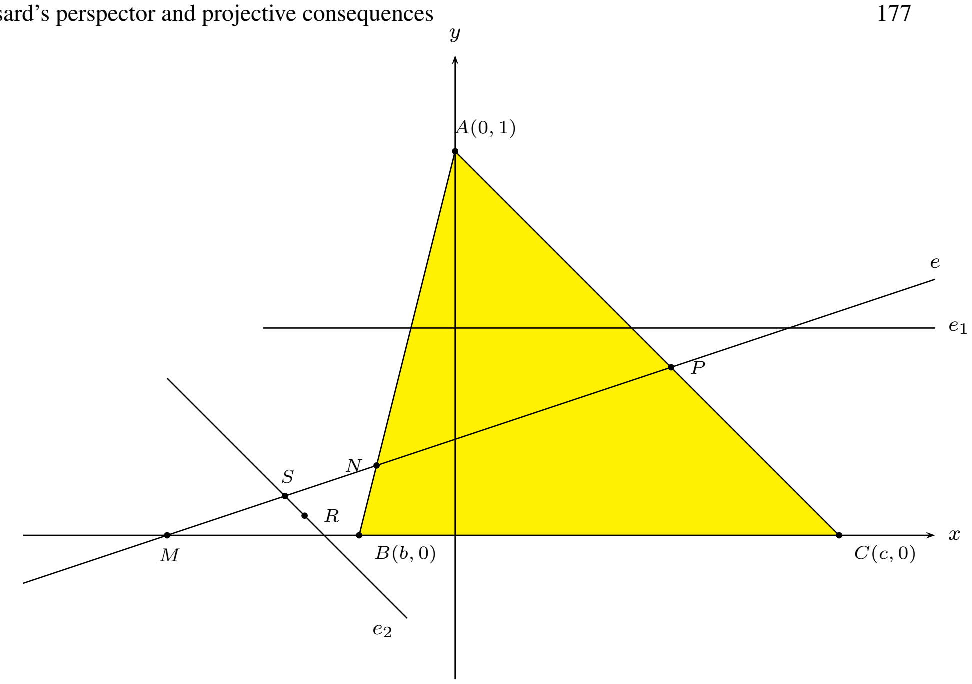

Figure 6

Proof. We choose coordinate axis such that the vertices of ΔABC have the coordinates $A(0,1), B(b,0), C(c,0)$ (see Figure 4). Let G be the gravity center of ΔABC; then $G(\frac{b+c}{3}, \frac{1}{3})$. The slope of Euler's line in ΔABC is given by

$$m_e = -\frac{m_2 m_3 + 3}{m_2 + m_3} = -\frac{\left(-\frac{1}{c}\right)\left(-\frac{1}{b}\right) + 3}{-\frac{1}{c} - \frac{1}{b}} = \frac{3bc+1}{b+c}.$$

Thus, the equation of Euler's line is $y = \frac{3bc+1}{b+c}\,x - bc$. The coordinates of the points M, N, and P are:

$$M\left(\frac{bc(b+c)}{3bc+1}, 0\right),$$
$$N\left(\frac{b(b+c)(bc+1)}{3b^2c+2b+c}, \frac{2b^2c - bc^2 + b}{3b^2c+2b+c}\right),$$
$$P\left(\frac{c(b+c)(bc+1)}{3bc^2+2c+b}, \frac{2bc^2 - b^2c + c}{3bc^2+2c+b}\right).$$

The line e_1 passes through the center of gravity of ΔANP and is parallel to BC, therefore it has the equation

$$(e_1): \qquad y = \frac{y_N + y_P + y_A}{3}.$$

At the intersection of lines e and e_1 we have the point Q whose coordinates are

$$Q\left(\frac{b+c}{3bc+1} \cdot \frac{1}{3}(E+3bc), \frac{1}{3}E\right),$$

where we have denoted by

$$E = \frac{2b^2c - bc^2 + b}{3b^2c + 2b + c} + \frac{2bc^2 - b^2c + c}{3bc^2 + 2c + b} + 1.$$

The center of gravity of ΔBMN, denoted R, has the coordinates

$$(x_R, y_R) = \left(\frac{1}{3}\left(\frac{bc(b+c)}{3bc+1} + b + \frac{b(b+c)(bc+1)}{3b^2c + 2b + c}\right), \frac{1}{3}\cdot\frac{2b^2c - bc^2 + b}{3b^2c + 2b + c}\right).$$

Euler's line in ΔBMN passes through R and is parallel to AC, thus it has the equation

$$(e_2): \qquad y - y_R = -\frac{1}{c}(x - x_R).$$

Denote by S the intersection of the lines e and e_2. We get

$$y_S = \frac{(3bc+1)(x_R + cy_R) - bc(b+c)}{3bc^2 + 2c + b}.$$

To emphasize the transformation by symmetry (as described in [15]), we claim that $y_S + y_p = y_Q + y_M$. This is equivalent to

$$\begin{aligned}&\frac{(3bc+1)(x_R + cy_R) - bc(b+c)}{3bc^2 + 2c + b} + \frac{2bc^2 - b^2c + c}{3bc^2 + 2c + b}\\ &= \frac{1}{3}\left(\frac{2b^2c - bc^2 + b}{3b^2c + 2b + c} + \frac{2bc^2 - b^2c + c}{3bc^2 + 2c + b} + 1\right).\end{aligned}$$

By replacing x_R and y_R and simplifying the relation, we obtain the desired equality. Therefore, the segments $[PS]$ and $[QM]$ have the same midpoint. (It is not necessary to check also that $x_P + x_S = x_Q + x_M$, since P, S, Q and M are collinear.)

Denote by I_G the common midpoint of those two segments. As above, one can prove that I_G is the midpoint of the segment $[NT]$, where $\{T\} = e_3 \cap e$. The analogy of the computation can be further seen since the coordinates of I_G are symmetric in b and c. Thus, with the above notation for E, I_G has the coordinates

$$(x_{I_G}, y_{I_G}) = \left(= \frac{1}{2}\left(\frac{bc(b+c)}{3bc+1} + \frac{b+c}{3bc+1}\cdot\frac{1}{3}(E + 3bc)\right), \frac{1}{6}E\right).$$

We can write the coordinates in the form

$$I_G\left(\frac{1}{6}\cdot\frac{b+c}{3bc+1}(E + 6bc), \frac{1}{6}E\right).$$

This is the point called *the Gossard perspector.* Denote S_{I_G} the symmetry of center I_G in the Euclidean plane. Since $e_1 \| BC$, $Q \in e_1$ $M \in BC$, and I_G is the midpoint of $[QM]$, we have $e_1 = S_{I_G}(BC)$. Similarly $e_2 = S_{I_G}(AC)$, $e_3 = S_{I_G}(AB)$.

Then, we have obtained the following:

$$\{A'\} = e_2 \cap e_3 = S_{I_G}(AC) \cap S_{I_G}(AB) = S_{I_G}(AC \cap AB) = S_{I_G}(\{A\}).$$

Similarly, $\{B'\} = S_{I_G}(\{B\})$, and $\{C'\} = S_{I_G}(C)$.

Consequently, $\Delta A'B'C' \equiv \Delta ABC$, and $\Delta A'B'C' = S_{I_G}(\Delta ABC)$.

Denoting G and G' the gravity centers of ΔABC and $\Delta A'B'C'$, we have $\{G'\} = S_{I_G}(\{G\})$. For the orthocenters we get a similar correspondence: $\{H'\} = S_{I_G}(\{H\})$. Thus, $e' = S_{I_G}(e)$, where e' is Euler's line of $\Delta A'B'C'$. But $I_G \in e$. Thus, Euler's line e passes through the center of symmetry. We deduce that $S_{I_G}(e) = e$, or $e' = e$. Finally, we proved that ΔABC and $\Delta A'B'C'$ have the same Euler's line. This completes the analytic proof of Gossard's prospector theorem, as mentioned in our introduction (see [15]). □

Example 3. We have seen in Example 1 (see [19], 244) that if H is the orthocenter of ΔABC, and O_a, O_b, O_c are the circumcenters of triangles BHC, CHA, AHB, then ΔABC and $\Delta O_aO_bO_c$ have the same Euler's line (see Figure 4). In fact, O_a, O_b, and O_c are the symmetric points of O with respect to the sides BC, AC and, respectively, AB. Denote by A_1, B_1, and C_1 the midpoints of the sides BC, AC and, respectively, AB.

Then H is the circumcenter of $\Delta O_aO_bO_c$. Actually, $\Delta O_aO_bO_c$ is the homothetic of $\Delta A_1B_1C_1$ by homothety of center O and ratio 2. Thus, $\Delta O_aO_bO_c$ has the sides parallel and congruent to the sides of ΔABC, and, furthermore, $OO_a \perp BC$, and also $OO_a \perp O_bO_c$, (and the similar relations). This proves that O is the orthocenter of $\Delta O_aO_bO_c$. Therefore ΔABC and $\widehat{}\Delta O_aO_bO_c$ interchanged among them the orthocenters and the circumcenters. This is the argument to see that the Euler's lines in the two triangles are the same and the two triangles have the same center of the nine-point circle, since O_9 is the midpoint of OH. Further, ΔABC and $\Delta O_aO_bO_c$ are symmetric with respect to O_9. Therefore, Gossard's perspector in $\Delta O_aO_bO_c$ is the symmetric of Gossard perspector in ΔABC with respect to O_9, the center of the nine-point circle.

4. Projective viewpoint

Consider now a projectivity $f : d_1 \to d_2$. (See also [7, pp.39 ff], [8, pp.9-11]) The geometric locus of the points from which the the projectivity is seen as an involution of pencils of lines is called axis of the projectivity.

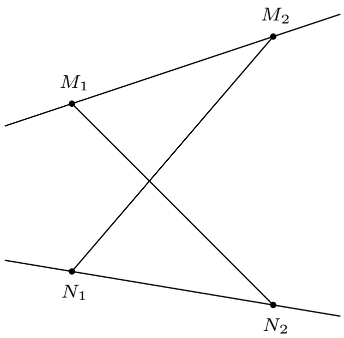

Figure 7.

More precisely (see Figure 7), any projectivity relating ranges on two distinct lines determines another special line, the axis of projectivity, which contains the intersection of the cross-joints of any pairs of corresponding points (see [8, pp.36-37]). This result is known as *the axis theorem.* To illustrate it, if $M_1 \to N_1$ and $M_2 \to N_2$, then the point $\{P\} = M_1N_2 \cap M_2N_1$ lies on the axis of the projectivity, since we have the mapping $r_1 = PM_1 \to PN_1 = r_2$ and $r_2 = PM_2 \to PN_1 = r_1$. Thus, $r_1 \to r_2$ and $r_2 \to r_1$, which means that the projectivity $f : d_1 \to d_2$ is seen as an involution. As a consequence, we remind here the well-known geometric structure called *Pappus' line*.

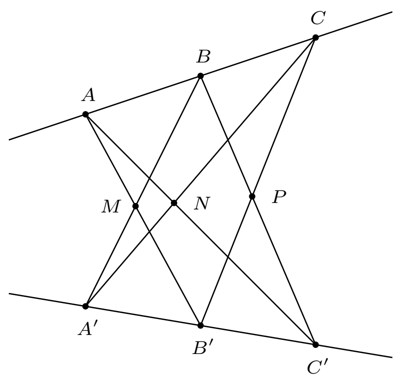

Figure 8.

Let $A, B, C \in d_1$ and $A', B', C' \in d_2$. Then the points $\{M\} = AB' \cap BA'$, $\{N\} = AC' \cap AC' \cap CA'$, $\{P\} = BC' \cap CB'$, are collinear (see Figure 8). This result can be viewed as an immediate consequence of the axis theorem. Indeed, consider the projectivity $f : d_1 \to d_2$ uniquely determined by $A \to A'$, $B \to B'$, $C \to C'$. By the axis theorem, we get immediately that the points $\{M\} = AB' \cap BA'$, $\{N\} = AC' \cap AC' \cap CA'$, $\{P\} = BC' \cap CB'$ are collinear. With this preparation, we are able to show that *the Euler's line of a triangle ABC can be regarded as the axis of projectivity for three suitable projectivities between the sides of* ΔABC (see Figure 9).

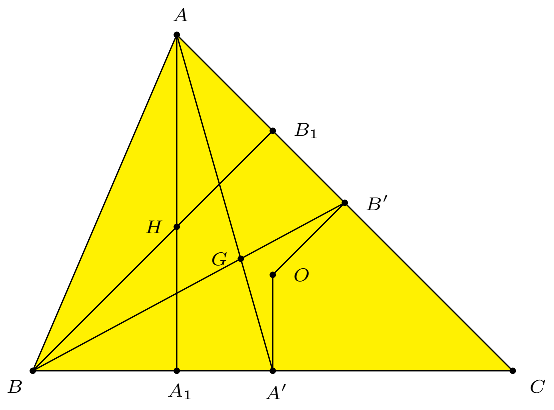

Figure 9.

Denote by A', B', C' the midpoints of the sides BC, AC, and respectively AB. Denote by A_1, B_1, C_1 the feet of altitudes from A, B, C. We use the standard notations for O, the circumcircle, G the center of gravity, and H the orthocenter of ΔABC. There are three projectivities, each one between two sides of ΔABC. One of them is $f_C : BC \to AC$, the projectivity determined by $B \to A$, $A_1 \to B_1$, $A' \to B'$. Since H and G appear as cross-joints points, they lie on the axis of projectivity of f_C. Specifically, $\{H\} = AA_1 \cap BB_1$, $\{G\} = BB' \cap AA'$. Since two points determine uniquely a line, and since G and H determine Euler's line, this means that *the Euler's line is identified with the axis of projectivity* f_C. Furthermore, on the Euler's line we get a new point: $\{\Omega_{AB}\} = A_1B' \cap A'B_1$. We can also emphasize the pair of homologous points that determine O, the circumcenter, in this projectivity. Extend the line determined by the vertex A and by O and denote $\{X\} = AO \cap BC$. Similarly, $\{Y\} = BO \cap AC$. Since in our projectivity $B \to A$, then $X \to Y$. Thus, on the axis of projectivity we obtain $\{O\} = AX \cap BY$.

Considering similar constructions for the projectivities f_A and f_B, we obtain the following fact.

Proposition 4 (Barbilian [4])**.** *In ΔABC, let A', B', C' be the midpoints of the sides BC, AC, and respectively AB. Denote by A_1, B_1, C_1 the feet of altitudes from A, B, C. Then the points $\{\Omega_{AB}\} = A_1B' \cap A'B_1$, $\{\Omega_{CB}\} = C_1B' \cap C'B_1$, $\{\Omega_{AC}\} = A_1C' \cap A'C_1$ are collinear and they lie on Euler's line of ΔABC.*

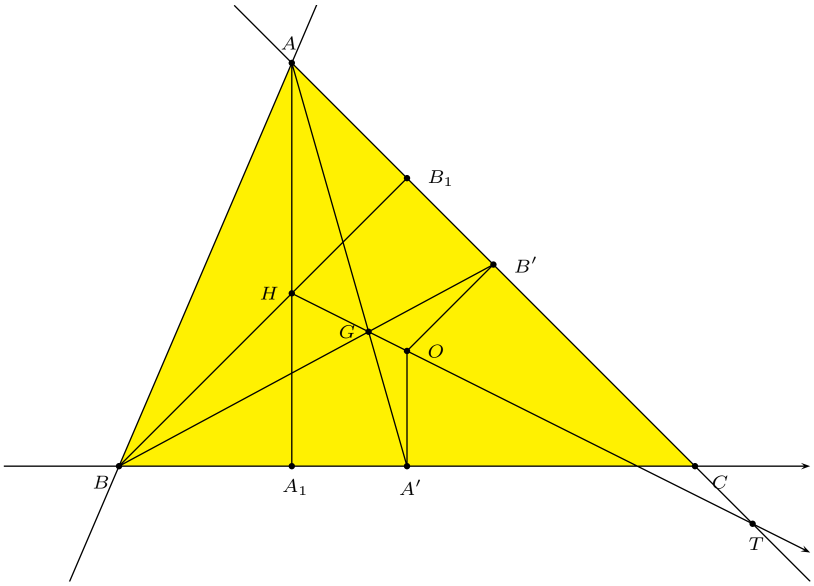

Figure 10.

In the first part, we have presented Applications 3 and 4, where we give synthetic and trigonometric characterizations of the fact that Euler's line is parallel to a side of the triangle. We study here the following question: What is projective condition that the projectivity $f_C : BC \to AC$ must satisfy such that Euler's line is parallel to BC? Denote by (e) Euler's line in ΔABC. (See Figure 10.) Let $\{T\} = AC \cap (e)$, $\{U\} = BC \cap (e)$. We need to determine the pairs of straight lines that characterize in a projectivity the points T and U. Recall that the projectivity f_C has as homologous points $B \to A$. To get T, consider the pair $C \to (e) \cap AC$. Similarly, we get U by the pair $(e) \cap BC \to C$. Therefore we have obtained the projective characterization of the fact that the Euler line is parallel to a side of the triangle. Thus, we are able to state the projective counterpart of Application 3, which is the trigonometric characterization of this parallelism.

Proposition 5. *In ΔABC, let (e) be the Euler's line. The sufficient condition that $(e) \| BC$, is that the projectivity f_C has $\infty \to C$ as pair of homologous points. Similarly, to have $(e) \| AC$, it is sufficient that f_C has $C \to \infty$ as pairs of homologous points.*

Four our next step, we need to recall here Pappus' Theorem on the circle. Let A, B, C and A', B', C' six points on the circle C. Then the intersection points $AB' \cap A'B$, $AC' \cap A'C$ and $BC' \cap B'C$ are collinear. To recall the idea of the most direct proof, consider the projectivity $f : C \to C$ uniquely determined by $A \to A'$, $B \to B'$, $C \to C'$. Then, the intersection points mentioned in the statement lie precisely on the axis of the projectivity. With this observation, we obtain that Euler's line is the axis of projectivity of a certain projectivity within the nine-point circle. The result is the following.

Proposition 6. *Consider A', B', C' the midpoints of the sides BC, AC and respectively AB. Let A_1, B_1 and C_1 the feet of the altitudes. Consider the projectivity ϕ uniquely determined by $A_1 \to B_1$, $A' \to B'$, $B_2 \to A_2$. Then the points $A_1A_2 \cap B_1B_2 = \{H\}$ (the orthocenter of ΔABC), $A_1B' \cap A'B_1 = \{\Omega_{AB}\}$ and $A'A_2 \cap B'B_2 = \{O_9\}$ (the center of the nine-point circle) are collinear on the axis of projectivity of ϕ.*

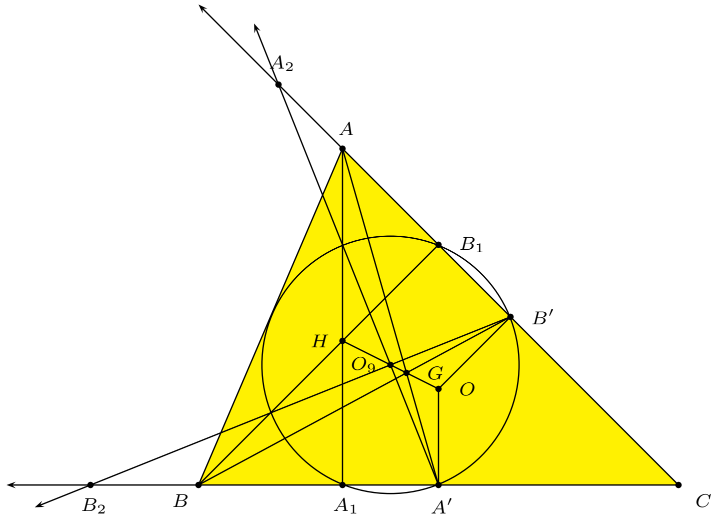

Figure 11.

The proof is just a direct application of Pappus' Theorem on the circle, for the geometric structure described in the statement. Since H and Ω_{AB} are on Euler's line, the axis of projectivity and Euler's line must be the same straight line. As a consequence, the third point, O_9, the center of the nine-point circle, must be on the axis of projectivity, thus on Euler's line.

Proposition 4 appears in [4, pp. 40]. Actually, Dan Barbilian collected in an undated note, published in the cited collection of posthumuous works, several projective properties of the nine-point circle and its connection with Euler's line. He focused mainly on the projective properties, which represent, as we can see, an important part of the more complex phenomenon whose overall picture we tried to present here.

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