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A Sangaku-Type Problem with Regular Polygons, Triangles, and Congruent Incircles

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Abstract. We consider a dissection problem of a regular n-sided polygon that generalizes Suzuki's problem of four congruent incircles in an equilateral triangle.

1. Introduction

The geometrical problem that is the starting point of our note is due to Denzaburo Suzuki. It was engraved on a wooden tablet and dedicated 1886 to the Miwatari Shrine in Fukushima prefecture [4, p. 6], [1, p. 24]. Referring to Figure 1 below we state the problem in the following equivalent form.

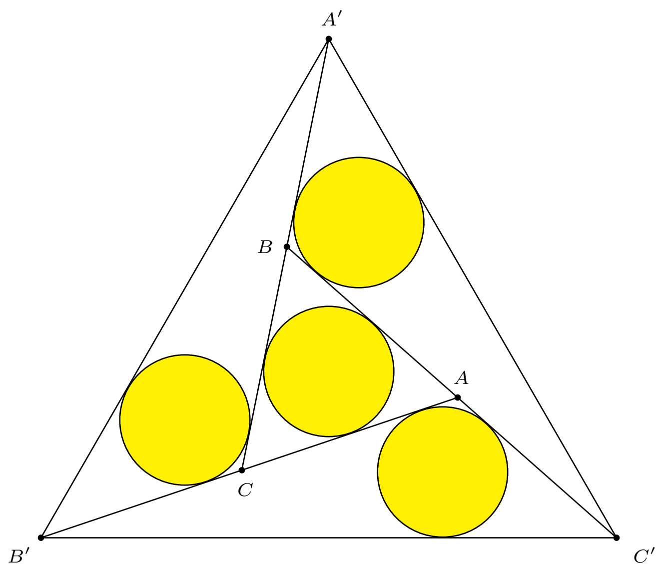

Figure 1. Four incircles in an equilateral triangle

Suzuki's problem of four congruent incircles in an equilateral triangle. Let ABC be an equilateral triangle. The side AC is extended to the point B', the side BA is extended to C', and CB to A', such that the triangles $AB'C'$, $BC'A'$, $CA'B'$ and ABC have congruent incircles. Find the length of $A'B'$ of the exterior equilateral triangle in terms of the length of AB.

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Suzuki's problem is an example of a unique brand of mathematics that flourished in Japan in the 18th and 19th century. Amateur mathematicians crafted geometric theorems on wooden tablets (called sangaku), which were displayed in the precincts of a shrine or temple. Remarkably, some of those theorems predate work of Western mathematicians (see [5]). In addition to [1] we also mention the monographs [2] and [3] as sources of sangaku problems. An excellent survey of Japanese temple geometry is Rothman's article [6] in the Scientific American.

In this note we generalize Suzuki's four-congruent-incircles problem. Instead of an equilateral triangle we now consider a regular n-sided polygon. To illustrate the general case we choose $n = 5$. Figure 2 shows the configuration of six congruent incircles in a regular pentagon.

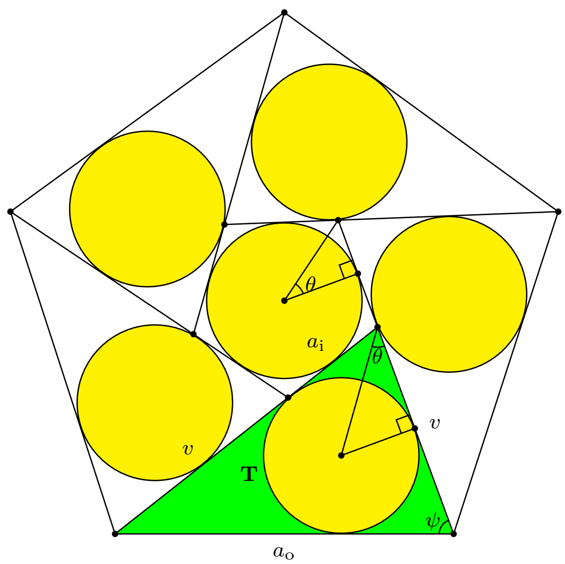

Figure 2. Six congruent incircles in a regular pentagon

Let $\mathbf{P}_o$ be the regular n-sided polygon that forms the exterior boundary of Figure 2 and let $\mathbf{P}_i$ be the regular n-sided polygon in the interior. Our main result is the following.

Theorem 1. *If a_o and a_i are the lengths of the sides of $\mathbf{P}_o$ and $\mathbf{P}_i$ respectively, then*

$$a_o = a_i \left(1 + \sqrt{1 + \left(\sin\frac{\pi}{n}\right)^{-2}}\right). \tag{1}$$

The proof of Theorem 1 will be given in Section 2. In Section 4 we assume that the exterior polygon $\mathbf{P}_o$ is given. We derive a result that leads to a simple construction of the interior polygon $\mathbf{P}_i$. The special cases with $3 \leq n \leq 6$ will be considered in Section 5.

2. Incircles

The proof of Theorem 1 is based on a relation between inradius and area of a triangle.

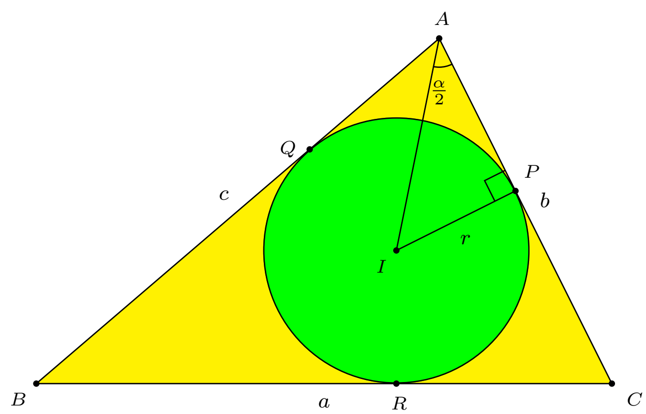

Figure 3. A triangle and its incircle

Consider a triangle ABC in Figure 3 with sides $a = BC$, $b = CA$, $c = AB$, and $\angle BAC = \alpha$. We denote the area, the incenter, and the inradius of the triangle by Δ, I, and r, respectively. Then

$$\Delta = r^2 \cot \frac{\alpha}{2} + ra. \tag{2}$$

This can be seen as follows.

Suppose the incircle touches CA, AB and BC at P, Q and R, respectively. Since $AP = r \cot \frac{\alpha}{2}$, the area of the kite $AQIP$ is given by $r \cdot AP = r^2 \cot \frac{\alpha}{2}$. The area of the triangle BIC is equal to $\frac{1}{2}ra$. Thus the area of the polygon $QBCPIQ$ is ra. Combining the areas of the kite and the polygon, we obtain (2).

The following notation refers to Figure 2. Let $\mathbf{T}$ be one of the n triangles that bound the interior polygon $\mathbf{P}_\text{i}$. Then $\mathbf{T}$ has sides a_o, v, $v + a_\text{i}$. Let ψ be the angle between v and a_o. We set $\theta = \frac{\pi}{n}$. Then the angle opposite to a_o is equal to $2\theta = \frac{2\pi}{n}$.

3. Proof of Theorem 1

If r_i is the radius of the incircle of $\mathbf{P}_\text{i}$, then $r_\text{i} = \frac{1}{2} a_\text{i} \cot \theta$. Let F_o, F_i and F be the areas of $\mathbf{P}_\text{o}$, $\mathbf{P}_\text{i}$ and $\mathbf{T}$ respectively. Then

$$F_\text{o} = \frac{n\, a_\text{o}^2}{4} \cot \theta \quad \text{and} \quad F_\text{i} = \frac{n\, a_\text{i}^2}{4} \cot \theta,$$

and $nF + F_\text{i} = F_\text{o}$. Therefore,

$$F = \frac{1}{n}(F_\text{o} - F_\text{i}) = \frac{1}{4}(a_\text{o}^2 - a_\text{i}^2) \cot \theta.$$

Let ρ be the inradius of $\mathbf{T}$. Then (2) implies $F = \rho^2 \cot\theta + a_{\rm o}\rho$. Hence,

$$\frac{a_{\rm o}^2 - a_{\rm i}^2}{4}\cot\theta = \rho^2\cot\theta + a_{\rm o}\rho.$$

Now assume that the incircles of $\mathbf{P}_{\rm i}$ and $\mathbf{T}$ are equal, i.e.,

$$r_{\rm i} = \rho = \frac{1}{2}a_{\rm i}\cot\theta.$$

We obtain $a_{\rm o}^2 - a_{\rm i}^2 = a_{\rm i}^2\cot^2\theta + 2a_{\rm o}a_{\rm i}$. This yields the quadratic equation

$$a_{\rm o}^2 - 2a_{\rm i}a_{\rm o} = a_{\rm i}^2(1+\cot^2\theta) = a_{\rm i}^2(\sin\theta)^{-2}.$$

Then $a_{\rm o} > 0$ implies (1).

4. The triangle T

Suppose the exterior polygon $\mathbf{P}_{\rm o}$ is given. How can one construct the triangle $\mathbf{T}$ and subsequently the interior polygon $\mathbf{P}_{\rm i}$? The angle opposite to $a_{\rm o}$ is equal to $\frac{2\pi}{n}$. Hence it suffices to know how to obtain the angle ψ between $a_{\rm o}$ and v. For that purpose we derive the following result due to P. Yiu.

Theorem 2. *The angle ψ between $a_{\rm o}$ and v is given by*

$$\cos\psi = \sin^2\frac{\pi}{n}. \tag{3}$$

Proof. It follows from (2) that the area of the triangle $\mathbf{T}$ is given by

$$F = \rho^2\cot\frac{\psi}{2} + \rho(v + a_{\rm i}) \tag{4}$$

and

$$F = \rho^2\cot\frac{\pi - (2\theta+\psi)}{2} + \rho v.$$

Substituting $a_{\rm i} = 2\rho\tan\theta$ in (4) leads to

$$\cot\frac{\psi}{2} + 2\tan\theta = \tan(\theta + \frac{\psi}{2}) = \frac{\tan\theta + \tan\frac{\psi}{2}}{1 - \tan\theta\tan\frac{\psi}{2}},$$

which is equivalent to

$$\cot\frac{\psi}{2} - 2\tan^2\theta\tan\frac{\psi}{2} = \tan\frac{\psi}{2},$$

and

$$\tan^2\frac{\psi}{2} = \frac{1}{1+2\tan^2\theta}. \tag{5}$$

From this,

$$\cos\psi = \frac{1-\tan^2\frac{\psi}{2}}{1+\tan^2\frac{\psi}{2}} = \frac{1 - \frac{1}{1+2\tan^2\theta}}{1 + \frac{1}{1+2\tan^2\theta}} = \frac{\tan^2\theta}{1+\tan^2\theta} = \sin^2\theta.$$

□

5. Special cases

We apply Theorem 1 and Theorem 2 to numbers $n \leq 6$. Note that the righthand side of (1) becomes infinitely large as n goes to infinity. In the case $n = 6$ (and also in a less conspicuous form with $n = 5$) we encounter the golden ratio $\phi = \frac{1}{2}(1+\sqrt{5})$. To check the case $n = 5$ we recall

$$\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4} \quad\text{and}\quad \sin\frac{2\pi}{5} = \frac{\sqrt{10+2\sqrt{5}}}{4}.$$

In the case $n = 3$ we obtain the known solution of Suzuki's 4-congruent-incircles problem of Section 1. The following table summarizes the data for $n = 3, 4, 5, 6$.

n	$\cos\psi$	$\frac{a_o}{a_i}$
3	$\frac{3}{4}$	$\frac{3+\sqrt{21}}{3}$
4	$\frac{1}{2}$	$1+\sqrt{3}$
5	$\frac{5-\sqrt{5}}{8} = \frac{\sqrt{5}}{4\phi}$	$1+\sqrt{3+\frac{2}{\sqrt{5}}}$
6	$\frac{1}{4}$	$1+\sqrt{5} = 2\phi$

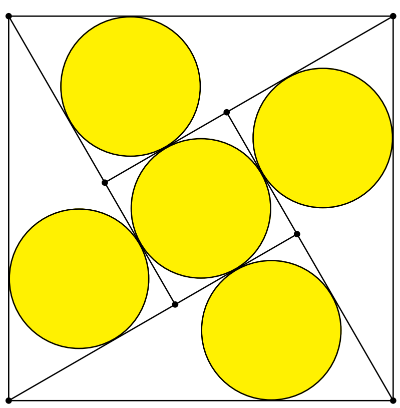

Figure 4. Five congruent incircles in a square

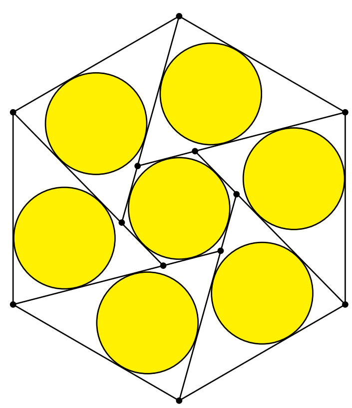

Figure 5. Seven congruent incircles in a regular hexagon

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