Forum Geometricorum
Volume 13 (2013) 219–225.

FORUM GEOM
ISSN 1534-1178

Intersecting Equilateral Triangles

Colleen Nielsen and Christa Powers

Abstract. In 1980, J. Fickett proposed the following problem: Assume two congruent rectangles R_1 and R_2 intersect in at least one point. Let a be the length of the part of the boundary of R_1 that lies inside R_2 and let b be the length of the part of the boundary of R_2 that lies inside R_1. The conjecture was that the ratio $\frac{a}{b}$ is no smaller than $\frac{1}{3}$ and no larger than 3. This paper presents the solution to the problem when R_1 and R_2 are replaced by equilateral triangles of the same size. We have proved that the ratio $\frac{a}{b}$ is no smaller than $\frac{1}{2}$ and no larger than 2.

In [1], Fickett proposed a problem involving congruent rectangles R_1 and R_2 (including their interiors) in the Euclidean plane and their boundaries ∂R_1 and ∂R_2. The conjecture was that

$$\frac{1}{3} \leq \frac{\text{length } \partial R_1 \cap R_2}{\text{length } \partial R_2 \cap R_1} \leq 3.$$

We show a similar result for equilateral triangles and include a generalization for regular polygons.

Theorem 1. *Let P and Q (including their interiors) be congruent regular $n-$gons, $3 \leq n$, in the Euclidean plane with respective boundaries ∂P and ∂Q.*

(a) *If P and Q intersect in exactly $2n-1$ or $2n$ boundary points, then* length $\partial P \cap Q =$ length $\partial Q \cap P$.

(b) *If $n = 3$, then* $\frac{1}{2} \leq \frac{\text{length } \partial P \cap Q}{\text{length } \partial Q \cap P} \leq 2$.

We begin with a useful Lemma.

Lemma 2. *If a_i, b_i are positive real numbers, $2 \leq n$ and $1 \leq i \leq n-1$, and* $\frac{a_i}{b_i} = \frac{a_{i+1}}{b_{i+1}}$ *then* $\frac{a_i}{b_i} = \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n}$.

Proof. $\frac{a_i}{b_i} = \frac{a_{i+1}}{b_{i+1}}$ implies that $\frac{a_i}{b_i} = \frac{a_j}{b_j}$ for $1 \leq i, j \leq n$ and $a_i b_j = b_i a_j$. Thus,

Publication Date: November 19, 2013. Communicating Editor: Paul Yiu.

$$\begin{aligned} a_i b_1 + a_i b_2 + \cdots + a_i b_n &= b_i a_1 + b_i a_2 + \cdots + b_i a_n, \\ a_i (b_1 + b_2 + \cdots + b_n) &= b_i (a_1 + a_2 + \cdots + a_n), \\ \frac{a_i}{b_i} &= \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n}. \end{aligned}$$

□

We now prove part (a) of Theorem 1 for $2n$ points. Only an adjustment in the subscripts is needed for the $2n-1$ case. Assume P and Q intersect in $2n$ boundary points with sides labeled as in Figure 1. We must show that $b_0 + b_2 + \cdots + b_{2n-2} = b_1 + b_3 + \cdots + b_{2n-1}$.

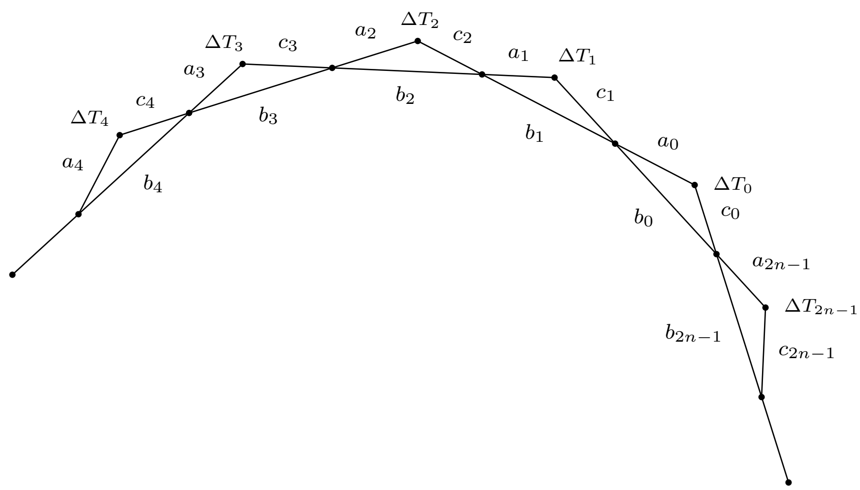


Figure 1

ΔT_i and ΔT_{i+1} both have a $\dfrac{180\,(n-2)}{n}$ degree angle and a pair of congruent vertical angles and so they are similar and more generally, $\Delta T_i \sim \Delta T_k$. This implies that $\dfrac{a_i}{a_j} = \dfrac{b_i}{b_j} = \dfrac{c_i}{c_j}$ and therefore $\dfrac{a_i}{b_i} = \dfrac{a_1}{b_1} = \dfrac{a_3}{b_3} = \cdots = \dfrac{a_{2n-1}}{b_{2n-1}}$ and $\dfrac{a_i}{b_i} = \dfrac{a_0}{b_0} = \dfrac{a_2}{b_2} = \cdots = \dfrac{a_{2n-2}}{b_{2n-2}}$. Using Lemma 1 yields

$$\begin{aligned} \frac{a_1 + a_3 + \cdots + a_{2n-1}}{b_1 + b_3 + \cdots + b_{2n-1}} &= \frac{a_i}{b_i} = \frac{a_0 + a_2 + \cdots + a_{2n-2}}{b_0 + b_2 + \cdots + b_{2n-2}} \\ \frac{a_1 + a_3 + \cdots + a_{2n-1}}{a_0 + a_2 + \cdots + a_{2n-2}} &= \frac{b_1 + b_3 + \cdots + b_{2n-1}}{b_0 + b_2 + \cdots + b_{2n-2}} \end{aligned}$$

Similarly, $\dfrac{a_1 + a_3 + \cdots + a_{2n-1}}{a_0 + a_2 + \cdots + a_{2n-2}} = \dfrac{b_1 + b_3 + \cdots + b_{2n-1}}{b_0 + b_2 + \cdots + b_{2n-2}} = \dfrac{c_1 + c_3 + \cdots + c_{2n-1}}{c_0 + c_2 + \cdots + c_{2n-2}}$. Again, applying Lemma 1,

$$\frac{b_1 + b_3 + \cdots + b_{2n-1}}{b_0 + b_2 + \cdots + b_{2n-2}} = \frac{a_1 + a_3 + \cdots + a_{2n-1} + c_1 + c_3 + \cdots + c_{2n-1}}{a_0 + a_2 + \cdots + a_{2n-2} + c_0 + c_2 + \cdots + c_{2n-2}}. \quad (1)$$

Assume the sides of the regular polygon are of length 1 and so $a_i + b_{i+1 \text{ (mod n)}} + c_{i+2 \text{ (mod n)}} = 1$ for $1 \leq i \leq n-1$. We find the sum of the sides of each polygon.

$$\begin{aligned} a_0 + b_1 + c_2 + a_2 + b_3 + c_4 + \cdots + a_{2n-2} + b_{2n-1} + c_0 = \ & n, \\ a_0 + a_2 + \cdots + a_{2n-2} + c_0 + c_2 + \cdots + c_{2n-2} = \ & n - (b_1 + b_3 + \cdots + b_{2n-1}) ; \\ a_1 + b_2 + c_3 + a_3 + b_4 + c_5 + \cdots + a_{2n-1} + b_0 + c_1 = \ & n, \\ a_0 + a_2 + \cdots + a_{2n-1} + c_1 + c_3 + \cdots + c_{2n-1} = \ & n - (b_0 + b_2 + \cdots + b_{2n-2}) . \end{aligned}$$

These along with (1) yield,

$$\begin{aligned} \frac{b_1 + b_3 + \cdots + b_{2n-1}}{b_0 + b_2 + \cdots + b_{2n-2}} = \ & \frac{a_1 + a_3 + \cdots + a_{2n-1} + c_1 + c_3 + \cdots + c_{2n-1}}{a_0 + a_2 + \cdots + a_{2n-2} + c_0 + c_2 + \cdots + c_{2n-2}} \\ = \ & \frac{n - (b_0 + b_2 + \cdots + b_{2n-2})}{n - (b_1 + b_3 + \cdots + b_{2n-1})}. \end{aligned}$$

Let $X = b_1 + b_3 + \cdots + b_{2n-1}$ and $Y = b_0 + b_2 + \cdots + b_{2n-2}$. Substituting $\frac{X}{Y} = \frac{n-Y}{n-X}$, we obtain $nX - X^2 = nY - Y^2$, $(X - Y)(X + Y - n) = 0$.

Now, for each ΔT_i, $b_i < a_i + c_i$, and so

$$2(X + Y) = 2 \sum_{k=0}^{2n-1} b_i < \sum_{k=0}^{2n-1} a_i + b_i + c_i = 2n,$$

which implies that $X + Y < n$. Thus, $X + Y - n \neq 0$, and $X - Y = 0$ or $X = Y$.

Later, we shall need Theorem 1(a) in the special case where $n = 3$ and the two equilateral triangles intersect in exactly five points. We now prove part (b) of Theorem 1 and begin with two preliminary results.

Lemma 3. *If* $0 \leq x \leq \frac{\pi}{3}$ *then* $f(x) = \sin\left(x + \frac{\pi}{3}\right) + \sin x$ *has a minimum value of* $\frac{\sqrt{3}}{2}$ *at* $x = 0$ *and a maximum value of* $\sqrt{3}$ *at* $x = \frac{\pi}{3}$.

Proof. For $0 \leq x \leq \frac{\pi}{3}$,

$$f'(x) = \cos\left(x + \frac{\pi}{3}\right) + \cos x = \sqrt{3} \cos\left(x + \frac{\pi}{6}\right) \geq 0,$$

which implies that f is an increasing function on $\left[0, \frac{\pi}{3}\right]$. Thus, $f(0) = \frac{\sqrt{3}}{2}$ is the minimum value of f, and $f\left(\frac{\pi}{3}\right) = \sqrt{3}$ is the maximum value. □

Proposition 4. *Assume triangle* ABC *has sides* a, b, c *and opposite vertices* A, B, C , *respectively. If* $\angle C = 60°$, *then* $\frac{1}{2} \leq \frac{c}{a+b} < 1$.

Proof. Using the law of cosines, $c^2 = a^2 + b^2 - 2ab\cos 60^\circ = a^2 + b^2 - ab$. Now, $(2c)^2 - (a+b)^2 = 4(a^2 - ab + b^2) - (a^2 + 2ab + b^2) = 3a^2 - 6ab + 3b^2 = 3(a-b)^2 \geq 0$. Therefore, $2c \geq a + b$. Since also $c < a + b$, we have $\dfrac{1}{2} \leq \dfrac{c}{a+b} < 1$. □

To prove part (b) of Theorem 1, we must consider the number of intersection points that are not vertices of a triangle. An intersection point that is a vertex of a triangle will be called a *vertex intersection*. These points will not affect the underlying geometry of what is to follow. If the triangles intersect in fewer than five points, we have four cases to consider (Figure 2).

Case 1: Two intersection points forming a triangle; possibly one vertex intersection.

Case 2: Two intersection points forming a quadrilateral; possibly one or two vertex intersections.

Case 3: Four intersection points forming a quadrilateral.

Case 4: Four intersection points forming a pentagon; possibly two vertex intersections.

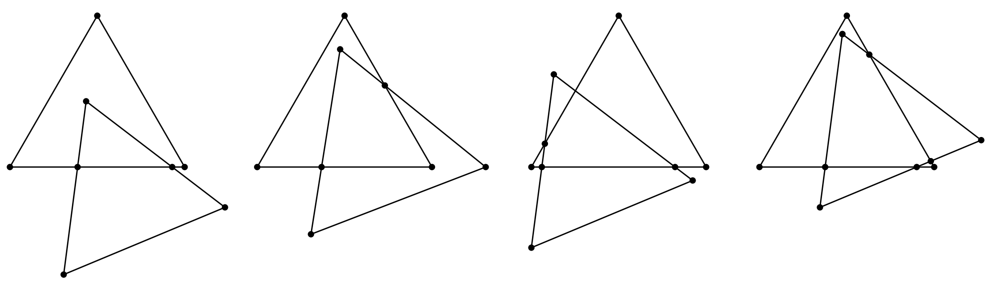


Figure 2

Case 1: In Figure 3, the result follows directly from Proposition 4.

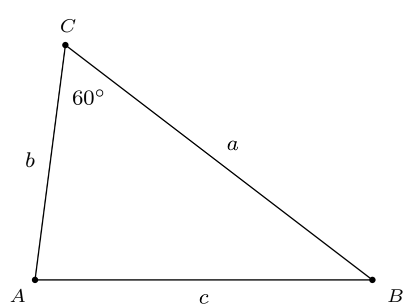


Figure 3

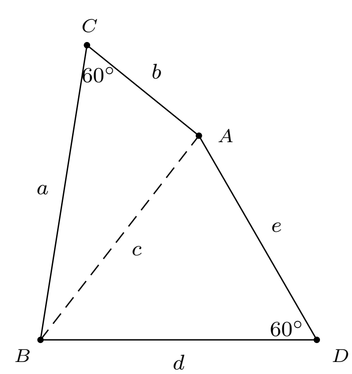


Figure 4

Case 2: We apply the proposition to ΔABC and ΔABD in Figure 4.

Case 3: This case is represented by Figure 5.

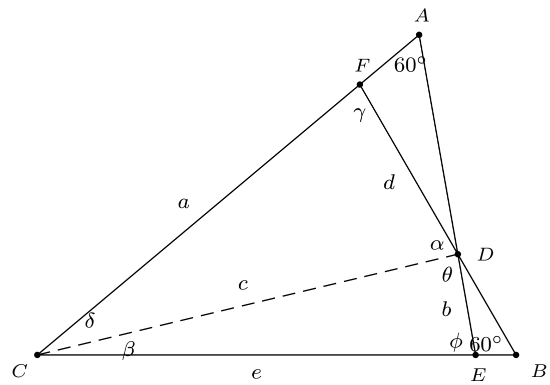


Figure 5

We must show that $\frac{1}{2} \leq \frac{a+b}{d+e} \leq 2$. The angles γ and ϕ are external angles of triangles AFD and BED respectively. Since $\angle FDA = \angle BDE$, $\gamma = \phi$. Applying the law of sines and Lemma 1,

$$\frac{a}{\sin\alpha} = \frac{d}{\sin\delta} = \frac{c}{\sin\gamma} = \frac{c}{\sin\phi} = \frac{b}{\sin\beta} = \frac{e}{\sin\theta},$$
$$\frac{a+b}{\sin\alpha + \sin\beta} = \frac{c}{\sin\gamma} = \frac{d+e}{\sin\delta + \sin\theta}$$
$$\frac{a+b}{d+e} = \frac{\sin\alpha + \sin\beta}{\sin\delta + \sin\theta}.$$

Since α is an external angle of triangle CDB, $\alpha = \beta + 60^\circ$. Similarly, $\theta = \delta + 60^\circ$. Substitution yields

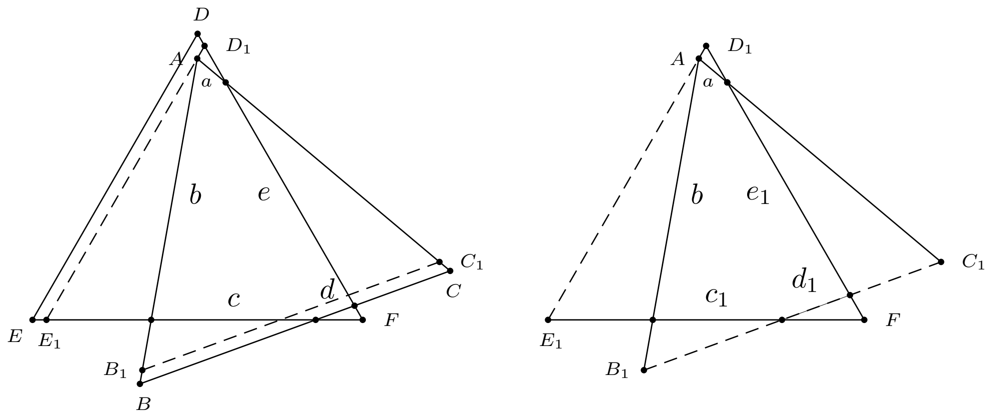
$$\frac{a+b}{d+e} = \frac{\sin(\beta + 60^\circ) + \sin\beta}{\sin\delta + \sin(\delta + 60^\circ)}.$$

Since $\alpha + \delta < 180^\circ$, $\beta + \delta \leq 60^\circ$, and by Lemma 2, $\frac{\sin(\beta+60)+\sin\beta}{\sin\delta+\sin(\delta+60)}$ will have a minimum value when $\beta = 0$ and $\delta = 60^\circ$ and a maximum value when $\beta = 60^\circ$ and $\delta = 0$. Thus,

$$\frac{\frac{\sqrt{3}}{2}}{\sqrt{3}} \leq \frac{\sin(\beta + 60^\circ) + \sin\beta}{\sin\delta + \sin(\delta + 60^\circ)} \leq \frac{\sqrt{3}}{\frac{\sqrt{3}}{2}} \implies \frac{1}{2} \leq \frac{a+b}{d+e} \leq 2.$$

Case 4: We must now show that $\frac{1}{2} \leq \frac{a+b+d}{c+e} \leq 2$ and there are two subcases to consider. The first is illustrated by Figure 6.

In Figure 6(a), construct a segment through point A that is parallel to ED and that intersects FD and FE in points D_1 and E_1 , respectively. Similarly, construct a segment that is parallel to BC and that intersects AB and AC in points B_1 and C_1 , respectively so that triangles AB_1C_1 and D_1E_1F are congruent (Figure 6(b)). By construction, $c_1 < c$, $d_1 > d$, and $e_1 < e$. We apply Theorem 1(a) for

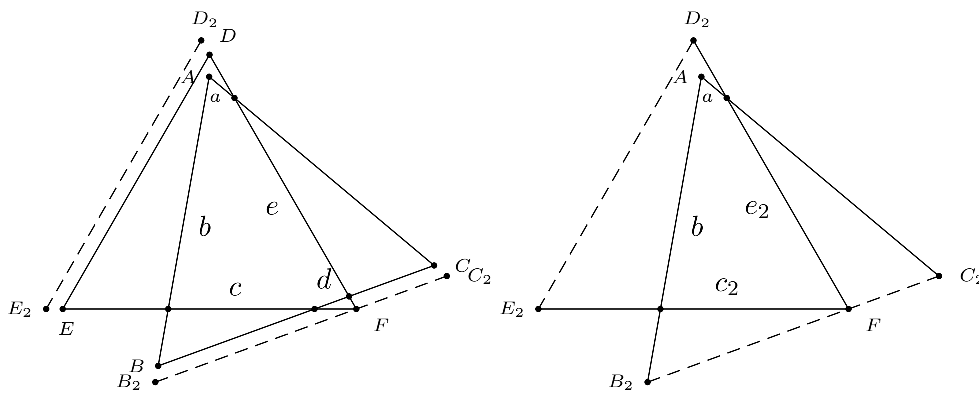


(a) (b)

Figure 6

equilateral triangles having five intersection points to Figure 6(b), and obtain

$$\frac{a+b+d}{c+e} < \frac{a+b+d_1}{c_1+e_1} = 1 \leq 2.$$



(a) (b)

Figure 7

The second case is shown in Figure 7(a). Extend AB and AC so that each intersects the line through F parallel to BC at points B_2 and C_2 , respectively. Construct a segment that is parallel to ED and that intersects FE and FD in points E_2 and D_2 respectively, so that triangles AB_2C_2 and D_2E_2F are congruent (Figure 7(b)). By construction, $c < c_2$ and $e < e_2$. We apply Case 2 to Figure 7(b) and obtain

$$\frac{1}{2} \leq \frac{a+b}{c_2+e_2} < \frac{a+b+d}{c+e}.$$

Combining the inequalities yields $\frac{1}{2} \leq \frac{a+b+d}{c+e} \leq 2$. This completes the proof of Theorem 1.

Fickett's rectangle problem also appeared in a conjecture that if the congruent polygons were triangles then the maximum ratio would be $\csc\frac{\theta}{2}$ where θ is the smallest angle of the triangle. For equilateral triangles, $\theta = 60^\circ$ and $\csc\frac{\theta}{2} = 2$. This corresponds to our result.

References

[1] J. Fickett, Overlapping congruent convex bodies, *Amer. Math. Monthly*, 87 (1980) 814–815.
[2] H. Croft, K. Falconer, and R. Guy, *Unsolved Problems in Geometry*, Springer-Verlag, New York, 1991. p. 25.

Colleen Nielsen: 15 Parkside Place, Apt. 231, Revere, MA 02151, USA
E-mail address: Cnielsen1022@gmail.com

Christa Powers: Timberlane Regional High School, 36 Greenough Road, Plaistow, NH 03865, USA
E-mail address: cody.christa@yahoo.com