Forum Geometricorum
Volume 13 (2013) 227–231.

FORUM GEOM
ISSN 1534-1178

Some Simple Results on Cevian Quotients

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Abstract. We find the loci of the cevian quotients P/Q and Q/P when one of the points is fixed and the other moves along a given line. We also show that, for a given point P, the locus of Q for which the line joining P/Q and Q/P is parallel to PQ is a conic through P and G/P, and give two simple constructions of the conic.

The term cevian quotient was due to John Conway [1]. Given two points $P = (u : v : w)$ and $Q = (x : y : z)$ in homogeneous barycentric coordinates with reference to a triangle ABC, the cevian quotient P/Q is the perspector of the cevian triangle of P and the anticevian triangle of Q. It is the point

$$P/Q = \left(x\left(-\frac{x}{u}+\frac{y}{v}+\frac{z}{w}\right) : y\left(\frac{x}{u}-\frac{y}{v}+\frac{z}{w}\right) : z\left(\frac{x}{u}+\frac{y}{v}-\frac{z}{w}\right)\right).$$

A most basic property of cevian quotient is the following theorem.

Theorem 1 ([2, §2.12], [5, §8.3])**.** *$P/Q = Q'$ if and only if $P/Q' = Q$.*

This is equivalent to $P/(P/Q) = Q$. It can be proved by direct verification with coordinates. We offer an indirect proof, with the advantage of an explicit construction, for given Q and Q', of a point P with $P/Q = Q'$ and $P/Q' = Q$.

For $Q = (x : y : z)$ and $Q' = (x' : y' : z')$ with anticevian triangles XYZ and $X'Y'Z'$, it is easy to check that the lines QX' and $Q'X$ intersect on the sideline BC, at the point $(0 : xy' + x'y : zx' + z'x) = \left(0 : \frac{1}{zx'+z'x} : \frac{1}{xy'+x'y}\right)$ (see Figure 1). Similarly, the lines QY' and $Q'Y$ intersect on CA at $\left(\frac{1}{yz'+y'z} : 0 : \frac{1}{xy'+x'y}\right)$, and the lines QZ' and $Q'Z$ intersect on AB at $\left(\frac{1}{yz'+y'z} : \frac{1}{zx'+z'x} : 0\right)$. These form the cevian triangle of the point $P = \left(\frac{1}{yz'+y'z} : \frac{1}{zx'+z'x} : \frac{1}{xy'+x'y}\right)$. It is clear that $P/Q = Q'$ and $P/Q' = Q$.

Remark. The point $P = \left(\frac{1}{yz'+y'z} : \frac{1}{zx'+z'x} : \frac{1}{xy'+x'y}\right)$ is called the cevian product $Q * Q'$ of Q and Q'. Clearly, $Q * Q' = Q' * Q$.

Proposition 2. *Let P be a fixed point. If Q moves along a line $\mathscr{L}$, then the quotient P/Q traverses the bicevian conic of P and the trilinear pole of $\mathscr{L}$.*

Publication Date: December 3, 2013. Communicating Editor: Paul Yiu.
The author thanks Paul Yiu for his help in improving this paper.

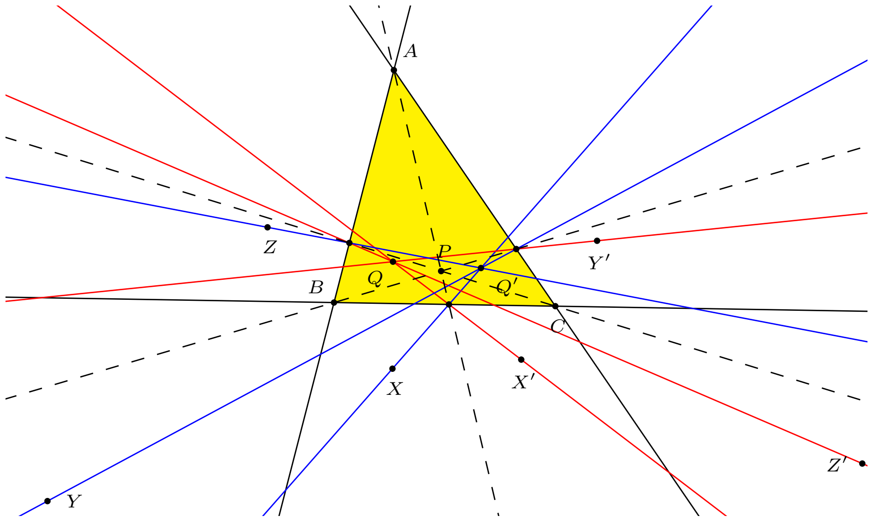

Figure 1.

Proof. Let $P = (u : v : w)$ and Q move along the line $\mathscr{L}$ with line coordinates $[p : q : r]$. If $Q' = P/Q = (x : y : z)$, then $Q = P/Q'$ is on the line $\mathscr{L}$, and

$$px\left(-\frac{x}{u}+\frac{y}{v}+\frac{z}{w}\right)+qy\left(\frac{x}{u}-\frac{y}{v}+\frac{z}{w}\right)+rz\left(\frac{x}{u}+\frac{y}{v}-\frac{z}{w}\right)=0.$$

Clearing denominators and simplifying, we obtain

$$pvwx^2+qwuy^2+ruvz^2-u(qv+rw)yz-v(rw+pu)zx-w(pu+qv)xy=0.$$

If $x = 0$, this becomes $u(qy - rz)(wy - uz) = 0$. The conic intersects the line BC at $(0 : v : w)$ and $(0 : r : q)$. Similarly, it intersects CA at $(u : 0 : w)$ and $(r : 0 : p)$, and AB at $(u : v : 0)$ and $(q : p : 0)$. This is the bicevian conic through the traces of P and $\left(\frac{1}{p} : \frac{1}{q} : \frac{1}{r}\right)$, the trilinear pole of $\mathscr{L}$. □

Corollary 3 ([4])**.** *Let P be a fixed point. The locus of Q for which the cevian quotient P/Q lies on the tripolar of P is the inscribed conic with perspector P.*

Proposition 4. *Let P be a fixed point. If Q moves along a line $\mathscr{L}$, then the cevian quotient Q/P traverses the circumconic of the anticevian triangle of P with perspector $P_{\mathscr{L}}/P$, where $P_{\mathscr{L}}$ is the trilinear pole of $\mathscr{L}$.*

Proof. Let $P = (u : v : w)$ and Q move along the line $\mathscr{L}$ with line coordinates $[p : q : r]$. If $Q'' = Q/P = (x : y : z)$, then $Q = \left(\frac{1}{wy+vz} : \frac{1}{uz+wx} : \frac{1}{vx+uy}\right)$ is on the line $\mathscr{L}$, and

$$\frac{p}{wy+vz}+\frac{q}{uz+wx}+\frac{r}{vx+uy}=0.$$

Clearing denominators and simplifying, we obtain

$$pvwx^2+qwuy^2+ruvz^2+(pu+qv+rw)(uyz+vzx+wxy)=0.$$

It is easy to verify that this conic passes through $A' = (-u : v : w)$, $B' = (u : -v : w)$, $C' = (u : v : -w)$. It is a circumconic of the anticevian triangle of P. The tangents to the conic at A', B', C' are the lines $L_a : (qv+rw)x + quy + ruz = 0$, $L_b : pvx + (pu+rw)y + rvz = 0$, $L_c : pwx + qwy + (pu+qv)z = 0$ which intersects $\mathscr{L}$ on BC, CA, AB respectively. This is the conic tangent to the lines $A'X'$, $B'Y'$, $C'Z'$ at A', B', C' respectively. These lines bound a triangle with vertices

$$\left(\frac{qv+rw}{u} : q : r\right), \quad \left(p : \frac{rw+pu}{v} : r\right), \quad \left(p : q : \frac{pu+qv}{w}\right).$$

These form a triangle perspective with the anticevian triangle of P at

$$(u(-pu+qv+rw) : v(pu-qv+rw) : w(pu+qv-rw)),$$

the cevian quotient of $\left(\frac{1}{p} : \frac{1}{q} : \frac{1}{r}\right)$ (the trilinear pole of $\mathscr{L}$) by P. □

Let $\mathscr{L}/P$ be the conic in Proposition 4. This conic is a circle if and only if it is the circumcircle of the anticevian triangle of P. The line $\mathscr{L}$ is the one containing the intercepts of the tangents to this circle at A', B', C' on the respective sidelines of triangle ABC (see Figure 2). This has line coordinates

$$\begin{aligned} p : q : r = (&u(v+w-u)(c^2v^2 - (b^2+c^2-a^2)vw + b^2w^2) \\ &v(w+u-v)(a^2w^2 - (c^2+a^2-b^2)wu + c^2u^2) \\ &w(u+v-w)(b^2u^2 - (a^2+b^2-c^2)uv + a^2v^2). \end{aligned}$$

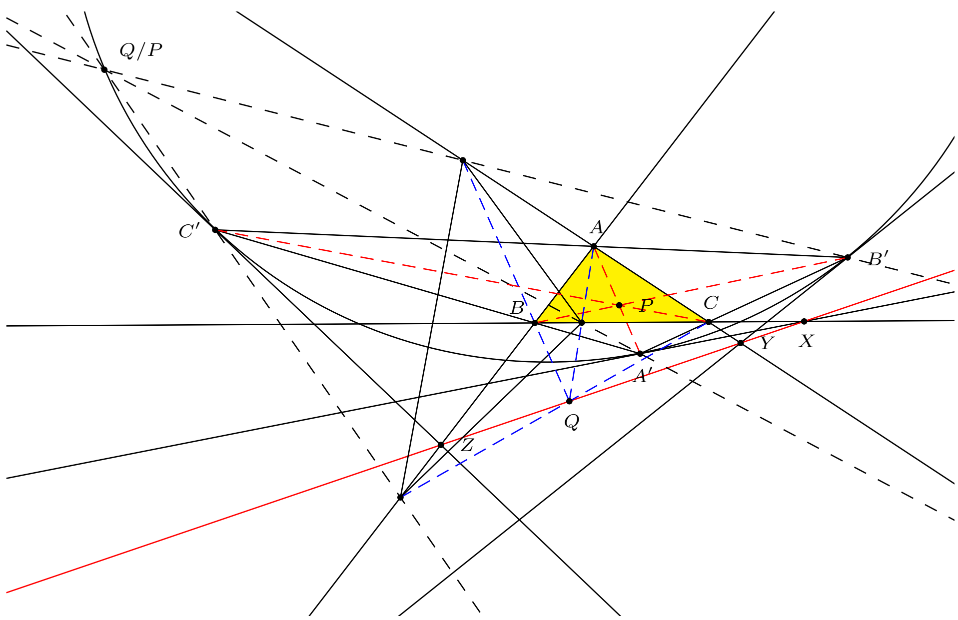

Figure 2.

Here are some simple examples in which $\mathscr{L}/P$ is the circumcircle of the anticevian triangle of P:

P	$\mathscr{L}$
centroid	$a^2x+b^2y+c^2z=0$
incenter	line at infinity
symmedian point	$\sum_{\text{cyclic}} a^2(b^2+c^2-a^2)x=0$

Proposition 5. *Let P be a fixed point. The locus of Q for which the line joining (P/Q) to (Q/P) is parallel to PQ is the union of the cevian lines AP, BP, CP and a conic $\Gamma(P)$*
(1) *homothetic to the circumconic with perspector P,*
(2) *passing through P and the cevian quotient G/P, and has*
(3) *the midpoint of P and G/P as center.*

Proof. If $P=(u:v:w)$ and $Q=(x:y:z)$, the line joining P/Q and Q/P contains the infinite point of PQ if and only if

$$\begin{vmatrix} x\left(-\frac{x}{u}+\frac{y}{v}+\frac{z}{w}\right) & y\left(\frac{x}{u}-\frac{y}{v}+\frac{z}{w}\right) & z\left(\frac{x}{u}+\frac{y}{v}-\frac{z}{w}\right) \\ u\left(-\frac{u}{x}+\frac{v}{y}+\frac{w}{z}\right) & v\left(\frac{u}{x}-\frac{v}{y}+\frac{w}{z}\right) & w\left(\frac{u}{x}+\frac{v}{y}-\frac{w}{z}\right) \\ (v+w)x-u(y+z) & (w+u)y-v(z+x) & (u+v)z-w(x+y) \end{vmatrix}=0.$$

Clearing denominators and simplifying, we obtain

$$2(wy-vz)(uz-wx)(vx-uy)(vwx^2+wuy^2+uvz^2-u^2yz-v^2zx-w^2xy)=0.$$

Therefore Q lies on one of the lines AP, BP, CP or the conic $\Gamma(P)$ defined by

$$vwx^2+wuy^2+uvz^2-u^2yz-v^2zx-w^2xy=0.$$

Rewriting this as

$$\Gamma(P):\ (u+v+w)(uyz+vzx+wxy)-(x+y+z)(vwx+wuy+uvz)=0,$$

it is clear that $\Gamma(P)$ is homothetic to the circumconic with perspector P, and it is routine to verify that it contains P and the cevian quotient $G/P=(u(-u+v+w):v(u-v+w):(u+v-w)w)$. The center of the conic is the midpoint of P and G/P, namely,

$$\left(u(u^2-uv-uw-2vw):v(v^2-uv-2uw-vw):w(w^2-2uv-uw-vw)\right).$$

□

Remarks. (1) If P is the symmedian point, then $\Gamma(P)$ is the Brocard circle with diameter OK.

(2) If the line PQ contains A, then both cevian quotients P/Q and Q/P are on the same line.

It is easy to note that the conic $\Gamma(P)$ contains the points

$$A_1=(-u+v+w:v:w),\quad B_1=(u:u-v+w:w),\quad (u:v:u+v-w)$$

We present two simple constructions of these points, one by Peter Moses [3], and another by Paul Yiu [6].

Construction (Moses)**.** Intersect the cevians AP, BP, CP with the parallels through the centroid G to BC, CA, AB, at X, Y, Z respectively. A_1, B_1, C_1 are the harmonic conjugates of P in AX, BY, CZ respectively.

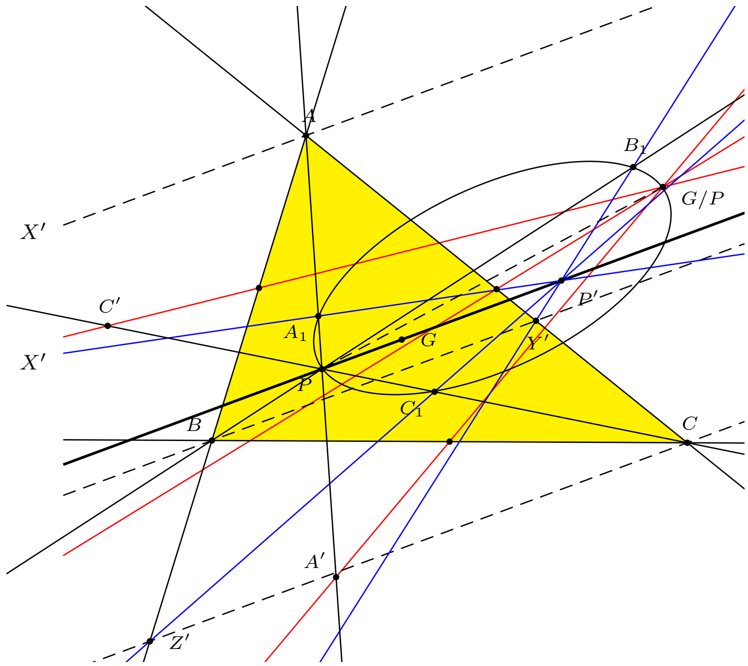

Figure 3.

Construction (Yiu)**.** Let P' be the superior of P, i.e., the point dividing PG in the ratio $PP' : P'G = 3 : -2$. Construct the parallels of the line PG through the vertices A, B, C, to intersect the sidelines BC, CA, AB at X', Y', Z' respectively. Then $A_1 = AP \cap X'P'$, $B_1 = BP \cap Y'P'$, and $C_1 = CP \cap Z'P'$. See Figure 3.

References

[1] J. H. Conway, Hyacinthos message 1018, June 14, 2000.
[2] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.
[3] P. Moses, Private communication, October 19, 2013.
[4] P. Yiu, Hyacinthos message 1030, June 16, 2000.
[5] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University Lecture Notes, 2001; with corrections, 2013, available at
`http://math.fau.edu/Yiu/Geometry.html`
[6] P. Yiu, ADGEOM message 728, October 19, 2013.

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