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A Vector-based Proof of Morley's Trisector Theorem

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Abstract. A proof is given of Morley's trisector theorem using elementary vector analysis and trigonometry. The known expression for the side of Morley's equilateral triangle is also obtained.

Since its formulation in 1899, many proofs of Morley's trisector theorem have appeared, typically based on plane geometry or involving trigonometry; a historical overview of this theorem with numerous references up to the year 1977 can be found in [4]. Some of the more recent geometric proofs are of the "backward" type [2, 5]; a group-theoretic proof was also given [3]. About fifteen different methods that were used to prove Morley's theorem are described in detail in [1], with comments on their specific characteristics. The website [1] also provides the related references, which span from the year 1909 to 2010.

In this note we prove the theorem in two stages. First a lemma is proved by use of the dot product of vectors and trigonometry, then the theorem itself easily follows from elementary geometry.

Morley's Theorem. *In any triangle, the three points of intersection of the adjacent angle trisectors form an equilateral triangle.*

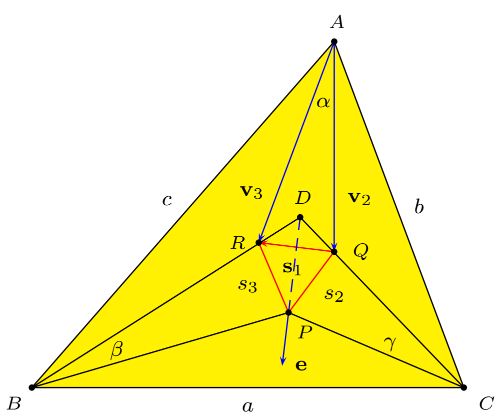

Figure 1

Let the angles of triangle ABC be of amplitude 3α, 3β, 3γ, then $\alpha + \beta + \gamma = 60°$. The adjacent trisectors meet to form Morley's triangle PQR; the line extensions of BR and CQ intersect at D. In triangle BDC the bisector of angle

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D is concurrent with the other two bisectors BP and CP at P, the incenter of the triangle. First, a lemma is proved, from which the theorem easily follows.

Lemma. *The line DP is perpendicular to the line RQ.*
Proof. Here use is made of the vector method in conjunction with trigonometry. Let $\mathbf{e}$ be the unit vector along DP and $\mathbf{s}_1$ the vector representing the side QR of triangle PQR. Then the lemma can be restated as saying that the scalar product $\mathbf{s}_1 \cdot \mathbf{e}$ vanishes. Figure 1 shows that $\mathbf{s}_1 = \mathbf{v}_3 - \mathbf{v}_2$ so that we must prove that

$$(\mathbf{v}_3 - \mathbf{v}_2) \cdot \mathbf{e} = 0.$$

In triangle BDC we have $2\beta + 2\gamma = 120^\circ - 2\alpha$, therefore $\angle D = 60^\circ + 2\alpha$. This angle is bisected by the line DP, hence $\angle QDP = \angle RDP = 30^\circ + \alpha$. By the exterior angle theorem $\angle AQD = \alpha + \gamma$ and $\angle ARD = \alpha + \beta$. The angle between the vectors $\mathbf{v}_3$ and $\mathbf{e}$, being the difference between angles RDP and ARD, is $30^\circ - \beta$. Similarly, the angle between $\mathbf{v}_2$ and $\mathbf{e}$ is $30^\circ - \gamma$. From these,

$$\begin{aligned}(\mathbf{v}_3 - \mathbf{v}_2) \cdot \mathbf{e} &= \mathbf{v}_3 \cdot \mathbf{e} - \mathbf{v}_2 \cdot \mathbf{e} \\ &= v_3 \cos(30^\circ - \beta) - v_2 \cos(30^\circ - \gamma) \\ &= v_3 \sin(60^\circ + \beta) - v_2 \sin(60^\circ + \gamma). \qquad (1)\end{aligned}$$

The magnitudes of $\mathbf{v}_3$ and $\mathbf{v}_2$ can be found by applying the law of sines to triangles ARB and AQC respectively:

$$v_3 = \frac{c \sin\beta}{\sin(\alpha+\beta)} = \frac{c \sin\beta}{\sin(60^\circ - \gamma)}, \qquad v_2 = \frac{b \sin\gamma}{\sin(\alpha+\gamma)} = \frac{b \sin\gamma}{\sin(60^\circ - \beta)}.$$

Substituting these expressions into (1), we obtain

$$\begin{aligned}(\mathbf{v}_3 - \mathbf{v}_2) \cdot \mathbf{e} &= \frac{c \sin\beta \sin(60^\circ + \beta)}{\sin(60^\circ - \gamma)} - \frac{b \sin\gamma \sin(60^\circ + \gamma)}{\sin(60^\circ - \beta)} \\ &= \frac{c \sin\beta \sin(60^\circ + \beta) \sin(60^\circ - \beta) - b \sin\gamma \sin(60^\circ + \gamma) \sin(60^\circ - \gamma)}{\sin(60^\circ - \beta) \sin(60^\circ - \gamma)} \\ &= \frac{1}{4} \cdot \frac{c \sin 3\beta - b \sin 3\gamma}{\sin(60^\circ - \beta) \sin(60^\circ - \gamma)}\end{aligned}$$

with the aid of the identity

$$\sin x \sin(60^\circ + x) \sin(60^\circ - x) = \frac{1}{4} \sin 3x, \qquad (2)$$

which can be easily proved through the product-to-sum trigonometric formulas.

The law of sines for triangle ABC yields $c \sin 3\beta - b \sin 3\gamma = 0$. Therefore, $(\mathbf{v}_3 - \mathbf{v}_2) \cdot \mathbf{e} = 0$. □

Proof of Morley's Theorem. Knowing that $DP \perp RQ$, we see that DP divides DQR into two congruent right triangles (with a common leg and a pair of equal acute angles) so that $DQ = DR$. Consequently, triangles DPQ and DPR are also congruent (by SAS), and $s_2 = s_3$. The whole procedure can be used to prove that $s_1 = s_2$. It follows that $s_1 = s_2 = s_3$, and triangle PQR is equilateral. This completes the proof of Morley's theorem.

Remark. Since triangle DQR is composed of two congruent right triangles, and $\angle QDR = 30^\circ + \alpha$, its complement $\angle DQR = 60^\circ - \alpha$.

The side of Morley's triangle. The side length s of the equilateral triangle PQR can be calculated by applying the law of sines to triangle AQR, whose angles are now known. Since $\angle RAQ = \alpha$, and $\angle AQR = \angle AQD + \angle DQR = (\alpha + \gamma) + (60^\circ - \alpha) = 60^\circ + \gamma$, we find that

$$s = \frac{v_3 \sin\alpha}{\sin(60^\circ + \gamma)} = \frac{c \sin\alpha \sin\beta}{\sin(60^\circ + \gamma)\sin(60^\circ - \gamma)}.$$

By multiplying both terms of the last fraction by $\sin\gamma$, and using in the denominator the identity (2), we get the known expression for the side of Morley's triangle

$$s = \frac{4c \sin\alpha \sin\beta \sin\gamma}{\sin 3\gamma} = 8R \sin\alpha \sin\beta \sin\gamma,$$

where $R = \frac{c}{2\sin 3\gamma}$ is the radius of the circumcircle of triangle ABC.

References

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