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Soddyian Triangles

Frank M. Jackson

Abstract. A Soddyian triangle is a triangle whose outer Soddy circle has degenerated into a straight line. This paper examines some properties of Soddyian triangles, including the facts that no Soddyian triangle can be right angled and all integer Soddyian triangles are Heronian. A generating formula is developed to produce all primitive integer Soddyian triangles. A ruler and compass construction of a Soddyian triangle concludes the paper.

1. The outer Soddy circle

In 1936 the chemist Frederick Soddy re-discovered the Descartes’ theorem that relates the radii of two tangential circles to the radii of three touching circles and applied the problem to the three contact circles of a general triangle.

The tangential circles with centers $S$ and $S'$ are called the inner and outer Soddy circles of the reference triangle $ABC$. If $r_i$ and $r_o$ are the radii of the inner and outer Soddy circles, then

$$r_i = \frac{\Delta}{4R + r + 2s} \quad \text{and} \quad r_o = \frac{\Delta}{4R + r - 2s}.$$

(1)
where $\Delta$ is the area of the triangle, $R$ its circumradius, $r$ its inradius, $s$ its semi-perimeter, and $s-a$, $s-b$, $s-c$ the radii of the touching circles (see [1]). By adjusting the side lengths of the reference triangle it is possible to fashion a triangle with contact circles such that the outer Soddy circle degenerates into a straight line. This occurs when $4R + r = 2s$ and is demonstrated in Figure 2 below, where it is assumed that $a \leq b \leq c$.

Note that for a Soddyian triangle, the radius of the inner Soddy circle is $\frac{r}{4}$. This follows from

$$r_i = \frac{\Delta}{4R + r + 2s} = \frac{\Delta}{4s} = \frac{r}{4}$$

provided $4R + r = 2s$.

![Figure 2](image_url)

Now the common tangent to the three circles is the outer Soddy circle. Consequently a class of triangles can be defined as Soddyian if their outer Soddy radius is infinite. However from the above diagram it is possible to derive a relationship that is equivalent to the condition that the outer Soddy circle is infinite by considering the length of the common tangents between pairs of touching circles. Given two touching circles of radii $u$ and $v$, their common tangent has a length of $2\sqrt{uv}$ and applying this to the three touching circle with a common tangent gives

$$\frac{1}{\sqrt{s-c}} = \frac{1}{\sqrt{s-a}} + \frac{1}{\sqrt{s-b}}. \quad (2)$$

### 2. Can a Soddyian triangle be right angled?

If triangle $ABC$ has a right angle at $C$, then

$$R = \frac{c}{2} \quad \text{and} \quad r = s - c.$$ 

If $4R + r = 2s$, then $2c + s - c = 2s$. This resolves to $c = a + b$, an impossibility. Therefore, no Soddyian triangle is right angled.
3. Can a Soddyian triangle be isosceles?

A Soddy triangle with side lengths $a \leq b \leq c$ is isosceles only if $a = b$. Since $s = a + \frac{c}{2}$ and $\frac{1}{\sqrt{s-c}} = \frac{2}{\sqrt{s-a}}$, we have $\frac{c}{2} = 4a - 2c$. Hence $a : c = 5 : 8$, and the only primitive integer isosceles Soddyian triangle has sides 5, 5, 8. Note that this has integer area 12.

4. Are all integer Soddyian triangles Heronian?

Now consider the Soddyian constraint $4R + r = 2s$ expressed in terms of the area $\Delta$:

$$\frac{abc}{\Delta} + \frac{\Delta}{s} = 2s.$$  

This is quadratic in $\Delta$ and

$$\Delta = s^2 \pm \sqrt{s^4 - abc}.$$  

Since $s$ is greater than any of the sides, $\Delta < s^2$ and we must have

$$\Delta = s^2 - \sqrt{s^4 - abc}.$$  

By the Heron formula, $16\Delta^2 = (a + b + c)(b + c - a)(c + a - b)(a + b - c)$ is an integer. This can only happen if $s^4 - abc$ is also a square integer. Hence all integer Soddyian triangles are Heronian.

5. Construction of integer Soddyian triangles

It is well known that for a Heronian triangle, the semiperimeter $s$ is an integer. From (2),

$$s - c = \frac{(s - a)(s - b)}{(s - a) + (s - b) + 2\sqrt{(s - a)(s - b)}}.$$  

This requires $\sqrt{(s - a)(s - b)}$ to be an integer. We write $s - a = km^2$ and $s - b = kn^2$ for integers $k, m, n$, and obtain

$$s - c = \frac{km^2n^2}{(m + n)^2}.$$  

Therefore,

$$s - a : s - b : s - c : s = m^2(m + n)^2 : n^2(m + n)^2 : m^2n^2 : (m^2 + mn + n^2)^2,$$

and we may take

$$a = n^2((m + n)^2 + m^2),$$  

$$b = m^2((m + n)^2 + n^2),$$  

$$c = (m + n)^2(m^2 + n^2).$$
For this triangle,
\[
\Delta = m^2n^2(m + n)^2(m^2 + mn + n^2),
\]
\[
R = \frac{(m^2 + n^2)((m + n)^2 + m^2)((m + n)^2 + n^2)}{4(m^2 + mn + n^2)},
\]
\[
r = \frac{m^2n^2(m + n)^2}{m^2 + mn + n^2},
\]
\[
s = (m^2 + mn + n^2)^2.
\]

Here are some examples of integer Soddyian triangles.

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6. Soddyian triangles with a given side

Consider Soddyian triangles with a given base AB in a rectangular coordinate system with origin at A and B = (c, 0) on the x-axis. Suppose the vertex C has coordinates (x, y). Using the expressions in §5 in terms of m and n, allowing them to take on positive real values, we have

\[
\frac{x}{c} = \frac{b^2 + c^2 - a^2}{2c^2} = \frac{m^2(m^2 + mn + 2n^2)}{(m + n)(m^2 + 2n^2)},
\]
\[
\frac{y}{c} = \frac{2\Delta}{c^2} = \frac{m^2n^2(m^2 + mn + n^2)}{(m + n)^2(m^2 + n^2)^2}.
\]

Writing n = tm, we obtain a parametrization of the locus of C as follows (see Figure 3).

\[
(x, y) = c \left( \frac{1 + t + 2t^2}{(1 + t)(1 + t^2)^2}, \frac{2t^2(1 + t + t^2)}{(1 + t)^2(1 + t^2)^2} \right).
\]
In fact, given \( s - a \) and \( s - b \), there is a simple ruler and compass construction for the Soddyian triangle.

**Construction 1.** Given a segment \( AB \) and a point \( Z \) on it (with \( AZ = s - a \) and \( BZ = s - b \)),
1. construct the perpendicular to \( AB \) at \( Z \), to intersect the semicircle with diameter \( AB \) at \( P \);
2. let \( A' \) and \( B' \) be points on the same side of \( AB \) such that \( AA' \parallel AB \) and \( AA' = AZ \), \( BB' = BZ \);
3. join \( PA' \) and \( PB' \) to intersect \( AB \) at \( X \) and \( Y \) respectively;
4. construct the circle through \( P \), \( X \), \( Y \) to intersect the line \( PZ \) again at \( Q \);
5. let \( X' \) and \( Y' \) be point on \( AZ \) and \( ZB \) such that \( X'Z = ZY' = ZQ \);
6. construct the circles centers \( A \) and \( B \), passing through \( Y' \) and \( X' \) respectively, to intersect at \( C \).

The triangle \( ABC \) is Soddyian with incircle touching \( AB \) at \( Z \) (see Figure 4).

**Proof.** Let \( AZ = u \) and \( BZ = v \). From (1), \( ZP = \sqrt{uv} \). From (2),

\[
ZX = ZA \cdot \frac{ZP}{ZP + AA'} = u \cdot \frac{\sqrt{uv}}{u + \sqrt{uv}} = \frac{u\sqrt{v}}{\sqrt{u} + \sqrt{v}}.
\]
Similarly, \( ZY = \frac{u\sqrt{v}}{\sqrt{u} + \sqrt{v}} \). By the intersecting chords theorem,
\[
ZQ = \frac{ZX \cdot ZY}{ZP} = \frac{uv}{(\sqrt{u} + \sqrt{v})^2}.
\]

It follows that
\[
\frac{1}{\sqrt{ZQ}} = \frac{\sqrt{u} + \sqrt{v}}{\sqrt{uv}} = \frac{1}{\sqrt{u}} + \frac{1}{\sqrt{v}} = \frac{1}{\sqrt{AZ}} + \frac{1}{\sqrt{ZB}}.
\]

Therefore, triangle \( ABC \) satisfies
\[
BC = BX' = BZ + ZX' = BZ + ZQ,
AC = AY' = AZ + ZY' = AZ + ZQ,
AB = AZ + ZB,
\]
with
\[
\frac{1}{\sqrt{s - c}} = \frac{1}{\sqrt{ZQ}} = \frac{1}{\sqrt{AZ}} + \frac{1}{\sqrt{ZB}} = \frac{1}{\sqrt{s - a}} + \frac{1}{\sqrt{s - b}}.
\]
It is Soddyian and with incircle tangent to \( AB \) at \( Z \).

Reference


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A Triad of Circles Tangent Internally to the Nine-Point Circle

Nikolaos Dergiades and Alexei Myakishev

Abstract. Given an acute triangle, we construct the three circles each tangent to two sides and to the nine point circle internally. We show that the centers of these three circles are collinear.

In this note we construct, for a given acute triangle, the three circles each tangent to two sides of the triangle and tangent to the nine point circle internally. We show that the centers of these three circles are collinear ([1, 3]).

Let $ABC$ be the given triangle with incenter $I$. For three points $A', B', C'$ on the respective angle bisectors, write the vectors

\[ IA' = pIA, \quad IB' = qIB, \quad IC' = rIC. \]

Since

\[ \left( \frac{a}{p} \right) IA' + \left( \frac{b}{q} \right) IB' + \left( \frac{c}{r} \right) IC' = aIA + bIB + cIC = 0, \]

the three points $A', B', C'$ are collinear if and only if $\frac{a}{p} + \frac{b}{q} + \frac{c}{r} = 0$.

Now consider the nine-point circle of triangle $ABC$. This is tangent to the incircle at the Feuerbach point $F_e$. The power of $A$ is $d^2 = \frac{1}{2} S_A$. If we apply
inversion with center $A$ and power $d^2$, the inverse of the incircle is a circle $(A')$ tangent to $AB$, $AC$ and the nine-point circle at the second intersection $F_1$ of the line $AF_e$. We have

$$\frac{AA'}{AI} = \frac{d^2}{(s-a)^2}.$$ 

Hence, $p = \frac{IA'}{IA} = \frac{2(a-b)(a-c)}{(b+c-a)}$.

Similarly for the other centers $B'$, $C'$ we have

$$q = \frac{2(b-c)(b-a)}{(a+b+c)} , \quad r = \frac{2(c-a)(c-b)}{(a+b+c)^2}.$$ 

It is easy to prove that

$$\frac{a}{p} + \frac{b}{q} + \frac{c}{r} = 0.$$ 

Therefore, the three centers $A'$, $B'$, $C'$ are collinear.

These centers are

$$A' = pA + (1-p)I = \left( \frac{p(a+b+c)}{1-p} + a : b : c \right),$$

$$B' = qB + (1-q)I = \left( a : \frac{q(a+b+c)}{1-q} + b : c \right),$$

$$C' = rC + (1-r)I = \left( a : b : \frac{r(a+b+c)}{1-r} + c \right).$$ 

If the line containing these centers has barycentric equation $ux + vy + wz = 0$ with reference to triangle $ABC$, then
A triad of circles tangent internally to the nine-point circle

\[ A' = \left( -\frac{bv + cw}{u} : b : c \right), \quad B' = \left( a : -\frac{au + cw}{v} : c \right), \quad C' = \left( a : b : -\frac{au + bv}{w} \right). \]

It follows that

\[ \frac{p - 1}{p} = \frac{(a + b + c)u}{au + bv + cw}, \quad \frac{q - 1}{q} = \frac{(a + b + c)v}{au + bv + cw}, \quad \frac{r - 1}{r} = \frac{(a + b + c)w}{au + bv + cw}, \]

and

\[ u : v : w = \frac{p - 1}{p} : \frac{q - 1}{q} : \frac{r - 1}{r} \]
\[ = \frac{b^2 + c^2 - a^2}{2(c - a)(a - b)} : \frac{c^2 + a^2 - b^2}{2(a - b)(b - c)} : \frac{a^2 + b^2 - c^2}{2(b - c)(c - a)} \]
\[ = (b - c)S_A : (c - a)S_B : (a - b)S_C. \]

The line containing these points has equation

\[ (b - c)S_Ax + (c - a)S_By + (a - b)S_Cz = 0. \]

This line contains the orthocenter \(\left( \frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C} \right)\) and the Spieker center. As such, it is the Soddy line of the inferior triangle. It is perpendicular to the Gergonne axis, and is the trilinear polar of \(X_{1897}\). Randy Hutson [2] has remarked that this is also the Brocard axis of the excentral triangle.

We conclude with two remarks about the constructions in this note.

1. If angle \(A\) is acute, then the circle \((A')\) is tangent internally to the nine-point circle, and the circle \((A'')\) inverse to the \(A\)-excircle is tangent externally to the nine-point circle (see Figure 3).
(2) The constructions apply also to obtuse triangles. If angle $A$ is obtuse, the points $A', A''$ are on the extension of $IA$. The circle $(A')$ is tangent externally to the nine-point circle, and the inverse of the $A$-excircle is a circle tangent internally to the nine-point circle (see Figure 4).

Figure 4.

References

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Bicentric Quadrilaterals through Inversion

Albrecht Hess

Abstract. We show that inversion is a delightful tool for making some recent and some older results on bicentric quadrilaterals more transparent and to smoothen their proofs. As a main result we give an illustrative interpretation of Yun’s inequality and derive a sharper form.

1. Introduction

Figure 1 shows a bicentric quadrilateral $ABCD$, its circumcircle $C$ with center $O$ and radius $R$, and its incircle $C'$ with center $Z$ and radius $r$, $OZ = d$. The sides of $ABCD$ are tangent to $C$ at $E, F, G, H$. Apply an inversion with respect to $C$.

The images $A', B', C', D'$ of the vertices lie on the circle $C'$ with center $M$ and radius $R'$, $MZ = d'$. The image $A'$ lies on the polar of $A$ with respect to $C$ and is therefore the midpoint of $EH$. The same applies to the other images. $A'B'C'D'$ is a rectangle, because being the quadrilateral of the midpoints of $EFGH$ it is a cyclic parallelogram. The diagonals $EG$ and $HF$ are orthogonal, since they are parallel to the sides of $A'B'C'D'$. Cf. [14, step 2] and [7, 837 ff.].

Publication Date: January 29, 2013. Communicating Editor: Paul Yiu.
2. Orthogonality of Newton lines

Theorem 1 (9, Theorem 6). A tangential quadrilateral $ABCD$ - without axes of symmetry - is cyclic if and only if its Newton line is perpendicular to the Newton line of its contact quadrilateral.

The restriction is included, since Newton lines do not exist in tangential quadrilaterals with several axes of symmetry, for isosceles tangential trapezoids the theorem is obvious and it is false for kites. Let $I$ and $J$ be the points of intersection of $AB$ and $CD$, respectively. $BC$ and $AD$. The midpoints $M_{AC}$, $M_{BD}$, $M_{IJ}$ are collinear in any quadrilateral. The line passing through these points is called the Newton line. To prove the collinearity one could use barycentric coordinates. For a visual proof, connect some midpoints of the quadrilateral sides and the appearing parallelograms will guide you. More information about Newton lines can be found in [1, pp. 116–118] and [2].

Proof of Theorem 1. Suppose that the Newton line of $ABCD$, i.e., the line $n_1$ through $M_{AC}$, $Z$, $M_{BD}$, $M_{IJ}$, and the Newton line of $EFGH$, i.e., the line $n_2$ through $M_{EG}$, $M_{FH}$, are perpendicular. Apply the inversion with respect to the incircle $C$. The images of $I$ and $J$ of $M_{EG}$ and $M_{FH}$ lie on the image of $n_2$, which is a circle through $Z$ orthogonal to $n_1$, whose center lies on $n_1$. If $M_{IJ} \in n_1$ is not the center of this circle, then $I$ and $J$ are symmetrical with respect to $n_1$ and $ABCD$ is a kite, which was excluded. Hence $M_{IJ}$ is the center of the image of $n_2$, $\angle IZJ = 90^\circ$, $EG \perp FH$, $A'B'C'D'$ is a rectangle and $ABCD$ cyclic. This argument can be reversed easily.

3. Fuss' formula

We derive Fuss’ theorem (cf. [3], [7, 837 ff.], [8, Theorem 125], [11, p.1],) by inversion. I found no other place in literature, except the quoted book [7], where Fuss’ theorem is proved with inversion. But the calculations in F. G.-M.’s book are somewhat cumbersome.

Observe - with Thales’ theorem or angle chasing - that $B'SD'Z$ is a parallelogram. $M$ being the midpoint of $ZS$, the parallelogram law says

$$4R^2 + 4d^2 = 4MD'^2 + 4d'^2 = 2ZD'^2 + 2SD'^2 = 2r^2.$$
The formulae for radius and midpoint distance of an inverted circle $R' = \frac{r^2 R}{R^2 - d^2}$ and $d' = \frac{r^2 d}{R^2 - d^2}$, [8, p. 51], substituted into $2R'^2 + 2d'^2 = r^2$ lead to Fuss’ formula

$$\frac{1}{(R - d)^2} + \frac{1}{(R + d)^2} = \frac{1}{r^2}.$$ 

4. Poncelet’s porism

**Theorem 2.** $ABCD$ is a bicentric quadrilateral with circumcircle $\mathcal{C}$ and incircle $\mathcal{C}$. Then bicentric quadrilaterals with circumcircle $\mathcal{C}$ and incircle $\mathcal{C}$ can be constructed starting from any point of the circumcircle $\mathcal{C}$ (cf. [4], [6], [12], [13]).

**Proof.** If $ABCD$ is bicentric (see Figure 1), $R$, $r$ and $d$ obey Fuss’ formula. Using inversion with respect to $\mathcal{C}$, the circumcircle $\mathcal{C}'$ of $ABCD$ is mapped onto the circle $\mathcal{C}'$ with center $M$ and $2R'^2 + 2d'^2 = r^2$, just reverse the substitutions above. Let $S$ be a point such that $M$ is the midpoint between the center $Z$ of the incircle and this point $S$. Choose any point $A'$ on $\mathcal{C}'$. This point $A'$ and its diametrically opposite point $C'$ form with $Z$ and $S$ a parallelogram. From the parallelogram law follows that $A'$ is the midpoint of a chord $HE$ of $\mathcal{C}$ which forms together with $S$ a right triangle. $G$ and $F$ are the endpoints of the chords from $E$ and $H$ through $S$ and $B'$, $C'$ and $D'$ the midpoints of the corresponding chords. Inversion with respect to $\mathcal{C}$ converts the circles with diameters $ZE, ZF, ZG, ZH$ into the sides of the bicentric quadrilateral whose vertices are the images of $A', B', C'$ and $D'$. $\square$
5. Carlitz’ inequality

Furthermore, from \(2R'^2 + 2d'^2 = r^2\) we get \(\sqrt{2}R' \leq r\). Substituted into \(R = \frac{r^2R'}{R'^2 - d'^2} \geq \frac{r^2}{R'} \geq \sqrt{2}r\), Carlitz’ inequality [5] is obtained.

6. Coaxial system of circles

Writing \(2R'^2 + 2d'^2 = r^2\) as \(R'^2d' + d'^2 = r^2\), we see that in a bicentric quadrilateral \(ABCD\) the image \(S'\) of the point \(S\) of intersection of \(GE\) and \(FH\) - and also of \(AC\) and \(BD\) - and also of \(AC\) and \(BD\) by Pascal’s theorem applied to a degenerated hexagon - is the same when inverted with respect to \(C\) or when inverted with respect to \(C'\). This means that the circle with diameter \(SS'\) is orthogonal to \(C\) and to \(C'\)- and also to \(C\) by inversion with respect to \(C\). This reveals \(C, C'\) as members of a coaxial system of circles with limiting points \(S\) and \(S'\). The perpendicular bisector of \(SS'\) is the radical axis of this coaxial system, [8, chapter III].

7. Yun’s inequality revisited

With \(\frac{A+B}{2} = E, \frac{B+C}{2} = F\) and the law of sines \(2r \sin E = FH, 2r \sin F = EG\), Yun’s inequality

\[
\frac{\sqrt{2}r}{R} \leq \frac{1}{2} \left( \sin \frac{A}{2} \cos \frac{B}{2} + \sin \frac{B}{2} \cos \frac{C}{2} + \sin \frac{C}{2} \cos \frac{D}{2} + \sin \frac{D}{2} \cos \frac{A}{2} \right) \leq 1,
\]

[10, [15], is converted by multiplication with \(2r\) into \(2\sqrt{\frac{2r^2}{R}} \leq \frac{EG+FH}{2} \leq 2r\). The right hand side is obvious. We increase the left hand side applying the formula for the radius of an inverted circle \(R = \frac{r^2R'}{R'^2 - d'^2} \geq \frac{r^2}{\sqrt{R'^2 - d'^2}} \leq \frac{2\sqrt{2r^2}}{R} \leq 2\sqrt{2R'^2 - 2d'^2}. \) From \(2R'^2 + 2d'^2 = r^2\) we get \(2\sqrt{\frac{2r^2}{R}} \leq 2\sqrt{r^2 - (d')^2}\). But \(2\sqrt{r^2 - (d')^2}\) is the length of the minimum chord of the circle \(C\) through \(S\) and \(\frac{EG+FH}{2}\) is the mean of any two orthogonal chords through \(S\), which is obviously greater, equality occurs only for squares \(ABCD\) when \(S = Z\).

Comparing one chord instead of the mean of two orthogonal chords with the minimum chord we get the inequality

\[
\frac{\sqrt{2}r}{R} \leq \sin \frac{A+B}{2} = \sin \frac{A}{2} \sin \frac{D}{2} + \sin \frac{B}{2} \sin \frac{C}{2},
\]

of which Yun’s inequality is a consequence.

References


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Five Proofs of an Area Characterization of Rectangles

Martin Josefsson

Abstract. We prove in five different ways a necessary and sufficient condition for a convex quadrilateral to be a rectangle regarding its area expressed in terms of its sides.

There are a handful of well known characterizations of rectangles, most of which concerns one or all four of the angles of the quadrilateral (see [8, p.34]). One example is that a parallelogram is a rectangle if and only if it has (at least) one right angle. Here we shall prove that a convex quadrilateral with consecutive sides \(a, b, c, d\) is a rectangle if and only if its area \(K\) satisfies

\[
K = \frac{1}{2} \sqrt{(a^2 + c^2)(b^2 + d^2)}.
\]

We give five different proofs of this area characterization.

First proof. For the area of a convex quadrilateral, we have (see the left half of Figure 1)

\[
K = \frac{1}{2}ab \sin B + \frac{1}{2}cd \sin D \leq \frac{1}{2}(ab + cd),
\]

where there is equality if and only if \(B = D = \frac{\pi}{2}\). Using the following algebraic identity due to Diophantus of Alexandria

\[
(ab + cd)^2 + (ad - bc)^2 = (a^2 + c^2)(b^2 + d^2)
\]

directly yields the two dimensional Cauchy-Schwarz inequality

\[
ab + cd \leq \sqrt{(a^2 + c^2)(b^2 + d^2)}
\]
with equality if and only if \( ad = bc \). Hence the area of a convex quadrilateral satisfies

\[
K \leq \frac{1}{2} \sqrt{(a^2 + c^2)(b^2 + d^2)}
\]

with equality if and only if \( B = D = \frac{\pi}{2} \) and \( ad = bc \). The third equality is equivalent to \( \frac{a}{c} = \frac{b}{d} \), which together with \( B = D \) yields that triangles \( ABC \) and \( CDA \) are similar. But these triangles have the side \( AC \) in common, so they are in fact congruent right triangles (since \( B = D = \frac{\pi}{2} \)). Then the angles at \( A \) and \( C \) in the quadrilateral must also be right angles, so \( ABCD \) is a rectangle. Conversely it is trivial, that in a rectangle \( B = D = \frac{\pi}{2} \) and \( ad = bc \). Hence there is equality in (2) if and only if the quadrilateral is a rectangle.

![Figure 2. Congruent right triangles ABC and CDA](image)

**Second proof.** A diagonal can divide a convex quadrilateral into two triangles in two different ways (see Figure 1). Adding these four triangle areas yields that the area \( K \) of the quadrilateral satisfies

\[
2K = \frac{1}{2}ab \sin B + \frac{1}{2}bc \sin C + \frac{1}{2}cd \sin D + \frac{1}{2}da \sin A
\]

\[
\leq \frac{1}{2}ab + \frac{1}{2}bc + \frac{1}{2}cd + \frac{1}{2}da = \frac{1}{2}(a + c)(b + d)
\]

where there is equality if and only if \( A = B = C = D = \frac{\pi}{2} \). Thus

\[
K \leq \frac{1}{2}(a + c)(b + d),
\]

which is a known inequality for the area of a quadrilateral (see [2, p.129]), with equality if and only if it is a rectangle.\(^1\) According to the AM-GM inequality,

\[
(a + c)^2 = a^2 + c^2 + 2ac \leq 2(a^2 + c^2)
\]

\(^1\)An interesting historical remark is that the formula \( K = \frac{a + c}{2} \cdot \frac{b + d}{2} \) (this is another area characterization of rectangles) was used by the ancient Egyptians to calculate the area of a quadrilateral, but it’s only a good approximation if the angles of the quadrilateral are close to being right angles. In all quadrilaterals but rectangles the formula gives an overestimate of the area, which the tax collectors probably didn’t mind.
with equality if and only if \( a = c \). Similarly, \((b + d)^2 \leq 2(b^2 + d^2)\). Using these two inequalities in (3), which we first rewrite, we get
\[
K \leq \frac{1}{4} \sqrt{(a+c)^2(b+d)^2} \\
\leq \frac{1}{4} \sqrt{2(a^2 + c^2) \cdot 2(b^2 + d^2)} = \frac{1}{2} \sqrt{(a^2 + c^2)(b^2 + d^2)}.
\]
There is equality if and only if \( a = c, b = d, \) and \( A = B = C = D = \frac{\pi}{2} \), that is, only when the quadrilateral is a rectangle.

\[
\begin{align*}
&\text{Figure 3. The Varignon parallelogram and the bimedians} \\
&\text{Third proof.} \quad \text{The area of a convex quadrilateral is twice the area of its Varignon parallelogram [3, p.53]. The diagonals in that parallelogram are the bimedians } m \text{ and } n \text{ in the quadrilateral, that is, the line segments connecting the midpoints of opposite sides (see Figure 3). Using that the area } K \text{ of a convex quadrilateral is given by one half the product of its diagonals and sine for the angle between the diagonals (this was proved in [5]), we have that} \\
&K = mn \sin \phi \quad \text{(4)} \\
&\text{where } \phi \text{ is the angle between the bimedians. In [7, p.19] we proved that the diagonals of a convex quadrilateral are congruent if and only if the bimedians are perpendicular. Hence the area of a convex quadrilateral is} \\
&K = mn \quad \text{(5)} \\
&\text{if and only if the diagonals are congruent (it is an equidiagonal quadrilateral). The length of the bimedians in a convex quadrilateral can be expressed in terms of two opposite sides and the distance } v \text{ between the midpoints of the diagonals as} \\
&m = \frac{1}{4} \sqrt{2(b^2 + d^2) - 4v^2}, \\
n = \frac{1}{4} \sqrt{2(a^2 + c^2) - 4v^2} \quad \text{(6)} \\
&\text{(see [6, p.162]). Using these expressions in (5), we have that the area of a convex quadrilateral is given by} \\
&K = \frac{1}{4} \sqrt{(2(a^2 + c^2) - 4v^2)(2(b^2 + d^2) - 4v^2)}
\end{align*}
\]
if and only if the diagonals are congruent. Now solving the equation

\[ \frac{1}{2} \sqrt{(a^2 + c^2)(b^2 + d^2)} = \frac{1}{4} \sqrt{(2(a^2 + c^2) - 4v^2)(2(b^2 + d^2) - 4v^2)} \]

yields \( 8v^2 = 0 \) or \( a^2 + b^2 + c^2 + d^2 = 2v^2 \). The second equality is not satisfied in any quadrilateral, since according to Euler’s extension of the parallelogram law, in all convex quadrilaterals

\[ a^2 + b^2 + c^2 + d^2 = p^2 + q^2 + 4v^2 > 2v^2 \]

where \( p \) and \( q \) are the lengths of the diagonals [1, p.126]. Thus we conclude that \( v = 0 \) is the only valid solution. Hence a convex quadrilateral has the area given by (1) if and only if the diagonals are congruent and bisect each other. A parallelogram, the quadrilateral characterized by bisecting diagonals \( v = 0 \), has congruent diagonals if and only if it is a rectangle.

**Fourth proof.** Combining equations (4) and (6) yields that the area of a convex quadrilateral with consecutive sides \( a, b, c, d \) is given by

\[ K = \frac{1}{4} \sqrt{(2(a^2 + c^2) - 4v^2)(2(b^2 + d^2) - 4v^2)} \sin \phi \]

where \( v \) is the the distance between the midpoints of the diagonals and \( \phi \) is the angle between the bimedians. Since parallelograms are characterized by \( v = 0 \), we have that the area is

\[ K = \frac{1}{2} \sqrt{(a^2 + c^2)(b^2 + d^2)} \sin \phi \]

if and only if the quadrilateral is a parallelogram. In a parallelogram, \( \phi \) is equal to one of the vertex angles since each bimedian is parallel to two opposite sides (see Figure 4). A parallelogram is a rectangle if and only if one of the vertex angles is a right angle. The equation \( \sin \phi = 1 \) only has one possible solution \( \phi = \frac{\pi}{2} \); hence we have that the area of a convex quadrilateral is given by (1) if and only if it is a rectangle.

---

Figure 4. The angle between the bimedians in a parallelogram
Fifth proof. A convex quadrilateral with consecutive sides \(a, b, c, d\) and diagonals \(p, q\) has the area [4, p.27]

\[
K = \frac{1}{4} \sqrt{4p^2q^2 - (a^2 - b^2 + c^2 - d^2)^2}.
\]

Now solving the equation

\[
\frac{1}{2} \sqrt{(a^2 + c^2)(b^2 + d^2)} = \frac{1}{4} \sqrt{4p^2q^2 - (a^2 - b^2 + c^2 - d^2)^2}
\]

we get

\[
(2pq)^2 - (a^2 + b^2 + c^2 + d^2)^2 = 0
\]

with only one positive solution \(a^2 + b^2 + c^2 + d^2 = 2pq\). Using again Euler’s extension of the parallelogram law

\[
a^2 + b^2 + c^2 + d^2 = p^2 + q^2 + 4v^2,
\]

where \(v\) is the distance between the midpoints of the diagonals \(p\) and \(q\), yields

\[
p^2 + q^2 + 4v^2 = 2pq \iff (2v)^2 = -(p - q)^2.
\]

Here the left hand side is never negative, whereas the right hand side is never positive. Thus for equality to hold, both sides must be zero. Hence \(v = 0\) and \(p = q\). This is equivalent to that the quadrilateral is a parallelogram with congruent diagonals, i.e., a rectangle.

References


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Characterizations of Trapezoids

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Abstract. We review eight and prove an additional 13 necessary and sufficient conditions for a convex quadrilateral to be a trapezoid. One aim for this paper is to show that many of the known properties of trapezoids are in fact characterizations.

1. Introduction

A trapezoid (in British English it is called a trapezium) is a quadrilateral with a pair of opposite parallel sides. But there is some disagreement if the definition shall state exactly one pair or at least one pair. The former is called an exclusive definition and the latter an inclusive definition. The exclusive seems to be common in textbooks at lower levels of education, whereas the inclusive is common among mathematicians and at higher levels of education (beyond high school) [10, p. xiii]. What is the reason for and significance of the two possible definitions?

One likely explanation for the exclusive definition is that when students first encounter shapes like a trapezoid or a rhombus, they could get confused if a rhombus also can be called a trapezoid. When proving properties of a trapezoid it is important to actually draw it with only one pair of opposite parallel sides, so the proof covers the general case. Here the exclusive definition has its merits. But when students are to progress in their mathematical education, the exclusive definition has some drawbacks.

First of all, the main strength of the inclusive definition is the fact that a property that is proved to hold for a trapezoid automatically also holds for all quadrilaterals with two pairs of opposite parallel sides, that is, for parallelograms, rhombi, rectangles, and squares. This is a major advantage, since then we do not have to repeat arguments for those classes. Other benefits are that the taxonomy for quadrilaterals is more perspicuous within the inclusive definition, and features like symmetry and duality becomes more prominent. Also, there is the trapezoid rule for calculating integrals. But these trapezoids do not always just have one pair of opposite parallel sides; sometimes they are in fact rectangles. That would make the name of the rule confusing if a rectangle was not considered to be a special case of a trapezoid. These are some of the reasons why mathematicians nowadays prefer the inclusive definition, that is, a trapezoid is a quadrilateral with at least one pair of opposite parallel sides.

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We claim that many geometry textbooks do not put much effort into summarizing even the most basic characterizations of trapezoids. The trapezoid is one of the six simplest types of quadrilaterals, so it is usually covered in books at lower levels of education. In those texts the authors often covers quite extensively methods for proving that a quadrilateral is one of the other five types: parallelograms, rhombi, rectangles, squares, and isosceles trapezoids.\footnote{But not the general trapezoid. Why is that? One reason could be that authors consider the topic already covered in connection with the treatment of parallel lines. But if so, then why not instead take this opportunity to connect that theory with quadrilaterals to show how it all fits together?} But not the general trapezoid. Why is that? One reason could be that authors consider the topic already covered in connection with the treatment of parallel lines. But if so, then why not instead take this opportunity to connect that theory with quadrilaterals to show how it all fits together?

Anyway, we will now summarize a handful of the simplest characterizations of trapezoids. These are the ones that rely only upon the theory of parallel lines or similarity. Then we shall prove a dozen of other characterizations, and in doing so we will demonstrate that most of the well known properties of trapezoids are in fact necessary and sufficient conditions for a quadrilateral to be a trapezoid.

First a comment on notations. The consecutive sides of a convex quadrilateral $ABCD$ will be denoted $a = AB$, $b = BC$, $c = CD$, and $d = DA$. In most of the characterizations we only consider the case when $a \parallel c$ and $a \geq c$. We trust the reader can then reformulate the characterizations in the other main case $b \parallel d$ using symmetry.

Figure 1. Two altitudes to the side $CD$

If the extensions of opposite sides $AB$ and $CD$ in a convex quadrilateral intersect at an angle $\xi$, then the quadrilateral is a trapezoid if and only if $\xi = 0$. A second characterization is that the quadrilateral $ABCD$ is a trapezoid with parallel sides $AB$ and $CD$ if and only if $\angle ABD = \angle CDB$, see Figure 1. An equivalent necessary and sufficient condition is that a convex quadrilateral is a trapezoid if and only if two pairs of adjacent angles are supplementary, that is

$$A + D = \pi = B + C.$$\hspace{1cm} (1)

\footnote{The kite must also be considered to be one of the basic quadrilaterals. Perhaps since there are only a few known characterizations of these, it usually don’t get that much attention. If we are to include the possibility of a tangential quadrilateral as well (i.e. one that has an incircle), then there are a further dozen of (less well known) characterizations of kites, see [6]. We note that Theorem 2 (ix) in that paper contained a misprint. It should state: The incenter lies on the diagonal that is a line of symmetry (which is not necessarily the longest one).}
Characterizations of trapezoids

From the theory of parallel lines we also have that the line segments \( AB \) and \( CD \) are the bases of a trapezoid \( ABCD \) if and only if the triangles \( ACD \) and \( BCD \) have equal altitudes to the common side \( CD \) (\( h_1 = h_2 \) in Figure 1).

Two characterizations concerning similarity are the following. A convex quadrilateral \( ABCD \) is a trapezoid if and only if the diagonals divide each other in the same ratio, that is

\[
\frac{AP}{CP} = \frac{BP}{DP},
\]

where \( P \) is the intersection of the diagonals. A closely related necessary and sufficient condition states that the diagonals divide a convex quadrilateral into four non-overlapping triangles, of which two opposite are similar if and only if the quadrilateral is a trapezoid (\( ABP \sim CDP \) in Figure 1).

2. Trigonometric characterizations

As part of the proof of Theorem 2 in [7] we have already proved two trigonometric characterizations of trapezoids, so we just restate them here. A convex quadrilateral \( ABCD \) is a trapezoid if and only if

\[
\sin A \sin C = \sin B \sin D.
\]

An equivalent necessary and sufficient condition is

\[
\cos (A - C) = \cos (B - D).
\]

In fact, both of these conditions incorporate the possibility for either pair of opposite sides to be parallel, not just \( a \parallel c \).

The first theorem and the subsequent proposition are trigonometric versions of the adjacent angle characterization (1).

**Theorem 1.** A convex quadrilateral \( ABCD \) is a trapezoid with parallel sides \( AB \) and \( CD \) if and only if

\[
\cos A + \cos D = \cos B + \cos C = 0.
\]

**Proof.** (\( \Rightarrow \)) If the quadrilateral is a trapezoid, then \( A + D = \pi \). Hence

\[
\cos A + \cos D = \cos A + \cos (\pi - A) = \cos A - \cos A = 0.
\]

The second equality is proved in the same way.

(\( \Leftarrow \)) We do an indirect proof of the converse. Assume the quadrilateral is not a trapezoid and without loss of generality that \( A > \pi - D \). Since \( 0 < A < \pi \) and the cosine function is decreasing on that interval, we get \( \cos A < \cos (\pi - D) \). Hence

\[
\cos A + \cos D < \cos (\pi - D) + \cos D = 0.
\]

From the sum of angles in a quadrilateral we also have that

\[
A > \pi - D \quad \Rightarrow \quad B < \pi - C \quad \Rightarrow \quad \cos B + \cos C > 0.
\]

So if the quadrilateral is not a trapezoid, then \( \cos A + \cos D \neq \cos B + \cos C \), and neither side is equal to 0. This completes the indirect proof. \( \Box \)
Proposition 2. A convex quadrilateral $ABCD$ is a trapezoid with parallel sides $AB$ and $CD$ if and only if
\[
\cot A + \cot D = \cot B + \cot C = 0.
\]

Proof. Since the cotangent function is decreasing on the interval $0 < x < \pi$ and $\cot (\pi - x) = -\cot x$, the proof is identical to that of Theorem 1.  

So far we have characterizations with sine, cosine and cotangent. Next we prove one for the tangents of the half angles.

Theorem 3. A convex quadrilateral $ABCD$ is a trapezoid with parallel sides $AB$ and $CD$ if and only if
\[
\tan \frac{A}{2} \tan \frac{D}{2} = \tan \frac{B}{2} \tan \frac{C}{2} = 1.
\]

Proof. (⇒) If the quadrilateral is a trapezoid, then $A + D = \pi = B + C$. Using these, the equalities in the theorem directly follows since $\tan \frac{B}{2} = \cot \frac{A}{2}$ and $\tan \frac{C}{2} = \cot \frac{B}{2}$.

(⇐) Assume the quadrilateral is not a trapezoid and without loss of generality that $A + D > \pi$ and $B + C < \pi$. From the addition formula for tangent, we get
\[
0 > \tan \left( \frac{A}{2} + \frac{D}{2} \right) = \frac{\tan \frac{A}{2} + \tan \frac{D}{2}}{1 - \tan \frac{A}{2} \tan \frac{D}{2}}.
\]

The angles $\frac{A}{2}$ and $\frac{C}{2}$ are acute, so the numerator is positive. Then the denominator must be negative, so $\tan \frac{A}{2} \tan \frac{D}{2} > 1$. In the same way $\tan \frac{B}{2} \tan \frac{C}{2} < 1$. Hence
\[
\tan \frac{A}{2} \tan \frac{D}{2} \neq \tan \frac{B}{2} \tan \frac{C}{2}
\]
and neither side is equal to 1.  

3. Characterizations concerning areas

The first proposition about areas concerns a bimedian, that is, a line segment that connects the midpoints of two opposite sides.

Proposition 4. A convex quadrilateral is a trapezoid if and only if one bimedian divide it into two quadrilaterals with equal areas.

Proof. (⇒) In a trapezoid, the bimedian between the bases (see the left half of Figure 2) divide it into two quadrilaterals with equal altitudes and two pairs of equal bases. Hence these two quadrilaterals, which are also trapezoids, have equal areas according to the well known formula for the area of a trapezoid.\(^2\)

(⇐) If $T_1 + T_2 = T_3 + T_4$ in a convex quadrilateral with notations as in the right half of Figure 2, then we have $T_1 = T_3$ since $T_2$ and $T_3$ are equal due to equal bases and equal altitudes. But $T_1$ and $T_3$ also have equal bases, so then their altitudes must be equal as well. This means that the quadrilateral is a trapezoid.  

\(^2\)The area of a trapezoid is the arithmetic mean of the bases times the altitude.
We will need the next proposition in the proofs of the following two characterizations. A different proof was given as the solution to Problem 4.14 in [9, pp.80, 89].

**Proposition 5.** If the diagonals in a convex quadrilateral $ABCD$ intersect at $P$, then it is a trapezoid with parallel sides $AB$ and $CD$ if and only if the areas of the triangles $APD$ and $BPC$ are equal.

**Proof.** We have that the sides $AB$ and $CD$ are parallel if and only if (see Figure 1)

$$h_{ACD} = h_{BCD} \iff T_{ACD} = T_{BCD} \iff T_{APD} = T_{BPC}$$

where $h_{XYZ}$ and $T_{XYZ}$ stands for the altitude and area of triangle $XYZ$ respectively. \hfill $\Box$

The following theorem was proved by us using trigonometry as Theorem 2 in [7]. Here we give a different proof using the previous characterization.

**Theorem 6.** A convex quadrilateral is a trapezoid if and only if the product of the areas of the triangles formed by one diagonal is equal to the product of the areas of the triangles formed by the other diagonal.

**Proof.** We use notations on the subtriangle areas as in Figure 3. Then we have

$$(S + U_1)(T + U_2) = (S + U_2)(T + U_1)$$

$$\iff SU_2 + TU_1 = SU_1 + TU_2$$

$$\iff S(U_2 - U_1) = T(U_2 - U_1)$$

$$\iff (S - T)(U_2 - U_1) = 0.$$ 

The last equality is equivalent to $S = T$ or $U_2 = U_1$, where either of these equalities is equivalent to that the quadrilateral is a trapezoid according to Proposition 5. \hfill $\Box$

In the proof of the next theorem we will use the following lemma about a property that all convex quadrilaterals have. Observe that the triangles in this lemma are not the same as the ones in Theorem 6.

**Lemma 7.** The diagonals of a convex quadrilateral divide it into four non-overlapping triangles. The product of the areas of two opposite triangles is equal to the product of the areas of the other two triangles.
Figure 3. The diagonal parts and subtriangle areas

Proof. We denote the diagonal parts by \( w, x, y, z \) and the consecutive subtriangle areas by \( S, U_1, T, U_2 \), see Figure 3. These areas satisfy

\[
ST = \frac{1}{4}wxyz \sin^2 \theta = U_1 U_2
\]

where \( \theta \) is the angle between the diagonals. \( \square \)

Our last characterization concerning areas is a beautiful formula for the area of a trapezoid. It can be proved using similarity as in [1, p.50]. We give a short proof establishing it to be both a necessary and sufficient condition.

**Theorem 8.** The diagonals of a convex quadrilateral divide it into four non-overlapping triangles. If two opposite of these have areas \( S \) and \( T \), then the quadrilateral has the area

\[
K = \left( \sqrt{S} + \sqrt{T} \right)^2
\]

if and only if it is a trapezoid whose parallel sides are the two sides in the triangles in question that are not parts of the diagonals.

Proof. A convex quadrilateral has the area (see Figure 3)

\[
K = S + T + U_1 + U_2
\]

\[
= S + T + 2\sqrt{ST} - 2\sqrt{U_1 U_2} + U_1 + U_2
\]

\[
= \left( \sqrt{S} + \sqrt{T} \right)^2 + \left( \sqrt{U_1} - \sqrt{U_2} \right)^2
\]

where we in the second equality used that \( ST = U_1 U_2 \) according to the Lemma. We have that the quadrilateral is a trapezoid if and only if \( U_1 = U_2 \) (by Proposition 5), so it is a trapezoid if and only if it has the area \( K = \left( \sqrt{S} + \sqrt{T} \right)^2 \). \( \square \)

As a corollary we note that the area \( K \) of a convex quadrilateral satisfies the inequality \(^4\)

\[
\sqrt{K} \geq \sqrt{S} + \sqrt{T},
\]

\(^3\)This equality can also be proved without trigonometry. If the altitudes in the two triangles on respective side of the diagonal \( w + y \) are \( h_1 \) and \( h_2 \), then we have that \( ST = \frac{1}{2}w y h_1 h_2 = U_1 U_2 \).

\(^4\)We have seen this inequality before, but we cannot recall a reference.
where there is equality if and only if the quadrilateral is a trapezoid.

Theorem 8 was formulated as if there could be only one pair of opposite parallel sides (a general trapezoid). If there are two pairs of opposite parallel sides, then the two triangles with areas $S$ and $T$ could be any one of the two pairs of opposite triangles formed by the diagonals.

4. Characterizations concerning sides and distances

The following simple characterization concerns the ratio of two opposite sides and the ratio of the sine of two adjacent angles.

**Proposition 9.** The convex quadrilateral $ABCD$ is a trapezoid with parallel sides $AB$ and $CD$ if and only if

$$\frac{DA}{BC} = \frac{\sin C}{\sin D}.$$  

**Proof.** The quadrilateral is a trapezoid if and only if the triangles $ACD$ and $BCD$ have equal altitudes to the side $CD$, which is equivalent to that the areas of these two triangles are equal. This in turn is equivalent to

$$\frac{1}{2}CD \cdot DA \sin D = \frac{1}{2}CD \cdot BC \sin C,$$

which is equivalent to the equality in the theorem. □

The parallelogram law states that in a parallelogram, the sum of the squares of the four sides equals the sum of the squares of the two diagonals. Euler generalized this to a convex quadrilateral with sides $a, b, c, d$ and diagonals $p, q$ as

$$a^2 + b^2 + c^2 + d^2 = p^2 + q^2 + 4v^2$$  \hspace{1cm} (3)

where $v$ is the distance between the midpoints of the diagonals. A proof can be found in [2, p.126]. We shall now derive another generalization of the parallelogram law, that will give us a characterization of trapezoids as a special case. This equality was stated in [4, p.249], but Dostor’s derivation was very scarce.

**Theorem 10.** If a convex quadrilateral has consecutive sides $a, b, c, d$ and diagonals $p, q$, then

$$p^2 + q^2 = b^2 + d^2 + 2ac \cos \xi$$

where $\xi$ is the angle between the extensions of the sides $a$ and $c$.

**Proof.** In a convex quadrilateral $ABCD$, let the extensions of $AB$ and $CD$ intersect at $J$. Other notations are as in Figure 4, where $AC = p$, $BD = q$, $AB = a$, and $AE = x$. We construct $GC$ parallel to $AB$. Then $\angle DCG = \angle BJC$. We also have $EF = GC = c \cos \xi$, $DG = c \sin \xi$, $ED = h - c \sin \xi$, and $FB = a - c \cos \xi - x$. 

Applying the Pythagorean theorem in triangles \( ACF, BDE, BCF, AED \), we get respectively

\[
p^2 = h^2 + (x + c \cos \xi)^2, \tag{4}
\]

\[
q^2 = (a - x)^2 + (h - c \sin \xi)^2, \tag{5}
\]

\[
b^2 = h^2 + (a - c \cos \xi - x)^2, \tag{6}
\]

\[
d^2 = x^2 + (h - c \sin \xi)^2. \tag{7}
\]

Expanding the parentheses and adding (4) and (5), we get

\[
p^2 + q^2 = 2(h^2 + x^2) + 2x(c \cos \xi - a) + a^2 + c^2 - 2hc \sin \xi. \tag{8}
\]

From (6) and (7),

\[
b^2 + d^2 = 2(h^2 + x^2) + a^2 + c^2 - 2hc \sin \xi - 2ac \cos \xi + 2x(c \cos \xi - a). \tag{9}
\]

Comparing (9) and (8), we see that

\[
b^2 + d^2 = p^2 + q^2 - 2ac \cos \xi
\]

and the equation in the theorem follows. \(\square\)

**Corollary 11.** A convex quadrilateral with consecutive sides \( a, b, c, d \) and diagonals \( p, q \) is a trapezoid with parallel sides \( a \) and \( c \) if and only if

\[
p^2 + q^2 = b^2 + d^2 + 2ac.
\]

**Proof:** This characterization is a direct consequence of Theorem 10, since the quadrilateral is a trapezoid if and only if \( \xi = 0 \). \(\square\)

The next two theorems concerns the distances between the midpoints of the diagonals and the midpoints of two opposite sides (a bimedian).

**Theorem 12.** A convex quadrilateral is a trapezoid with parallel sides \( a \) and \( c \) if and only if the distance \( v \) between the midpoints of the diagonals has the length

\[
v = \frac{|a - c|}{2}.
\]
Proof. Inserting the equation in Corollary 11 into (3), we get that a convex quadrilateral is a trapezoid if and only if
\[ a^2 + b^2 + c^2 + d^2 = b^2 + d^2 + 2ac + 4v^2 \iff (a - c)^2 = 4v^2. \]
Hence we get the characterization \( v = \frac{1}{2} |a - c| \).
\( \square \)

Remark. According to the formula, the diagonals bisect each other \( (v = 0) \) if and only if \( a = c \). In this case the quadrilateral is a parallelogram, which is a special case of a trapezoid within the inclusive definition.

**Theorem 13.** A convex quadrilateral with consecutive sides \( a, b, c, d \) is a trapezoid with parallel sides \( a \) and \( c \) if and only if the bimedian \( n \) that connects the midpoints of the sides \( b \) and \( d \) has the length
\[ n = \frac{a + c}{2}. \]

Proof. The length of the bimedian \( n \) that connects the midpoints of the sides \( b \) and \( d \) in a convex quadrilateral is given by
\[ 4n^2 = p^2 + q^2 + a^2 - b^2 + c^2 - d^2 \]
according to [3, p.231] and post no 2 at [5] (both with other notations). Substituting \( p^2 + q^2 \) from Corollary 11, we get that a convex quadrilateral is a trapezoid if and only if
\[ 4n^2 = b^2 + d^2 + 2ac + a^2 - b^2 + c^2 - d^2 \iff 4n^2 = (a + c)^2. \]
Hence \( n = \frac{1}{2} (a + c) \).
\( \square \)

The last characterization on sides and distances is about formulas for the length of the diagonals.

**Theorem 14.** A convex quadrilateral \( ABCD \) with consecutive sides \( a, b, c, d \) is a trapezoid with \( a \parallel c \) and \( a \neq c \) if and only if the length of the diagonals \( AC \) and \( BD \) are respectively
\[ p = \sqrt{\frac{ac(a - c) + ad^2 - cb^2}{a - c}}, \]
\[ q = \sqrt{\frac{ac(a - c) + ab^2 - cd^2}{a - c}}. \]

Proof. We prove the second formula first. Using the law of cosines in the two triangles formed by diagonal \( BD = q \) in a convex quadrilateral, we have \( d^2 = a^2 + q^2 - 2aq \cos u \) and \( b^2 = c^2 + q^2 - 2cq \cos v \) (see Figure 5). Thus
\[ \cos u = \frac{a^2 + q^2 - d^2}{2aq} \]
and
\[ \cos v = \frac{c^2 + q^2 - b^2}{2cq}. \]
Figure 5. A diagonal and two alternate angles

The quadrilateral is a trapezoid with \( a \parallel c \) if and only if \( u = v \), which is equivalent to \( \cos u = \cos v \). This in turn is equivalent to

\[
\frac{a^2 + q^2 - d^2}{2aq} = \frac{c^2 + q^2 - b^2}{2cq}
\]

which we can rewrite as

\[
ac(a - c) + ab^2 - cd^2 = (a - c)q^2.
\]

Now if \( a \neq c \), the second formula follows.

The first formula can be proved in the same way, or we can use symmetry and need only to make the change \( b \leftrightarrow d \) in the formula we just proved. \( \square \)

Remark. The quadrilateral is a trapezoid with \( a \parallel c \) and \( a = c \) if and only if it is a parallelogram. In that case the sides alone do not uniquely determine neither the quadrilateral nor the length of the diagonals.

5. A collinearity characterization

The following theorem has been stated as a collinearity, but another possibility is to state it as a concurrency: a convex quadrilateral is a trapezoid if and only if the two diagonals and one bimedian are concurrent, in which case the two sides that the bimedian connects are parallel. The proof of the converse is cited from [11].

Theorem 15. Two opposite sides in a convex quadrilateral are parallel if and only if the midpoints of those sides and the intersection of the diagonals are three collinear points.

Proof. (⇒) In a trapezoid, let \( E \) and \( G \) be the midpoints of the sides \( AB \) and \( CD \) respectively, and \( P \) the intersection of the diagonals. Triangles \( CDP \) and \( ABP \) are similar due to two pairs of equal angles (see Figure 6). Note that \( PG \) and \( PE \) are medians in those triangles, but we do not yet know that \( \angle DPG = \angle BPE \).

This is what we shall prove. From the similarity, we get

\[
\frac{PD}{PB} = \frac{CD}{AB} = \frac{2GD}{2EB} = \frac{GD}{EB}.
\]
Figure 6. Are $E$, $P$ and $G$ collinear?

Also $\angle PDG = \angle PBE$, so triangles $PDG$ and $PBE$ are similar. Hence $\angle DPG = \angle BPE$, and since $BPD$ is a straight line, then so is $EPG$.

Figure 7. Is $ABCD$ a trapezoid?

$(\Leftarrow)$ In a convex quadrilateral $ABCD$ where $E$ and $G$ are the midpoints of $AB$ and $CD$ and $P$ is the intersection of the diagonals, we know that $E$, $P$ and $G$ are collinear. We shall prove that $AB$ and $CD$ are parallel. Extend $AD$ and $EG$ to intersect at $Q$ (see Figure 7). We apply Menelaus’ theorem to triangles $ABD$ and $ACD$ using the transversal $EPGQ$. Then

$$\frac{AE}{EB} \cdot \frac{BP}{PD} \cdot \frac{DQ}{QA} = 1 \quad (10)$$

and

$$\frac{AP}{PC} \cdot \frac{CG}{GD} \cdot \frac{DQ}{QA} = 1. \quad (11)$$

Since $AE = EB$ and $CG = GD$, equations (10) and (11) yields

$$\frac{BP}{PD} = \frac{AP}{PC}.$$

This equality states that the diagonals divide each other in the same ratio, which is the well known sufficient condition (2) for the sides $AB$ and $CD$ to be parallel. \(\square\)
6. Can all convex quadrilaterals be folded into a trapezoid?

A convex quadrilateral is not uniquely determined by its sides alone. This means that there can be different types of quadrilaterals having the same consecutive sides.\(^5\) Let us make a model of a convex quadrilateral as four very thin rods connected by hinges at their endpoints. We assume that the length of any rod is shorter than the sum of the other three, which ensures that the rods can be the sides of a convex quadrilateral. What we shall explore is if it's always possible to fold the model into a trapezoid?

![Figure 8. AECD is a parallelogram](image)

In a general trapezoid where \(a \parallel c\) and \(a \neq c\), we construct the triangle \(BCE\) in Figure 8 such that \(CE \parallel DA\). This triangle exists whenever its sides satisfy the three triangle inequalities \(a - c < b + d\), \(d < a - c + b\), and \(b < a - c + d\). The first of these is always satisfied if the quadrilateral exists. The second and third can be merged into

\[
|a - c| > |b - d|
\]

which is a necessary condition for \(a \parallel c\) when \(a \neq c\). But it is also a sufficient condition, since if it is satisfied, it is possible to construct the triangle \(BCE\) and then the trapezoid. In the same way, we have that

\[
|a - c| < |b - d|
\]

is a necessary and sufficient condition for \(b \parallel d\) when \(b \neq d\). Thus the only case when we can’t fold a convex quadrilateral into a trapezoid is when

\[
|a - c| = |b - d|
\]

This is the characterization for when the quadrilateral has an excircle (so it is an extangential quadrilateral) according to [8, p.64].

Let us examine why we can’t get a trapezoid in this case. We use the “semi factored” version of Heron’s formula for the area \(T\) of a triangle to get a formula for the altitude in a trapezoid. In a triangle with sides \(x, y, z\), the altitude \(h\) to the side \(x\) has the length

\[
h = \frac{2T}{x} = \sqrt{\frac{(y + z)^2 - x^2}{2x}} \frac{(x^2 - (y - z)^2)}{2x}
\]

\(^5\)For instance, a quadrilateral having the sides \(a, b, a, b\) can be either a (general) parallelogram or a rectangle.
The triangle $BCE$ has the same altitude as the trapezoid. Inserting $x = a - c$, $y = b$, and $z = d$ yields the trapezoid altitude

$$h = \frac{\sqrt{(b + d)^2 - (a - c)^2} \left( (a - c)^2 - (b - d)^2 \right)}{2|a - c|}$$

which is valid when $a \neq c$. Here we see that when $|a - c| = |b - d|$, the trapezoid altitude is zero. This means that the trapezoid has collapsed into a line segment. If we don’t consider that degenerate case to be a trapezoid, this is the reason why an extangential quadrilateral (with a finite exradius) can never be folded into a trapezoid.

Finally we have the case $|a - c| = |b - d| = 0$. Then both pairs of opposite sides have equal length, so the quadrilateral is a parallelogram. This is already a trapezoid (within the inclusive definition), so no folding is needed.

We conclude by stating the conclusions above in the following theorem.

**Theorem 16.** If four line segments $a, b, c, d$ have the property that any one of them is shorter than the sum of the other three, then they can always constitute the consecutive sides of a non-degenerate trapezoid except when $|a - c| = |b - d| \neq 0$. In that case they will be the sides of an extangential quadrilateral.

**References**


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6Note that parallelograms can be considered to be extangential quadrilateral with infinite exradius, see [8, p.76].
The Most Inaccessible Point of a Convex Domain

María Calvo and Vicente Muñoz

Abstract. The inaccessibility of a point $p$ in a bounded domain $D \subset \mathbb{R}^n$ is the minimum of the lengths of segments through $p$ with boundary at $\partial D$. The points of maximum inaccessibility $I_D$ are those where the inaccessibility achieves its maximum. We prove that for strictly convex domains, $I_D$ is either a point or a segment, and that for a planar polygon $I_D$ is in general a point. We study the case of a triangle, showing that this point is not any of the classical notable points.

1. Introduction

The story of this paper starts when the second author was staring at some workers spreading cement over the floor of a square place to construct a new floor over the existing one. The procedure was the following: first they divided the area into triangular areas (actually quite irregular triangles, of around 50 square meters of area). They put bricks all along the sides of the triangles and then poured the liquid cement in the interior. To make the floor flat, they took a big rod of metal, and putting it over the bricks on two of the sides, they moved the rod to flatten the cement. Of course, they had to be careful as they were reaching the most inner part of the triangle.

The question that arose in this situation is: What is the minimum size for the rod? Even more, which is the most inaccessible point, i.e. the one that requires the full length of the rod? Is it a notable point of the triangle?

The purpose of this paper is to introduce the concept of maximum inaccessibility for a domain. This is done in full generality for a bounded domain in $\mathbb{R}^n$. The inaccessibility function $r$ assigns to a point of the domain $D$ the minimum length of a segment through it with boundary in $\partial D$. We introduce the sets $D_r = \{x \mid r(x) > r\}$ and the most inaccessible set $I_D$ given by the points where the inaccessibility function achieves its maximum value (the notion has to be suitable modified for the case where $r$ only has suppremum).

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Then we restrict to convex domains to prove convexity properties of the sets $D_r$ and $I_D$. For strictly convex domains, $I_D$ is either a point or a segment. For planar convex domains not containing pairs of regular points with parallel tangent lines (e.g. polygons without parallel sides), $I_D$ is a point. In some sense, domains for which $I_D$ is not a point are of very special nature. When $I_D = \{ p_D \}$ is a point, we call $p_D$ the point of maximum inaccessibility of $D$.

In the final section, we shall study in detail the case of a polygonal domain in the plane, and more specifically the case of a triangle, going back to the original problem. One of the results is that the point $p_T$, for a triangle $T$, is not a notable point of $T$. It would be nice to determine explicitly this point in terms of the coordinates of the vertices. We do it in the case of an isosceles triangle.

### 2. Accessibility for domains

Let $D \subset \mathbb{R}^n$ be a bounded domain, that is an open subset such that $\overline{D}$ is compact. Clearly also $\partial D$ is compact. For a point $p \in D$, we consider the function:

$$f_p : S^{n-1} \to \mathbb{R}_+,$$

which assigns to every unit vector $v$ the length $l(\gamma)$ of the segment $\gamma$ given as the connected component of $(p + \mathbb{R}v) \cap D$ containing $p$.

**Lemma 1.** $f_p$ is lower-semicontinuous, hence it achieves its minimum.

**Proof.** Let us introduce some notation: for $p \in D$ and $v \in S^{n-1}$, we denote $\gamma_{p,v}$ the connected component of $(p + \mathbb{R}v) \cap D$ containing $p$. (So $\gamma_{p,v} = [P, Q]$ for some $P, Q \in \partial D$.) Now define the function

$$H : D \times S^{n-1} \to \mathbb{R}_+,$$

by $H(p, v) = f_p(v)$. Let us see that $H$ is lower-semicontinuous (see [3] for general definitions of continuity). Suppose that $(p_n, v_n) \to (p, v)$. Let $\gamma_{p_n,v_n} = [P_n, Q_n]$, where $P_n, Q_n \in \partial D$. As $\partial D$ is compact, then there are convergent subsequences (which we denote as the original sequence), $P_n \to P$, $Q_n \to Q$. Clearly $P, Q \in \partial D$. Let $\gamma$ be the open segment with $\overline{\gamma} = [P, Q]$. Then $p \in \gamma \subset (p + \mathbb{R}v)$. So $\gamma_{p,v} \subset \gamma$ and

$$H(p_n, v_n) = l(\gamma_{p_n,v_n}) = ||P_n - Q_n|| \to ||P - Q|| = l(\gamma) \geq l(\gamma_{p,v}) = H(p, v).$$

Clearly, $f_p(v) = H(p, v)$, obtained by freezing $p$, is also lower-semicontinuous. \qed

**Remark.** In Lemma 1, if $D$ is moreover convex, then $H$ is continuous. This follows from the observation that a closed segment $\sigma = [P, Q]$ with endpoints $P, Q \in \partial D$ either is fully contained in $\partial D$ or $\sigma \cap \partial D = \{ P, Q \}$. The segment $\gamma$ in the proof of Lemma 1 has endpoints in $\partial D$ and goes through $p$, therefore it coincides with $\gamma_{p,v}$. So $H(p_n, v_n) \to H(p, v)$, proving the continuity of $H$.

We say that a point $p \in D$ is $r$-accessible if there is a segment of length at most $r$ with boundary at $\partial D$ and containing $p$. Equivalently, let

$$r(p) = \min_{v \in S^{n-1}} f_p(v),$$
The most inaccessible point of a convex domain

which is called accessibility of $p$. Then $p$ is $r$-accessible if $r(p) \leq r$. Extend $r$ to $\overline{D}$ by setting $r(p) = 0$ for $p \in \partial D$.

**Proposition 2.** The function $r : \overline{D} \to \mathbb{R}_{\geq 0}$ is lower-semicontinuous.

**Proof.** We first study the function $r : D \to \mathbb{R}_+$. As $r(p) = \min_v H(p,v)$, the lower-semicontinuity of $H$ gives the lower-semicontinuity of $r$: If $p_n \to p$, take $v_n$ such that $r(p_n) = H(p_n, v_n)$. After taking a subsequence, we can assume that $(p_n, v_n) \to (p, v)$. So
\[
\lim r(p_n) = \lim H(p_n, v_n) \geq H(p, v) \geq r(p),
\]

as required.

Finally, as we define $r(p) = 0$ if $p \in \partial D$, those points give no problem to lower-semicontinuity. □

We have some easy examples where $f_p$ or $r$ are not continuous. For instance, if we consider the domain
\[
D = \{(x, y) | x^2 + y^2 < 1, x \leq 0\} \cup \{(x, y) | x^2 + y^2 < 4, x > 0\},
\]

and let $p = (0,0)$. Then $f_p : S^1 \to \mathbb{R}^+$ has constant value 3 except at the horizontal vectors where it has value 2. Also $r$ is not continuous, since $r(p) = 2$, but $r((\epsilon, 0)) \approx 3$, for $\epsilon > 0$ small.

**Remark.** If $D$ is convex, then $r : D \to \mathbb{R}_+$ is continuous. Let $p_n \to p$. Take $w$ so that $H(p, w) = r(p)$. Then $r(p) = H(p, w) = \lim H(p_n, w) \geq \lim r(p_n)$, using the continuity of $H$ and $H(p_n, w) \geq r(p_n)$. So $r$ is upper-semicontinuous, and hence continuous.

The function $r : \overline{D} \to \mathbb{R}_{\geq 0}$ may not be continuous, even for convex domains. Take a semicircle $\{(x, y) | x^2 + y^2 < 1, x > 0\}$. Then $r((\epsilon, 0)) = 1$, for $x > 0$ small, but $r((0,0)) = 0$.

We introduce the sets:
\[
D_r = \{p \in D | r(p) > r\},
E_r = \{p \in D | r(p) \geq r\}.
\]

$D_r$ is open by Proposition 2, and $E_r$ is compact. The function $r$ is clearly bounded, so it has a suppremum.

**Definition.** We call $R = \sup r$ the inaccessibility of $D$. We call
\[
I_D := \bigcap_{r<R} E_r
\]
the set of points of maximum inaccessibility of $D$.

The set $I_D$ may intersect the boundary of $D$. For instance, $D = \{(x, y) | x^2 + y^2 < 1, x > 0\}$. Then $R = 1$. It can be seen that $I_D = E_R = \{(x,0) | 0 \leq x \leq \sqrt{3}/2\}$.
Moreover, $I_D$ can be a point of the boundary. Take $D = \{(x, y) | \frac{x^2}{4} + y^2 < 1\} - \{(x, 0) | x \leq 0\}$. Then $R = 2$, and $I_D = \{(0, 0)\}$. The sets $D_r$, for $1 < r < 2$ are petals with vertex at the origin.

![Figure 1. The sets $D_r$ for the ellipse with a long axis removed. The set $I_D$ is in the boundary](image)

Note that $r$ does not achieve the maximum is equivalent to $I_D \subset \partial D$. This does not happen for convex $D$, as will be seen in the next section.

3. Convex domains

From now on, we shall suppose that $D$ is a convex bounded domain (see [4] for general results on convex sets). This means that if $x, y \in D$, then the segment $[x, y]$ is completely included in $D$. There are several easy facts: $\overline{D}$ is a compact convex set, the interior of $D$ is $\mathring{D}$, and $D$ is the convex hull of $\partial D$.

There is an alternative characterization for convex sets. Let $v$ be a unit vector in $\mathbb{R}^n$. Then the function $f(x) = \langle x, v \rangle$ achieves its maximum in $\partial D$, say $c$. Then $f(x) \leq c$ for $x \in D$. Consider the half-space

$$H_v^- = \{x \in \mathbb{R}^n | f(x) < c\}.$$ 

Then $D \subset H_v^-$. We call

$$H_v = \{x \in \mathbb{R}^n | f(x) = c\}$$

a supporting hyperplane for $D$ (see [2, p. 129]). Note that $\partial D \cap H_v \neq \emptyset$. Let also $H_v^+ = \{x \in \mathbb{R}^n | f(x) > c\}$.

**Lemma 3.** The convex set $D$ is the intersection

$$\bigcap_{|v|=1} H_v^-,$$

and conversely, any such intersection is a convex set. Moreover,

$$\overline{D} = \bigcap_{|v|=1} \overline{H_v^-}.$$
Proof. The second assertion is clear, since the intersection of convex sets is convex.

For the first assertion, we have the trivial inclusion $D \subset \bigcap_{|v|=1} H_v$. Now suppose $p \notin D$. We have two cases:

- $p \notin \overline{D}$. Then take $q \in \overline{D}$ such that $d(p, q)$ achieves its minimum, say $s > 0$. Let $v$ be the unit vector from $p$ to $q$. Let $H_v$ be the hyperplane through $q$ determined by $v$. It is enough to see that the half-space $H_v^+$ is disjoint from $D$, since $p \in H_v^+$. Suppose that $x \in D \cap H_v^+$. Then the segment from $q$ to $x$ should be entirely included in $\overline{D}$, but it intersects the interior of the ball of centre $p$ and radius $s$. This contradicts the choice of $q$.

- $p \in \partial D$. Consider $p_n \to p$, $p_n \notin \overline{D}$. By the above, there are $q_n \in \partial D$ and vectors $v_n$, such that $D \subset H_{v_n}^- = \{ (x - q_n, v_n) < 0 \}$. We take subsequences so that $q_n \to q \in \partial D$ and $v_n \to v$. So $D \subset \overline{H}_v^- = \{ (x - q, v) \leq 0 \}$. But $D$ is open, so $D \subset H_v^-$. Moreover, as $d(p_n, D) \to 0$, then $d(p_n, q_n) \to 0$, so $p = q$, and the hyperplane determining $H_v$ goes through $p$, so $p \notin H_v^-$ (actually $p \in H_v$).

Remark. The proof of Lemma 3 shows that if $p \in \partial D$, then there is a supporting hyperplane $H_v$ through $p$. We call it a supporting hyperplane at $p$, and we call $v$ a supporting vector at $p$. When a point $p$ has several supporting hyperplanes, it is called a corner point. The set

$$\mathbb{R}_+ : \{ v | v \text{ is supporting vector at } p \} \subset \mathbb{R}^n$$

is convex. Note that if $\partial D$ is piecewise smooth, and $p \in \partial D$ is a smooth point, then $p$ is non-corner and the tangent space to $\partial D$ is the supporting hyperplane.

Now we want to study the sets $D_r$ and $E_r$. First note that $r$ is continuous on $D$. Therefore

$$E_r \cap D = \{ x | r(x) \geq r \}$$

is closed on $D$.

**Proposition 4.** If $D$ is convex, then $r$ achieves its supremum $R$ at $D$. Moreover, $I_D \cap D = E_r \cap D = \{ p | r(p) = R \}$ and $I_D = E_R$ (which is the closure of $E_R \cap D$).

Proof. Let $p \in I_D \cap \partial D$, and take a supporting hyperplane $H_v$ at $p$. We claim that the open semiball $B_R(p) \cap H_v^- \subset D$. If not, then there is a point $q \in \partial D$, $d(p, q) < R$, $q \in H_v^-$. Then all the segments $[q, x]$, with $x \in B_\epsilon(p) \cap \partial D$, have length $\leq r_0 < (\epsilon$ for suitable small $\epsilon$). Therefore, there is neighbourhood $U$ of $p$ such that $r(x) \leq r_0$, $\forall x \in U \cap D$. Contradiction.

Now all points in the ray $p + tv$, $t \in (0, \epsilon)$ are not $r$-accessible for any $r < R$. Therefore they belong to $r^{-1}(R)$. So $p$ is in the closure of $r^{-1}(R)$, which is $E_R$.

Therefore $I_D \cap \partial D \subset E_R$. Also, by continuity of $r$ on $D$, we have that $E_r \cap D = r^{-1}(R, \infty)$, thus $I_D \cap D = \bigcap_{r < R} r^{-1}(R, \infty) = r^{-1}(R) = E_R \cap D$. All together, $I_D \subset E_R$. Obviously, $E_R \cap D = r^{-1}(R) \subset I_D$, and taking closures,
\( E_R = \overline{E_R \cap D} \subset I_D \). So \( I_D = E_R \). Finally, as \( I_D \) is always non-empty, we have that \( R \) is achieved by \( r \). \( \square \)

Now we prove a useful result. Given two points \( P, Q \), we denote \( \overrightarrow{PQ} = Q - P \) the vector from \( P \) to \( Q \).

**Lemma 5.** Let \( p \in D \) and \( r = r(p) \). Let \([P, Q]\) be a segment of length \( r \) with \( P, Q \in \partial D \) and \( p \in [P, Q] \). Let \( v_P, v_Q \) be supporting vectors at \( P, Q \) respectively. Then

1. If \( v_P, v_Q \) are parallel, then: \( v_P = -v_Q, P, Q \) are non-corner points, \( \overrightarrow{PQ} \parallel v_P \), and \( r = R \).
2. If \( v_P, v_Q \) are not parallel, then: \( \overrightarrow{PQ} \) is in the plane \( \pi \) spanned by them, there is a unit vector \( v \perp \overrightarrow{PQ}, v \in \pi \), such that for \( H_v = \{ \langle x - p, v \rangle = 0 \} \), it is \( E_r \subset \overrightarrow{PQ} \); and \( r(x) < r \) for \( x \in [P, Q] - \{p\} \) close to \( p \).

**Proof.** (1) Suppose first that \( v_P, v_Q \) are parallel. So \( D \) is inside the region between the parallel hyperplanes \( H_{v_P} \) and \( H_{v_Q} \). Clearly \( v_P = -v_Q \). Let \( x \in D \), and draw the segment parallel to \([P, Q]\) through \( x \) with endpoints in the hyperplanes. It has length \( r \). The intersection of this segment with \( D \) is of length \( \leq r \). Therefore \( r(x) \leq r \), for all \( x \in D \), so \( R = r \).

If \( \overrightarrow{PQ} \) is not parallel to \( v_P \), take a small vector \( w \) such that \( \langle w, v_P \rangle = 0 \), \( \langle w, \overrightarrow{PQ} \rangle > 0 \). Let \( t \in (0, 1) \) so that \( p = (1 - t)P + tQ \). Then \( P' = P + tw \in H_{v_P} \) and \( Q' = Q - (1 - t)w \in H_{v_Q} \), and \( p \in [P', Q'] \). First, \( ||\overrightarrow{PQ}'|| = ||\overrightarrow{PQ} - w|| < ||\overrightarrow{PQ}|| = r \). Also \( P', Q' \notin D \), so the segment \([P', Q'] \cap D \) is of length at most \( ||\overrightarrow{PQ}'|| \). Therefore \( r(p) < r \), a contradiction.

The assertion that \( P, Q \) are non-corner points is proved below.

(2) Suppose now that \( v_P, v_Q \) are not parallel. Again \( D \) is inside the region between the hyperplanes \( H_{v_P} \) and \( H_{v_Q} \). Let \( \pi \) be the plane spanned by \( v_P, v_Q \).

Let \( w \) be the projection of \( \overrightarrow{PQ} \) on the orthogonal complement to \( \pi \), and suppose \( w \neq 0 \). Clearly \( \langle w, \overrightarrow{PQ} \rangle > 0 \). Let \( t \in (0, 1) \) so that \( p = (1 - t)P + tQ \). Then \( P' = P + tw \in H_{v_P} \) and \( Q' = Q - (1 - t)w \in H_{v_Q} \), and \( p \in [P', Q'] \). So \( l([P', Q'] \cap D) \leq ||\overrightarrow{PQ}'|| < ||\overrightarrow{PQ}|| = r \), which is a contradiction. Therefore \( \overrightarrow{PQ} \in \pi \).

Let \( v \in \pi \) be a unit vector such that \( v \perp \overrightarrow{PQ} \). Now consider unit vectors \( e_1, e_2 \) in \( \pi \) so that \( e_1 \perp v, e_2 \perp v \). The vector

\[
  u = \frac{1}{\langle e_1, v \rangle \langle e_2, v \rangle} (\langle e_1, v \rangle e_2 - \langle e_2, v \rangle e_1)
\]

is perpendicular to \( v \), hence parallel to \( \overrightarrow{PQ} \). We arrange that \( \langle u, \overrightarrow{PQ} \rangle < 0 \) by changing the sign of \( v \) if necessary. Denote \( H_v = \{ \langle x - p, v \rangle = 0 \} \). Let us see that this satisfies the statement. Consider \( w \) so that \( \langle w, v \rangle > 0 \). Let \( w_1 = \frac{\langle w, v \rangle}{\langle e_1, v \rangle} e_1 \in \mathbb{R}^n \).
$H_{v_1}$ and $w_2 = \frac{\langle w, v \rangle}{\langle e_2, v \rangle} e_2 \in H_{v_Q}$. Then

$$w_2 - w_1 = \frac{\langle w, v \rangle}{\langle e_1, v \rangle} ((e_1, v) e_2 - (e_2, v) e_1) = \langle w, v \rangle u,$$

so $\langle w_2 - w_1, \overrightarrow{PQ} \rangle = \langle w, v \rangle \langle u, \overrightarrow{PQ} \rangle < 0$. Set $P' = P + w_1$, $Q' = Q + w_2$. So $[P', Q']$ is parallel to $[P, Q]$, it goes through $p + w$, and it is shorter than $[P, Q]$. So $H_v^+ \cap E_r = \emptyset$.

For the last assertion, we write $\overrightarrow{PQ} = a_1 e_1 + a_2 e_2$, where $a_1, a_2 \neq 0$. Let $P' = P + x e_1 \in H_{v_1}$, $Q' = Q + y e_2 \in H_{v_Q}$. The condition $p \in [P', Q']$ is equivalent to $p, P', Q'$ being aligned, which is rewritten as

$$x y + (1 - t) a_2 x - t a_1 y = 0.$$ (1)

Now, the condition $||\overrightarrow{PQ}|| = ||\overrightarrow{PQ} + y e_2 - x e_1|| < ||\overrightarrow{PQ}|| = r$ for small $r$ is achieved if $\langle \overrightarrow{PQ}, ye_2 - xe_1 \rangle < 0$. This is a linear equation of the form $\alpha_1 x + \alpha_2 y < 0$. The intersection of such half-plane with the hyperbola (1) is non-empty except if $\alpha_1 x + \alpha_2 y = 0$ is tangent to the hyperbola at the origin. So $(\alpha_1, \alpha_2)$ is a multiple of $((1 - t)a_2, -ta_1)$. This determines $t$ uniquely. So for $s \neq t$ (and close to $t$), we have that $p_s = (1 - s)P + sQ$ satisfies $r(p_s) < r$. (Note incidentally, that it cannot be $\overrightarrow{PQ} \parallel v_P$. If so, then $\alpha_1 = 0$, and then $(1 - t)a_2 = 0$, so $t = 1$, which is not possible.)

Now we finish the proof of (1). Suppose that $Q$ is a corner point. Then we can choose another supporting vector $v_Q'$. On the one hand $\overrightarrow{PQ} \parallel v_P$. On the other, as $v_P \not\parallel v_Q'$, we must have $\overrightarrow{PQ} \not\parallel v_P$, by the discussion above. Contradiction.

Theorem 6. The sets $D_r, E_r$ are convex sets, for $r \in [0, R]$, $R = \max r$. Moreover, $\partial D_r \cap D$ is $r^{-1}(r)$. for $r \in (0, R)$.

Proof. The assertion for $E_r$ follows from that of $D_r$: knowing that $D_r$ is convex, then

$$E_r = \bigcap_{\epsilon > 0} D_{r-\epsilon}$$

is convex since the intersection of convex sets is convex, and the closure of a convex set is convex.

Let $0 < r < R$, and let us see that $D_r$ is convex. Let $p \notin D_r$. Then $r(p) \leq r$. By Lemma 5, there is a segment $[P, Q]$ of length $r$, with $P, Q \in \partial D$, $v_P \not\parallel v_Q$, and a vector $v \perp \overrightarrow{PQ}$ such that $E_r \subset \overrightarrow{H_v^-}$. Then $D_r \subset H_v^-$, and $p \notin H_v^-$. So $D_r$ is the intersection of half-spaces, hence convex.

For the last assertion, note that the continuity of $r$ implies that $D \cap \partial D_r \subset r^{-1}(r)$. For the reversed inclusion, suppose that $r(p) = r$, but $p \notin \partial D_r$. Then there is some $\epsilon > 0$ so that $B_\epsilon(p) \subset r^{-1}(0, r]$. Now $r^{-1}[r, \infty)$ is convex, so it is the closure of its interior, call it $V$. Therefore $V \cap B_\epsilon(p)$ is open, convex, and contains $p$ in its adherence. Moreover $V \cap B_\epsilon(p) \subset r^{-1}(r)$. But this is
impossible, since an easy consequence of Lemma 5 is that \( r^{-1}(r) \) has no interior for any \( r \in (0, R) \).

**Proposition 7.** Suppose \( D \) is a convex planar set. Let \( r \in (0, R) \). Then \( \partial D_r \) is the envelope of the segments of length \( r \) with endpoints at \( \partial D \).

**Proof.** As we proved before, the boundary of \( D_r \) is \( r^{-1}(r) \), so the points of \( \partial D_r \) are \( r \)-accessible, but not \( r' \)-accessible for \( r' < r \). Let \( p \in \partial D_r \) be a smooth point. Then there is a segment of length \( r \) and \( D_r \) is at one side of it. Therefore the segment is tangent to \( \partial D_r \) at \( p \).

\[ \square \]

4. Strictly convex domains

Recall that \( D \) is strictly convex if there is no segment included in its boundary. We assume that \( D \) is strictly convex in this section. Therefore, for each unit vector \( v \), there is a unique point of contact \( H_v \cap \partial D \). We define the function

\[ g : S^{n-1} \to \partial D . \]

**Lemma 8.** If \( D \) is strictly convex, then \( g \) is continuous.

**Proof.** Let \( v_n \in S^{n-1} \), \( v_n \to v \). Consider \( p_n = g(v_n) \in \partial D \), and the supporting hyperplane \( \langle x - p_n, v_n \rangle \leq 0 \). Let \( p = g(v) \), with supporting hyperplane \( \langle x - p, v \rangle \leq 0 \). After taking a subsequence, we can suppose \( p_n \to q \in \partial D \). Now \( p \in \overline{D} \implies \langle p - p_n, v_n \rangle \leq 0 \), and taking limits, \( \langle p - q, v \rangle \leq 0 \). On the other hand, \( p_n \in \overline{D} \implies \langle p_n - p, v \rangle \leq 0 \), and taking limits, \( \langle q - p, v \rangle \leq 0 \). So \( \langle q - p, v \rangle = 0 \). By strict convexity, \( q = p \), so \( g(v_n) \to g(v) \), and \( g \) is continuous.

Now suppose that \( \partial D \) is \( C^1 \). Then for each point \( p \in \partial D \), there is a normal vector \( n(p) \). We have a well defined function

\[ \phi : \partial D \to S^{n-1}, \quad \phi(p) = n(p) . \]

Note that \( p \in H_{n(p)} \cap \overline{D} \). Therefore if \( D \) is \( C^1 \) and strictly convex, both \( \phi \) and \( g \) are defined and inverse to each other.

In general, for \( D \) convex, there are pseudo-functions \( g : S^n \to \partial D, \phi : \partial D \to S^n \). A pseudo-function assigns to each point \( v \in S^n \) a subset \( g(v) \subset \partial D \) in such a way that the graph \( \{(v, p) : p \in g(v)\} \) is closed. The inverse of a pseudo-function is well-defined, and \( g \) and \( \phi \) are inverse to each other. The set \( \phi(p) \) is the set of supporting vectors at \( p \) (see Remark 3).

**Lemma 9.** Suppose \( D \) strictly convex. For all \( 0 < r < R \), \( \partial D_r \cap \partial D = \emptyset \), so \( \partial D_r = r^{-1}(r) \).

**Proof.** Take a point \( p \in \partial D \), and let \( H_p \) be a supporting hyperplane. Consider a small ball \( B \) around \( p \) of radius \( \leq r/2 \). By strict convexity, \( d(\partial B \cap D, H) = \epsilon_0 > 0 \). Now we claim that \( B_{\epsilon_0}(p) \cap D \) does not intersect \( D_r \), so \( p \notin \overline{D}_r \). Let \( q \in B_{\epsilon_0}(p) \cap D \), and consider a line \( l \parallel H \) through \( q \). The segment \( l \cap B \) has endpoints \( P, Q \in \partial B \). But \( d(P, H) = d(Q, H) < \epsilon_0 \), so \( P, Q \notin D \). So the connected component \( [P, Q] \cap D \) has length \( < \|PQ\| \leq r \), and \( q \) is \( r' \)-accessible for some \( r' < r \).

\[ \square \]
Corollary 10. For $D$ strictly convex, $r : \overline{D} \to \mathbb{R}_{\geq 0}$ is continuous.

Proof. By Remark 2, $r$ is continuous on $D$. The continuity at $\partial D$ follows from the proof of Lemma 9.

Therefore, if $D$ is strictly convex, then

$$I_D = E_R = r^{-1}(R).$$

As $I_D \subset D$, we have that $I_D$ does not touch $\partial D$.

Theorem 11. Let $D$ be strictly convex. For all $0 < r < R$, $D_r$ is strictly convex.

Proof. Suppose that $\partial D_r$ contains a segment $l$. Let $p$ be a point in the interior of $l$. As it is $r$-accessible, there is a segment $[P, Q]$ of length $r$ through $p$, where $P, Q \in \partial D$. By Lemma 5, $v_P, v_Q$ are not parallel, and all points in $[P, Q]$ different from $p$ are $r'$-accessible for some $r' < r$. Therefore $l$ is transversal to $[P, Q]$. Let $H_v$ be the hyperplane produced by Lemma 5 (2). Then all points at one side of $H_v$ are $r'$-accessible for some $r' < r$, hence $l$ cannot be transversal to $H_v$, so $l \subset H_v$.

Now let $x \in l, x \neq p$. Consider the segment parallel to $[P, Q]$ through $x$, call it $\sigma$. It has length $r$ and endpoints at $H_{v_P}, H_{v_Q}$. But $D$ is strictly convex, so it only touches the supporting hyperplanes at one point. Hence $\sigma \cap D$ is strictly contained in $\sigma$. Therefore $r(x) < r$. Contradiction.

5. Set of maximum inaccessibility

In this section we suppose that $D$ is convex. Then $r$ is continuous on $D$ and it achieves its maximum $R$ on $D$. Then $I_D = E_R$ and $I_D \cap D = E_R \cap D = r^{-1}(R)$, by Proposition 4.

We want to characterize the case where $I_D$ contains interior. Let us see an example where this situation happens. Let $D$ be a rectangle. In this case $R$ is the length of the shortest edge of the rectangle, and we have an open set with $r(p) = R$ (see Figure 5). Note that it might happen that $\partial E_R$ intersects $\partial D$.

Proposition 12. If $I_D$ has non-empty interior, then $\partial D$ contains two open subsets which are included in parallel hyperplanes, which are at distance $R$.

Proof. Consider an interior point $p \in I_D$, so $r(p) = R$. Take a segment $l = [P, Q]$ of length $R$ with endpoints $P, Q \in \partial D$. Let $v_P, v_Q$ be vectors orthogonal to the supporting hyperplanes at $P, Q$. By Lemma 5, if they are not parallel, then there is a hyperplane through $p$ such that $E_R$ is contained in one (closed) half-space. This is not possible, as $p$ is an interior point of $E_R$. So $v_P, v_Q$ are parallel, and $\overline{PQ} \parallel v_P$. Now take any point $x$ close to $p$, and consider the segment $[P', Q']$ through $x$ parallel to $[P, Q]$. If $[P', Q'] \cap D$ is properly contained in $[P', Q']$, then $r(x) < R$, which contradicts that $x \in E_R$. So $P' \in H_{v_P}, Q' \in H_{v_Q}$, and $\partial D$ contains two open subsets in $H_{v_P}, H_{v_Q}$ around $P, Q$, respectively.

Theorem 13. Let $D$ be a strictly convex bounded domain, $R = \max r$. Then $I_D$ is a point or a segment.
Proof. Suppose that \( I_D \) is not a point. As it is convex by Theorem 6, it contains a maximal segment \( \sigma \). Let us see that it cannot contain two different (intersecting) segments. Let \( p \in \sigma \) be an interior point of the segment. By Lemma 5, if we draw the segment \([P, Q]\) of length \( R \) through \( p \), we have the following possibilities:

- \( v_P, v_Q \) are parallel. Then \( \overrightarrow{PQ} \parallel v_P \). Then any point \( x \notin [P, Q] \) lies in a segment \([P', Q']\) parallel to \([P, Q]\), with \( P' \in H_{v_P} \) and \( Q' \in H_{v_Q} \). By strict convexity, \( l([P', Q'] \cap D) < R \), so \( r(x) < R \). That is, \( E_R \subset [P, Q] \).

- \( v_P, v_Q \) are non-parallel. Then there is a hyperplane \( H_v \) through \( p \) such that \( E_R \subset H_{-v} \). As \( p \) is an interior point of \( \sigma \), \( \sigma \) does not cross \( H_v \), so \( \sigma \subset H_v \). Now let \( x \in \sigma \), and consider the segment \([P', Q']\) parallel to \([P, Q]\) through \( x \), with length \( R \), \( P' \in H_{v_P} \) and \( Q' \in H_{v_Q} \). If \( x \notin [P, Q] \) then strict convexity gives \( l([P', Q'] \cap D) < R \), so \( r(x) < R \). That is, \( E_R \subset [P, Q] \). Note that Lemma 5 (2) gives in this case that \( E_R = \{p\} \).

\[ \square \]

Let us see an example where \( E_R \) is a segment. Let \( D \) be the ellipse with equation \( \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \), where \( 2a > 2b \). Then \( R = 2b \) and \( E_R \) is a segment contained in the short axis, delimited by the intersection of the axis with the perpendicular segments of length \( R \) with endpoints in the ellipse.

![Figure 2. Ellipse](image)

All points \((x, y) \in D\) with \( x \neq 0 \) can be reached by vertical segments of length \( R = 2b \). Now let \( x_0 = b, y_0 = b\sqrt{a^2 - b^2}/a \). If \( y \in (-b, -y_0) \cup (y_0, b) \) then the point \((0, y)\) is \( r\)-accessible (with a horizontal segment) with \( r < R \). Now let \( y \in [-y_0, y_0] \), and consider a line through \((0, y)\). Let us parametrize it as

\[ r(s) = (sa \cos \theta, y + sb \sin \theta), \]

with \( \theta \) fixed. The intersection with the ellipse are given by \( s = -\frac{a}{b} \sin \theta \pm \sqrt{1 - \frac{a^2}{b^2} \cos^2 \theta} \). So the square of the distance between the two points is

\[ l(\theta)^2 = 4(1 - \frac{y^2}{b^2} \cos^2 \theta)(a^2 \cos^2 \theta + b^2 \sin^2 \theta) \]

\[ = 4(1 - \frac{y^2}{b^2} T)((a^2 - b^2)T + b^2), \]
where $T = \cos^2 \theta \in [0, 1]$. The minimum of this degree 2 expression on $T$ happens for a negative value of $T$, therefore, we only need to check the values $T = 0, 1$. For $T = 0$, we get $4b^2$; for $T = 1$, we get $4(1 - \frac{b^2}{2})a^2 \geq 4(1 - \frac{b^2}{2})a^2 = 4b^2$. So $l(\theta)^2 \geq 4b^2$.

A consequence of Theorem 13 is the following: for a strictly convex bounded domain $D$, if $I_D$ is not a point then there are two non-corner points $P, Q \in \partial D$ with parallel tangent hyperplanes which are moreover perpendicular to $\overrightarrow{PQ}$.

**Corollary 14.** Suppose $D$ is a planar convex bounded domain (not necessarily strictly convex). If $I_D$ is not a point then there are two non-corner points $P, Q \in \partial D$ with parallel tangent hyperplanes which are moreover perpendicular to $\overrightarrow{PQ}$.

**Proof.** Following the proof of Theorem 13, we only have to rule out case (2). As the hyperplane $H_v$ is now of dimension 1, we have $\sigma \subset [P, Q] = H_v \cap \overrightarrow{D}$. But Lemma 5 says also that $E_R \cap [P, Q] = \{p\}$. So $E_R$ does not contain a segment, i.e. it is a point. □

So, for a convex polygon $D$, if it does not have parallel sides, then $I_D$ is a point. Corollary 14 is not true in dimension $\geq 3$. Take a triangle $T \subset \mathbb{R}^2$ and consider $D = T \times [0, L]$ for large $L$. For $T$, denote $I_T = \{p\}$. Then $D$ has $I_D = \{p\} \times [a, b]$, for some $0 < a < b < L$. Certainly, there are two parallel faces (base and top), but we slightly move one of them to make them non-parallel, and $I_D$ is still a segment.

![Figure 3. $I_D$ can be positive dimensional](image)

One can make this construction to have $I_D$ of higher dimension (not just a segment), e.g. by considering $T \times [0, L]^N, N > 1$.

6. Polygons

In this section we want to study in detail the case of convex polygons in the plane, and to give some answers in the case of triangles. The starting point is the case of a sector.
Lemma 15. Fix $\lambda \in \mathbb{R}$. Let $D$ be the domain with boundary the half-lines $(x, 0)$, $x \geq 0$ and $(\lambda y, y)$, $y \geq 0$. Let $r > 0$. Then the boundary of $D_r$ is the curve:

$$\begin{align*}
x &= r(\cos^3 \theta + \lambda(\sin^3 \theta + 2\sin \theta \cos^2 \theta)) \\
y &= r(\sin^3 \theta - \lambda \sin^2 \theta \cos \theta)
\end{align*}$$

(2)

Proof. $D$ is not a bounded domain, but the theory works as well in this case. To find the boundary of $D_r$, we need to take the envelope of the segments of length $r$ with endpoints laying on the half-rays, according to Proposition 7. Two points at $(a, 0)$ and $(\lambda b, b)$ are at distance $r$ if

$$(\lambda b - a)^2 + b^2 = r^2.$$ 

So $\lambda b - a = -r \cos \theta$, $b = r \sin \theta$, i.e. $a = \lambda r \sin \theta + r \cos \theta$. The line which passes through $(\lambda b, b)$ and $(a, 0)$ is

$$r \sin \theta x + r \cos \theta y = r^2 \sin \theta \cos \theta + r^2 \lambda \sin^2 \theta.$$ 

We are going to calculate the envelope of these lines (see pp. 75-80 in [5]). Take the derivative and solve the system:

$$\begin{align*}
r \sin \theta x + r \cos \theta y &= r^2 \sin \theta \cos \theta + r^2 \lambda \sin^2 \theta \\
r \cos \theta x - r \sin \theta y &= -r^2 \sin^2 \theta + r^2 \cos^2 \theta + 2r^2 \lambda \sin \theta \cos \theta.
\end{align*}$$

We easily get the expression in the statement. The region $D_r$ is the unbounded region with boundary the curve (2) and the two half-rays. □

We call the curve in Lemma 15 a $\lambda$-bow (or just a bow). Let $\lambda = \cot \alpha$, $\alpha \in (0, \pi)$. If $\lambda < 0$, we are dealing with an obtuse angle, and $\theta \in [0, \pi - \alpha]$. If $\lambda = 0$, we have a right angle, and $\theta \in [0, \pi/2]$. Finally, an acute angle happens for $\lambda > 0$. In this case, $\theta \in [\pi/2 - \alpha, \pi/2]$. (Note that $\theta$ is the angle between the segment and the negative horizontal axis, in the proof of Lemma 15.)

As an application, we prove the following:

Corollary 16. Let $D$ be a planar convex polygon. Then the sets $D_r$, $0 < r < R$, and $I_D$ if it is not a point, have boundaries which are piecewise $C^1$, and whose pieces are $\lambda$-bows and (possibly) segments in the sides of $\partial D$. In particular, these domains are strictly convex when $\partial D_r$ does not intersect $\partial D$. 

Figure 4. $\lambda$-bows with $r = 1$
Proof. Let \( l_1, \ldots, l_k \) be the lines determined by prolonging the sides of the polygon. Consider \( l_i, l_j \). If they intersect, consider the sector that they determine in which \( D \) is contained. Lemma 15 provides us with a (convex) region \( D_{ij}^r \). If \( l_i, l_j \) are parallel and \( r < d(l_i, l_j) \) then set \( D_{ij}^r = D \), and if \( l_i, l_j \) are parallel and \( r \geq d(l_i, l_j) \) then set \( D_{ij}^r = \emptyset \). It is fairly clear that
\[
D_r = \bigcap_{i \neq j} D_{ij}^r.
\]

To see the last assertion, note that at any smooth point \( p \in \partial D_r \), we have strict convexity because of the shape of the bows given in Lemma 15. If \( p \in \partial D_r \) is a non-smooth point, then it is in the intersection of two such curves. This means that there are segments \( \sigma_1, \sigma_2 \) of length \( r \) where \( \sigma_1 \) has endpoints at lines \( l_{i_1}, l_{j_1} \) and \( \sigma_2 \) has endpoints at lines \( l_{i_2}, l_{j_2} \). Moreover, the endpoints should be actually in the sides of \( D \) (otherwise \( p \) would be \( r' \)-accessible for some \( r' < r \)). In particular, this means that \( \sigma_1, \sigma_2 \) cannot be parallel. As such segments are tangent to the bows, the curves intersect transversely at \( p \), and \( p \) is a corner point.

A similar statement holds for \( I_D = E_R \), when it is not a point, by doing the above reasoning for \( r = R \). \( \square \)

In particular, we see that \( I_D \) cannot be a segment for polygons.

For instance, when \( D \) is a rectangle of sides \( a \geq b \), then \( R = b \). We draw the bows at the vertices, to draw the set \( I_D = E_R \).

![Figure 5. For a rectangle, \( I_D \) has interior](image)

Note that \( I_D \) intersects \( \partial D \) if and only if \( a \geq 2b \).

It would be nice to have a function
\[
I_D = (I_1, I_2) = I_D((x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)) \in \mathbb{R}^2,
\]
which assigns the value of \( I_D \) given the vertices \( (x_i, y_i) \) of a \( k \)-polygon. Such function is only defined for polygons with non-parallel sides.

We shall produce the formula for \( I_D \) for the case of an isosceles triangle. Consider an isosceles triangle of height 1, and base \( 2\lambda > 0 \). Put the vertices at the points \( A = (0, 0) \), \( B = (2\lambda, 0) \) and \( C = (\lambda, 1) \). By symmetry, the point \( I_D \) must lie in the vertical axis \( x = \lambda \). Moreover, the segment of length \( R \) through
$I_D$ tangent to the bow corresponding to $C$ must be horizontal. This means that $I_D = (\lambda, I_2)$ where $\frac{R}{2} = \lambda(1 - I_2)$. So 

$$I_D = (I_1, I_2) = (\lambda, I_2) = \left(\lambda, 1 - \frac{R}{2\lambda}\right).$$

The sector corresponding to $A$ is that of Lemma 15, and the point $I_D$ should lie in its $\lambda$-bow, which is the curve given in Lemma 15 for the value $r = R$. Hence 

$$\lambda = R(\cos^3 \theta + \lambda(\sin^3 \theta + 2\sin \theta \cos^2 \theta)), \quad 1 - \frac{R}{2\lambda} = R(\sin^3 \theta - \lambda \sin^2 \theta \cos \theta).$$

Eliminating $R$, we get 

$$\lambda^2 \sin^2 \theta \cos \theta + 2\lambda \sin \theta \cos^2 \theta + \cos^3 \theta - \frac{1}{2} = 0$$

i.e. 

$$\lambda = \frac{-2 \cos^2 \theta + 2 \cos \theta}{2 \sin \theta \cos \theta}$$

(3)  

(the sign should be plus, since $\lambda > 0$). Note that for an equilateral triangle, $\lambda = \frac{1}{\sqrt{3}}$, $I_2 = \frac{1}{3}$, $\theta = \frac{\pi}{3}$ and $R = \frac{4}{3\sqrt{3}}$. Also 

$$R = \frac{\lambda}{\cos^3 \theta + \lambda(\sin^3 \theta + 2\sin \theta \cos^2 \theta)}. \quad (4)$$

One can check the following formula: 

$$I_2 = 1 - \frac{R}{2\lambda} = \lambda \frac{\sin^3 \theta - \lambda \sin^2 \theta \cos \theta}{\cos^3 \theta + \lambda(\sin^3 \theta + 2\sin \theta \cos^2 \theta)}. \quad (5)$$

This locates the point $I_D = (\lambda(\theta), I_2(\lambda(\theta)))$.

Remark. Do the change of variables $\cos \theta = \frac{1 - u^2}{1 + u^2}$, $\sin \theta = \frac{2u}{1 + u^2}$, to get algebraic expressions for $I_D$. It is to be expected that this algebraicity property holds for a general triangle.

Recall the position of the ortocentre, incentre, baricentre and circumcentre [1]  

$$H = (\lambda, \lambda^2),$$

$$I = \left(\lambda, \frac{\lambda}{\lambda + \sqrt{\lambda^2 + 1}}\right),$$

$$G = \left(\lambda, \frac{1}{3}\right),$$

$$O = \left(\lambda, \frac{1 - \lambda^2}{2}\right).$$

We draw the height of the point $H, I, G, O, I_D$ as a function of $\lambda$:

A simple consequence is that these 5 points are distinct for an isosceles triangle which is not equilateral. We conjecture that this is true for a non-isosceles triangle.
The most inaccessible point of a convex domain

Figure 6. Notable points of a triangle of height 1 and base $2\lambda$

Note the asymptotic for an isosceles triangle. For $\lambda \sim 0$, we have that (3) implies $\cos^3 \theta \sim \frac{1}{2}$. Now (4) and (5) give that $R \sim 2\lambda$ and

$$I_2(\lambda) \sim \frac{\sin^3 \theta}{\cos^3 \theta} \lambda \sim (2^{2/3} - 1)^{3/2} \lambda.$$ 

Rescale the triangle to have base $b = 2$ and height $h = \frac{1}{\lambda}$. Then when $h$ is large, the point $I_D$ approaches to be at distance $(2^{2/3} - 1)^{3/2} = 0.4502$ to the base, and $R \sim 2$. Also, for $\lambda \to \infty$, we have $I_2(\lambda) \to 1$.

Remark. Consider a rectangle $D$ with vertices $(\pm a, \pm 1)$, with $a \gg 1$. Then $I_D$ has interior (see Figure 5). Moving slightly the vertices at the left, we get an isosceles trapezoid $Z_\epsilon$, with vertices $(-a, \pm(1-\epsilon)), (a, \pm 1)$, for $\epsilon > 0$. Consider the triangle $T_\epsilon$ obtained by prolonging the long sides of $Z_\epsilon$, i.e. with vertices $(a - 2a/\epsilon, 0), (a, \pm 1)$. By the above, the point $I_{T_\epsilon} \sim (a - 0.4502, 0)$. As $R \sim 2$, we have that $I_{Z_\epsilon} = I_{T_\epsilon}$.

By symmetry, if we consider the isosceles trapezoid $Z'_\epsilon$ with vertices $(-a, \pm 1), (a, \pm(1-\epsilon))$, then $I_{Z'_\epsilon} \sim (-a + 0.4502, 0)$.

The polygons $Z_\epsilon$ and $Z'_\epsilon$ are nearby, but their points of maximum inaccessibility are quite far apart. So the map $D \mapsto I_D$ cannot be extended continuously (in any reasonable topology) to all polygons with 4 sides.

References

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Perpendicular Bisectors of Triangle Sides

Douglas W. Mitchell

Abstract. Formulas, in terms of the sidelengths and area, are given for the lengths of the segments of the perpendicular bisectors of the sides of a triangle in its interior. The ratios of perpendicular bisector segments to each other are given, and the ratios of the segments into which the perpendicular bisectors are divided by the circumcenter are considered. Then we ask whether a set of three perpendicular bisector lengths uniquely determines a triangle. The answer is no in general: depending on the set of bisectors, anywhere from zero to four (but no more than four) triangles can share the same perpendicular bisector segments.

1. Introduction

It is well-known that the perpendicular bisectors of the sides of a triangle meet at a single point, which is the center of the circumcircle. Bui [1] gives results for similar triangles associated with the perpendicular bisectors. In this paper we first find formulas for the lengths of the segments of the perpendicular bisectors in the interior of a given triangle. Then we study the question of existence of triangles with prescribed lengths of perpendicular bisector segments.

Lemma 1. The perpendicular bisector segment through the midpoint of one side terminates at a point on the longer of the remaining two sides (or at their intersection if these sides are equal).

Proof. Let the perpendicular bisector pass through the midpoint $F$ of the side $AB$ of triangle $ABC$. Consider the case when both angles $A$ and $B$ are acute. In Figure 1, $AF = FB$ and $FE \geq FD$. We show that $BC \geq AC$. Now,

$$\tan A = \frac{EF}{AF} \geq \frac{DF}{AF} = \frac{DF}{BF} = \tan B.$$
Since \( \tan A \geq \tan B \) and angles \( A \) and \( B \) are both acute, \( \angle A \geq \angle B \). It follows that \( BC \geq AC \).

The other cases (when one of angles \( A \) and \( B \) is obtuse or a right angle) are clear (see Figure 2 in the case of an obtuse angle \( A \)). □

Lemma 1 will be used in constructing the perpendicular bisectors in Figures 3-5. Henceforth we will adopt the notational convention that the sides \( a, b, \) and \( c \) opposite to angles \( A, B, \) and \( C \) are such that \( a \geq b \geq c \) (and hence \( A \geq B \geq C \)).

We denote by \( p_a, p_b, p_c \) the lengths of the perpendicular bisector segments on the sides \( BC, CA, AB \) respectively. Also, \( \Delta \) denotes the area of the triangle.

**Theorem 2.** Let \( a \geq b \geq c \).

\[
\begin{align*}
p_a &= \frac{2a\Delta}{a^2 + b^2 - c^2}, \\
p_b &= \frac{2b\Delta}{a^2 + b^2 - c^2}, \\
p_c &= \frac{2c\Delta}{a^2 - b^2 + c^2}.
\end{align*}
\]

**Proof.** (i) Figure 3 illustrates the case where \( \angle A \) is acute; the proof is identical when \( \angle A \) is right or obtuse. We have \( \tan C = \frac{p_a}{b} \) since by Lemma 1 bisector \( p_a \) meets side \( b \) because \( b \geq c \). We know from [2] (which applies since \( \angle C \) is oblique since \( \angle A \geq \angle C \)) that \( \Delta = \frac{\tan C}{4}(a^2 + b^2 - c^2) \). Combining these gives \( p_a = \frac{2a\Delta}{a^2 + b^2 - c^2} \).

(ii) Figure 4 illustrates the case where \( \angle A \) is acute; the proof is again identical when \( \angle A \) is right or obtuse. We have \( \tan C = \frac{p_b}{a} \) (since by Lemma 1 bisector \( p_b \) intersects side \( a \) because \( a \geq c \)). Again \( \Delta = \frac{\tan C}{4}(a^2 + b^2 - c^2) \). Combining these gives \( p_b = \frac{2b\Delta}{a^2 + b^2 - c^2} \).

(iii) Figure 5 illustrates the case where \( \angle A \) is acute; the proof is again identical when \( \angle A \) is right or obtuse. We have \( \tan B = \frac{p_c}{a} \) since by Lemma 1 bisector \( p_c \) intersects side \( a \) because \( a \geq b \). By [2] (which applies since \( \angle B \) is oblique since \( \angle A \geq \angle B \)) we have \( \Delta = \frac{\tan B}{4}(a^2 - b^2 + c^2) \). Combining these gives \( p_c = \frac{2c\Delta}{a^2 - b^2 + c^2} \). □

Since Heron’s well-known formula gives \( \Delta \) in terms of \( a, b, \) and \( c \), Theorem 2 gives each of the perpendicular bisectors in terms of the three sides. Moreover, we can apply the law of cosines for each angle to obtain these symmetric area formulas in terms of one side, another perpendicular bisector and a third angle.
Corollary 3. \[ \Delta = p_a b \cos C = p_b a \cos C = p_c a \cos B. \]

Theorem 4. Let \( a \geq b \geq c \).
(i) \( p_a \geq p_b \);
(ii) \( p_c \geq p_b \);
(iii) \( p_a \geq p_c \). \( p_a = p_c \) and \( p_a < p_c \) are all possible.

Proof. (i) By Theorem 2, \( \frac{p_a}{p_b} = \frac{a}{b} = \frac{\sin A}{\sin B} \). Since \( A \geq B \) and \( A + B < \pi \), \( \sin A \geq \sin B \). It follows that \( p_a \geq p_b \).
(ii) From Figures 4 and 5,
\[
\frac{p_b}{p_c} = \frac{\frac{b}{2} \tan C}{\frac{c}{2} \tan B} = \frac{b \sin C}{c \sin B} \cdot \frac{\cos B}{\cos C} = \frac{\cos B \cos C}{\cos C} \leq 1
\]

since \( B \geq C \) are acute angles.
(iii) We show that all scenarios are possible by examples. Let \( a = 6 \) and \( c = 4 \), so that \( 4 \leq b \leq 6 \).

<table>
<thead>
<tr>
<th>( b )</th>
<th>( \Delta )</th>
<th>( p_a )</th>
<th>( p_c )</th>
</tr>
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<td>( 3\sqrt{7} )</td>
<td>( \frac{6}{3} )</td>
<td>( \frac{6 \sqrt{7}}{9} )</td>
</tr>
<tr>
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<td>( 18 )</td>
<td>( \frac{5\sqrt{29}}{18} )</td>
<td>( \frac{5\sqrt{29}}{18} )</td>
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<tr>
<td>5</td>
<td>( \frac{15\sqrt{7}}{4} )</td>
<td>( \frac{4\sqrt{15}}{15} )</td>
<td>( \frac{8\sqrt{7}}{27} )</td>
</tr>
</tbody>
</table>

Since we also know [3] that the distances from the circumcenter - which is the intersection of the perpendicular bisectors - to the sides \( AC \) and \( AB \) are in the ratios \( \frac{\cos B}{\cos C} \), we have in Figure 6 that \( \frac{OE}{OF} = \frac{\cos B}{\cos C} = \frac{ED'}{FD'} \) (by Theorem 4(ii)) so \( \frac{OE}{ED'} = \frac{OF}{FD} \). Hence:

Corollary 5. In an acute triangle the circumcenter divides the perpendicular bisectors \( p_b \) and \( p_c \) in equal proportions.

A similar result applies when the triangle is obtuse (in which case the circumcenter lies outside the triangle):
Corollary 6. In an obtuse triangle, the perpendicular bisectors $p_b$ and $p_c$ extended to the circumcenter are divided by their respective intersecting triangle sides in equal proportions.

2. Do the perpendicular bisectors uniquely determine a triangle?

Theorem 7. Given positive $p_a$, $p_b$, $p_c$ satisfying $p_a \geq p_b$ and $p_c \geq p_b$, there are no more than two non-congruent triangles with $a \geq b \geq c$ and perpendicular bisector segments of lengths $p_a$, $p_b$, $p_c$.

Proof. By Theorem 2, $\frac{p_a}{p_b} = \frac{a}{b}$, and

$$\frac{p_c}{p_b} = \frac{c}{b} \cdot \left(\frac{a^2 + b^2 - c^2}{a^2 - b^2 + c^2}\right) = \frac{c}{b} \left(\frac{(\frac{a}{b})^2 + 1 - \left(\frac{c}{b}\right)^2}{(\frac{a}{b})^2 - 1 + \left(\frac{c}{b}\right)^2}\right)$$

Putting $\alpha := \frac{p_a}{p_b} \geq 1$, $\gamma := \frac{p_c}{p_b} \geq 1$, and $x := \frac{c}{b}$, we rearrange this as

$$f(x) = x^3 + \gamma x^2 - (\alpha^2 + 1)x + \gamma(\alpha^2 - 1) = 0. \quad (1)$$

Since $x$ is a ratio of sides, and $a - b < c \leq b$, we must have $x \in (\alpha - 1, 1]$.

If $\alpha - 1 = 0$ (the isosceles case of $\frac{a}{b} = \frac{p_a}{p_b} = 1$), (1) becomes $x(x^2 + \gamma x - 2) = 0$, and has exactly one solution in the interval $(\alpha - 1, 1]$.

If $\alpha - 1 > 0$, (1) exhibits two switches in the signs of parameters, so by Descartes’ rule the cubic has either 2 or 0 positive roots. Thus the number of non-similar triangles cannot be more than 2. Since similar but non-congruent triangles cannot share the same absolute sizes of perpendicular bisectors, the number of non-congruent triangles sharing the same $(p_a, p_b, p_c)$ can be no more than 2. \qed

Remark. Since $c > a - b$, we must have $x > \alpha - 1$. If $\gamma = 1$, (1) becomes

$$(x + \alpha + 1)(x - 1)(x - (\alpha - 1)) = 0.$$ 

It follows that $x = 1$ is the only admissible root. This results in an isosceles triangle.

Having already given an exact number of triangles above for the cases of $p_a = p_b$ and $p_c = p_b$, we next find specific parameter conditions that are necessary and sufficient for the number of triangles with specified $(p_a, p_b, p_c)$, i.e., the number of admissible solutions of (1) - to be 0, 1, or 2 when $\alpha$ and $\gamma$ both exceed 1, i.e., $p_a > p_b$ and $p_c > p_b$.

It is easy to see that the cubic $f(x)$ in (1) has a local minimum at

$$x_0 = \frac{-\gamma + \sqrt{\gamma^2 + 3(\alpha^2 + 1)}}{3} > 0,$$

and a local maximum at $x_1 = \frac{-\gamma - \sqrt{\gamma^2 + 3(\alpha^2 + 1)}}{3} < 0$.

It is routine to verify that the local minimum $x_0 \in (\alpha - 1, 1)$ if and only if $\gamma \in \left(\frac{\alpha^2 - 2}{2}, \frac{-3\alpha + 1}{\alpha - 1}\right)$. 
Theorem 8. Given \( p_a > p_b \) and \( p_c > p_b \), let \( \alpha = \frac{p_a}{p_b} \) and \( \gamma = \frac{p_c}{p_b} \). The number of triangles with perpendicular bisector segments of lengths \( p_a, p_b, p_c \) is

\[ 2 \text{ if and only if } \gamma \in \left( \frac{\alpha^2 - 2}{2}, \frac{\alpha^2 + 3\alpha - 1}{\alpha - 1} \right) \text{ and } f(x_0) < 0, \]

\[ 1 \text{ if and only if } \gamma \in \left( \frac{\alpha^2 - 2}{2}, \frac{\alpha^2 + 3\alpha - 1}{\alpha - 1} \right) \text{ and } f(x_0) = 0, \]

0 otherwise.

Proof. Note that \( f(\alpha - 1) = 2\alpha(\alpha - 1)(\gamma - 1) > 0 \) and \( f(1) = \alpha^2(\gamma - 1) > 0 \).

If the local minimum \( x_0 \in (\alpha - 1, 1) \), then the number of roots of \( f(x) \) in the interval 2, 1, or 0 according as \( f(x_0) < 0 = 0, or > 0 \).

If \( x_0 \notin (\alpha - 1, 1) \), then \( f(x) \) is monotonic and has no root in the interval. \( \square \)

While no more than two distinct triangles can share the same \( p_a, p_b, p_c \), one must also consider the possibility that up to two more triangles could share the same three perpendicular bisectors segments \( \{p_1, p_2, p_3\} \) with different assignments of the bisectors to the long, medium, and short sides of the triangle. This is because, by Theorem 4, the medium-length side \( b \) has the shortest perpendicular bisector but either the longest side \( a \) or the shortest side \( c \) can have the longest bisector. Therefore, given segments of lengths \( p_1 \geq p_2 \geq p_3 \), we seek triangles \( (a, b, c) \) with \( a \geq b \geq c \) and \( (p_a, p_b, p_c) = (p_1, p_3, p_2) \) or \( (p_2, p_3, p_1) \). Thus:

Corollary 9. There are a maximum of four triangles with the three segments of given lengths as perpendicular bisector segments.

We conclude this paper by giving explicit examples showing that the number of triangles in Corollary 9 can be any of 0, 1, 2, 3, 4.

(i) \( n = 4 \): Consider \( (p_1, p_2, p_3) = (20, 18, 15) \).

If \( (p_a, p_b, p_c) = (20, 15, 18), \alpha = \frac{4}{3}, \gamma = \frac{6}{5} \), and

\[ f_1(x) = x^3 + \frac{6}{5} x^2 - \frac{25}{9} x + \frac{14}{15} = \left(x - \frac{7}{15}\right) \left(x - \frac{\sqrt{97} - 5}{6}\right) \left(x + \frac{\sqrt{97} + 5}{6}\right). \]

This gives two triangles similar to \((20, 15, 7)\) and \((8, 6, \sqrt{97} - 5)\).

On the other hand, if \( (p_a, p_b, p_c) = (18, 15, 20), \alpha = \frac{6}{5}, \gamma = \frac{4}{3} \), and

\[ f_2(x) = x^3 + \frac{4}{3} x^2 - \frac{61}{25} x + \frac{44}{75} = \left(x - \frac{4}{5}\right) \left(x - \frac{\sqrt{421} - 16}{15}\right) \left(x + \frac{\sqrt{421} + 16}{15}\right). \]

This gives two triangles similar to \((6, 5, 4)\) and \((18, 15, \sqrt{421} - 16)\).

(ii) For \( n = 3 \), let \( (p_1, p_2, p_3) = (3, 2\sqrt{2}, \sqrt{5}) \).

If \( (p_a, p_b, p_c) = (3, \sqrt{5}, 2\sqrt{2}), \alpha = \frac{\sqrt{5}}{3}, \gamma = \frac{2\sqrt{2}}{\sqrt{5}} \), and

\[ f_3(x) = x^3 + \frac{2\sqrt{2}}{\sqrt{5}} x^2 - \frac{14}{5} x + \frac{8\sqrt{2}}{5\sqrt{5}} = \left(x + \frac{4\sqrt{2}}{\sqrt{5}}\right)^2 \left(x - \frac{\sqrt{2}}{\sqrt{5}}\right)^2. \]

This gives a triangle similar to \((3, \sqrt{5}, \sqrt{2})\).
If \((p_a, p_b, p_c) = (2\sqrt{2}, \sqrt{5}, 3)\), then
\[
f_4(x) = x^3 + \frac{3}{\sqrt{5}} x^2 - \frac{13}{5} x + \frac{9}{5\sqrt{5}} = \left(x - \frac{1}{\sqrt{5}}\right) \left(x - \frac{\sqrt{13} - 2}{\sqrt{5}}\right) \left(x + \frac{\sqrt{13} + 2}{\sqrt{5}}\right).
\]

The two roots in the interval \((0, 1)\) give the triangles similar to \((2\sqrt{2}, \sqrt{5}, 1)\) and \((2\sqrt{2}, \sqrt{5}, \sqrt{13} - 2)\).

(iii) For \(n = 2\), let \((p_1, p_2, p_3) = (39, 30, 25)\). If \((p_a, p_b, p_c) = (30, 25, 39)\), then \(\alpha = \frac{6}{5}, \gamma = \frac{39}{25}\), and
\[
f_5(x) = x^3 + \frac{39}{25} x^2 - \frac{61}{25} x + \frac{429}{625} = \left(x + \frac{13}{5}\right) \left(x - \frac{11}{25}\right) \left(x - \frac{3}{5}\right).
\]
The two positive roots are in \((\alpha - 1, 1) = (\frac{1}{5}, 1)\). These give two triangles similar to \((30, 25, 11)\) and \((6, 5, 3)\).

On the other hand, if \((p_a, p_b, p_c) = (39, 25, 30)\), the cubic
\[
f_6(x) = x^3 + \frac{6}{5} x^2 - \frac{2146}{625} x + \frac{5376}{3125}
\]
has only one real root which is negative (see Figure 7). There is no triangle with \((p_a, p_b, p_c) = (39, 25, 30)\).

(iv) For \(n = 1\), consider \((p_1, p_2, p_3) = (8, 5, \sqrt{19})\). If \((p_a, p_b, p_c) = (5, \sqrt{19}, 8)\), then \((\alpha, \gamma) = \left(\frac{5}{\sqrt{19}}, \frac{8}{\sqrt{19}}\right)\), and
\[
f_7(x) = x^3 + \frac{8}{\sqrt{19}} x^2 - \frac{44}{19} x + \frac{48}{19\sqrt{19}} = \left(x + \frac{12}{\sqrt{19}}\right) \left(x - \frac{2}{\sqrt{19}}\right)^2.
\]
This gives a single triangle similar to \((5, \sqrt{19}, 2)\).
On the other hand, with \((p_a, p_b, p_c) = (8, \sqrt{19}, 5)\), we have \((\alpha, \gamma) = \left(\frac{8}{\sqrt{19}}, \frac{5}{\sqrt{19}}\right)\), and
\[
f_8(x) = x^3 + \frac{5}{\sqrt{19}}x^2 - \frac{83}{19}x + \frac{225}{19\sqrt{19}}
\]
has only one real root which is negative (see Figure 7). There is no such triangle.

(v) Finally, for \(n = 0\), we take \((p_1, p_2, p_3) = (5, 4, 1)\). The two cubic polynomials are \(f_9(x) = x^3 + 5x^2 - 17x + 75\) and \(f_{10}(x) = x^3 + 4x^2 - 26x + 96\). Each of these has exactly one real root which is negative (see Figure 8). Therefore, there is no triangle with perpendicular bisector segments \((5, 4, 1)\).

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Convolution Filters for Triangles

Grégoire Nicollier

Abstract. The construction of a new triangle by erecting similar ears on the sides of a given triangle (as in Napoleon’s theorem) can be considered as the convolution of the initial triangle with another triangle. We use the discrete Fourier transformation and a shape function to give a complete and explicit description of such convolution filters and their iterates. Our method leads to many old and new results in a very direct way.

1. Introduction

Johnson [17, pp. 284, 294], citing Emmerich [5, p. 129], formulates and proves the following property: “If points divide the sides of a given triangle in equal ratios, they are vertices of a triangle having the same Brocard angle as the given triangle. [...] The triangles whose vertices divide in equal ratios the sides of a given triangle constitute all the different forms of triangle having the same Brocard angle.” (See also [29, 11] and our Theorem 2.) We analyze this kind of transformations of an initial triangle into a new triangle by considering convolutions of two triangles: with one exception, such a convolution simply erects three similar triangular ears on the sides of the initial triangle before transforming the triangle of the ears’ apices by a direct similarity. A circulant linear transformation of a triangle in the complex plane, given by the coefficients $c_0$, $c_1$, $c_2$ of the circulant linear combination of the vertices, is simply the convolution of the initial triangle and the triangle with vertices $c_0$, $c_2$, $c_1$.

Here is a sketch of our method. We use the spectral decomposition in the Fourier base of $\mathbb{C}^3$ to represent any triangle of the complex plane as the sum of its centroid and of two positively and negatively oriented equilateral triangles: the convolution with this triangle is then a diagonal linear map, and we call shape of the triangle the quotient of the eigenvalues belonging to the negatively and positively oriented equilateral base vectors; this shape is also the quotient of the corresponding spectral values of the convolving triangle, i.e., the ratio of the negatively and positively oriented equilateral quantities in this triangle. It is then immediate that the shape of a convolution product of two triangles is equal to the product of the triangles’ shapes. Two directly similar triangles (with vertices in order) have the same shape; moreover, when restricted to normalized triangles with vertices 0, 1, and $z$, the

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The author thanks Mowaffaq Hajja for drawing his attention to the shape function.
shape function is a Möbius transformation as function of $z$: the equivalence classes of directly similar triangles (with respect to the given order of their vertices) are thus parametrized by their shape. Every triangle transformation given by a convolution can be described by analyzing this Möbius transformation. Emmerich’s introducing result, for example, becomes almost immediate (see also [11]): the transformation is a convolution with a degenerate triangle; since degenerate triangles are characterized by shapes of modulus 1, the convolution acts on the shape of the initial triangle as a rotation and does not change the shape’s modulus; since triangles are equibrocardal and equally oriented exactly when their shapes have the same modulus, the Brocard angle is invariant under a convolution with a degenerate triangle; the converse follows from the fact that the shapes of the degenerate triangles needed for the Emmerich transformations cover the whole unit circle (−1 being half an exception). By iterating the convolution with a degenerate triangle, one simply rotates the shape by a constant angle at each step: the successive triangles are directly similar to triangles with a common base whose apices turn on the same Neuberg circle, and these apices are periodic or dense on the Neuberg circle according as the rotation angle for the shape is a rational or irrational multiple of $\pi$.

Many triangle transformations in the literature are in fact convolutions with a fixed triangle and could thus have been analyzed by the present method in a very efficient and standardized way. Moreover, if the convolving triangle is degenerate, i.e., if the shape of the convolving triangle lies on the unit circle, the transformation behaves as the Emmerich transformation with the same shape (as far as only the triangle’s form is concerned), and the work is done as soon as the rotation angle, i.e., the argument of the shape of the convolving degenerate triangle, is determined! The $s$-Rooth triangles of [11] for example, which already appear in [22], are in fact given by the convolution with a degenerate triangle, like any circulant transformation of a triangle where the new vertices are a real linear combination of the old ones, and such convolutions are included once for all in the Emmerich transformations (as far as only the triangle’s form is concerned).

The Fourier decomposition of a triangle or polygon and circulant matrices have been used for a long time for studying triangle and polygon transformations with a circulant structure, beginning with Darboux in 1878 [3], [1, 4, 6, 7, 8, 14, 25, 27, 28, 30, 31, 32, 33, 36]; they are in general more efficient than purely geometric and trigonometric approaches [16, 34, 35]. Our presentation is free from matrix algebra since we only need convolutions. The shape function we use seems to appear for the first time in 2003 in a paper by Nakamura and Oguiso [22] (from a sixteen years older preprint), and later independently in [10]: we found only in [10] an indirect relation between the shape function and some eigenfunction as tentative explanation of the nature of this shape function. As far as we know, the fact that the shape of a convolution product is the product of the shapes is noticed and exploited here for the first time, and this is the key point for the success of our method. Note that Hajja et al. [10, 11, 12, 13], and Nakamura and Oguiso [22] establish ad hoc for each analyzed transformation that the shape of the transformed triangle is the shape of the initial triangle multiplied by some function independent of the initial
triangle, without noticing that this function is in fact the shape of a triangle and that the transformation is a convolution. We extended this shape function to polygons in [24].

2. Fourier transform of a triangle

Consider a triangle \( \Delta \) of the complex plane as a point \( \Delta = (z_0, z_1, z_2) \in \mathbb{C}^3 \) representing the closed polygonal line \( z_0 \to z_1 \to z_2 \to z_0 \) starting at \( z_0 \); there are up to six triangles with the same vertices. A triangle is called degenerate when its vertices are collinear, trivial when it is reduced to a triple point, and proper when it is nondegenerate; a degenerate triangle is both positively and negatively oriented.

Endow \( \mathbb{C}^3 \) with the inner product \( \langle (z_0, z_1, z_2), (w_0, w_1, w_2) \rangle = \frac{1}{3} \sum_{k=0}^{2} z_k \overline{w_k} \) and set \( \zeta = e^{i2\pi/3} \). The vectors

\[
e_0 = (1, 1, 1), \quad e_1 = (1, \zeta, \zeta^2), \quad e_2 = (1, \zeta^2, \zeta^4) = (1, \zeta^2, \zeta) = e_1^2
\]

form the orthonormal Fourier base of \( \mathbb{C}^3 \): \( e_0 \) is a trivial triangle; \( e_1 \) and \( e_2 \) are a positively and a negatively oriented equilateral triangle centered at the origin, respectively. The discrete Fourier transform or spectrum of \( \Delta \) is the triangle \( \hat{\Delta} = (\hat{z}_0, \hat{z}_1, \hat{z}_2) \) given by the spectral representation of \( \Delta \) in the Fourier base:

\[
\Delta = \sum_{k=0}^{2} \hat{z}_k e_k \quad \text{with} \quad \hat{z}_k = \langle \Delta, e_k \rangle, \quad k = 0, 1, 2.
\]

\( \hat{z}_0 = \frac{1}{3}(z_0 + z_1 + z_2) \) is the centroid of \( \Delta \), and

\[
\hat{z}_1 = \frac{1}{3}(z_0 + \zeta^2 z_1 + \zeta z_2), \quad \hat{z}_2 = \frac{1}{3}(z_0 + \zeta z_1 + \zeta^2 z_2).
\]

A triangle is trivial if \( \hat{z}_1 = \hat{z}_2 = 0 \); it is pequilateral if it is equilateral and positively oriented, i.e., if \( \hat{z}_1 \neq 0, \hat{z}_2 = 0 \); it is nequilateral if it is equilateral and negatively oriented, i.e., if \( \hat{z}_1 \neq 0, \hat{z}_2 \neq 0 \); it is mixed if \( \hat{z}_1 \neq 0, \hat{z}_2 \neq 0 \), i.e., if it is neither trivial nor equilateral. A spectrum is full if all Fourier coefficients \( \hat{z}_k \) are different from 0.

3. Shape of a triangle

We define the shape of a nontrivial triangle \( \Delta \) by

\[
\sigma_\Delta = \sigma(z_0, z_1, z_2) = \frac{\hat{z}_2}{\hat{z}_1} = \frac{z_0 + \zeta z_1 + \zeta^2 z_2}{z_0 + \zeta^2 z_1 + \zeta z_2} \in \mathbb{C} \cup \{\infty\},
\]

the ratio of the contributions to \( \Delta \) of the nequilateral \( e_2 \) and of the pequilateral \( e_1 \) [22, 10]. Note that [22, 10] define this shape function without any reference to Fourier transforms or circulant matrices. The first properties of the shape function presented below can already be found in [10]: some errors are corrected in [13]. Since \( (w_0, w_1, w_2) = a(z_0, z_1, z_2) + b \zeta_0 \) for some \( b \) is equivalent to \( (\hat{w}_1, \hat{w}_2) = a(\hat{z}_1, \hat{z}_2) \), \( \Delta_1 \) has the shape of \( \Delta \) if and only if \( \Delta_1 = a \Delta + b \zeta_0 \) with \( a, b \in \mathbb{C}, a \neq 0; \Delta = (z_0, z_1, z_2) \) has thus the shape of a unique (directly similar) normalized triangle \( \Delta' = \Delta'(z) = (0, 1, z) \), namely \( \Delta'(\hat{z}_1, \hat{z}_2, \hat{z}_3) = (0, 1, \hat{z}_2 - \hat{z}_1, \hat{z}_3 - \hat{z}_0) \), where \( \Delta'(\infty) = (0, 1, \infty) \).
means $\Delta' = (0, 0, 1)$ for $z_0 = z_1$. The shape $f(z)$ of the normalized triangle $\Delta'(z)$ is the Möbius transformation

$$s = \sigma_{\Delta'}(z) = f(z) = \frac{\zeta z + 1}{z + \zeta} = \frac{\zeta - e^{i\pi/3}}{z - e^{-i\pi/3}} = \frac{1}{f(\hat{z})}$$

of the extended complex plane with inverse

$$z = f^{-1}(s) = \frac{\zeta s - 1}{\zeta - s} = -\frac{s - \zeta^2}{s - \zeta} = -f(-s) = \frac{1}{f^{-1}(\hat{z})}.$$

Triangles corresponding to different normalized triangles have different shapes: the equivalence classes of triangles are parametrized by their shape. $\sigma_{\Delta} = 0$ or $\infty$ if and only if $\Delta$ is equilateral or equilateral, respectively. A cyclic left shift $(z_0, z_1, z_2) \mapsto (z_1, z_2, z_0)$, i.e., a start from the next vertex of the triangle, causes a change $(\hat{z}_0, \hat{z}_1, \hat{z}_2) \mapsto (\hat{z}_0, \zeta \hat{z}_1, \zeta^2 \hat{z}_2)$ in the spectrum; an orientation reversing, i.e., a permutation of the last two vertices $\Delta = (z_0, z_1, z_2) \mapsto (z_0, z_2, z_1) = 3\Delta$, causes the same permutation of the Fourier coefficients:

$$\sigma(z_1, z_2, z_0) = \zeta \sigma(z_0, z_1, z_2), \quad \sigma(z_0, z_2, z_1) = \frac{1}{\sigma(z_0, z_1, z_2)}, \quad \sigma_\infty = \frac{1}{\sigma}.$$

A triangle $\Delta_1$ is thus directly similar to the proper $\Delta$ of shape $\sigma_{\Delta} = s$ with the same orientation if and only if $\sigma_{\Delta_1} = \zeta^k s, k \in \{0, 1, 2\}; \Delta_1$ is directly similar to the proper $\Delta$ with the inverse orientation if and only if $\sigma_{\Delta_1} = \zeta^k \frac{1}{s}; \Delta_1$ is inversely similar to the proper $\Delta$ with the same orientation if and only if $\sigma_{\Delta_1} = \zeta^k \bar{s}$; and $\Delta_1$ is inversely similar to the proper $\Delta$ with the inverse orientation if and only if $\sigma_{\Delta_1} = \zeta^k \frac{1}{\bar{s}}$.

Since $f(0) = \zeta^2, f(1) = 1, f(\infty) = \zeta$, and $f(e^{i\pi/3}) = 0$, $f$ maps the extended real axis (corresponding to the normalized degenerate triangles) to the unit circle and the extended upper half-plane to the unit disc: the modulus $|\sigma_{\Delta}|$ is thus $< 1$, $> 1$, or $= 1$ according as $\Delta$ is positively oriented and proper, negatively oriented and proper, or degenerate, respectively. Two nontrivial degenerate triangles $\Delta$ and $\Delta_1$ are similar if and only if $\sigma_{\Delta_1} = \zeta^k \sigma_{\Delta}$ or $\sigma_{\Delta_1} = \zeta^k \sigma_{\infty}, k \in \{0, 1, 2\}$. The degenerate normalized triangle $\Delta'(x), x \in \mathbb{R} \cup \{\infty\}$, and its shape $e^{i\varphi}$ are linked by

$$\varphi = \arg \sigma_{\Delta'}(x) = \frac{2\pi}{3} + 2 \arg(x - e^{i\pi/3}) = 2 \arctan \frac{\sqrt{3}}{1 - 2x}$$

and $x = \frac{\sin(\frac{\pi}{3} + \frac{\varphi}{2})}{\sin(\frac{\varphi}{2})}$: when $x$ runs rightwards over the whole extended real axis, $f(x) = \sigma_{\Delta'}(x)$ turns counterclockwise on the unit circle starting and ending at $\zeta$, which is the shape of $\Delta'(\infty) = (0, 0, 1)$.

For the normalized right-angled triangles (Figure 1), $f$ maps the extended imaginary axis to the circle $C$ of radius $\sqrt{3}$ centered at $-2$, the circle of radius $\frac{1}{2}$ centered at $\frac{1}{2}$ to the circle $C'$ of radius $\sqrt{3}$ centered at $1 - i\sqrt{3}$, and the extended vertical line through 1 to the circle $\zeta^2 C'$ of radius $\sqrt{3}$ centered at $1 + i\sqrt{3}$: the shape function $\sigma$ maps thus the right-angled triangles to the three circles $C, \zeta C$, and $\zeta^2 C$. 

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Convolution filters for triangles

Figure 1. Normalized right-angled triangles $(0, 1, z)$: loci of their vertex $z$ (dotted curves) and of their shape (plain circles)

Figure 2. Normalized isosceles (on the left) and automedian triangles $(0, 1, z)$: loci of their vertex $z$ (dotted curves) and of their shape (plain straight lines)

For the normalized isosceles triangles (Figure 2), $f$ maps the unit circle to the extended real axis, the extended vertical line through $\frac{1}{2}$ to the extended line through 0 and $\zeta$, and the circle of radius 1 centered at 1 to the extended line through 0 and $\zeta^2$: $\sigma$ maps thus the nonequilateral isosceles triangles to the punctured lines $\lambda \zeta^k$, $\lambda \in \mathbb{R} \setminus \{0\}$, $k = 0, 1, 2$. $s$ is the shape of an isosceles triangle if and only if $s \in \{s, \zeta s, \zeta^2 s\}$. The normalized isosceles $\Delta'(z)$ with apex $z = \frac{1 + i \tan \theta}{2}$, $|\theta| < \frac{\pi}{2}$, and base angles $\theta (< 0$ when $\text{Im} z < 0$) has the shape $f(z) = \frac{\tan \theta - \sqrt{3}}{\tan \theta + \sqrt{3}}$; the shape
Figure 3. Neuberg circles $C_R$: $|\sigma_{\Delta'}| = \left| \frac{z - e^{i\pi/3}}{z - e^{-i\pi/3}} \right| = R$ for the normalized triangles $\Delta' = (0, 1, z)$. $R = 0$, $\frac{4}{11}, \frac{4}{3}, 1, 2, 3, 10, \infty$

of the isosceles $\Delta'(\frac{1}{\sqrt{3}} e^{i\pi/6})$ with base angles $\frac{\pi}{6}$ is $-\zeta$. We set $\xi = -\frac{1}{\sqrt{3}} e^{i\pi/6} = \frac{1}{2} + \frac{i}{2\sqrt{3}} = \frac{1}{1-\zeta} = -\frac{1}{\sqrt{3}} i\zeta$.

A nontrivial triangle is automedian when it is (inversely) similar to its median triangle: by the median theorem, this is the case if and only if the vertex opposite to the middle side $u$ lies on the circle of radius $\frac{\sqrt{3}}{2} u$ centered at the midpoint of $u$ (as does the apex of the equilateral triangle erected on $u$), and also if and only if $2u^2$ is the sum of the other squared sides. The sides of a right-angled automedian triangle are proportional to $1 : \sqrt{2} : \sqrt{3}$, and an isosceles triangle is automedian exactly when it is equilateral. For the normalized automedian triangles (Figure 2), $f$ maps the circle of radius $\frac{\sqrt{3}}{2}$ centered at $\frac{1}{2}$ to the extended line through $0$ and $i\zeta = -e^{i\pi/6}$, the circle of radius $\sqrt{3}$ centered at $-1$ to the extended line through $0$ and $i\zeta^2$, and the circle $-C$ of radius $\sqrt{3}$ centered at $2$ to the extended imaginary axis: $\sigma$ maps thus the nonequilateral automedian triangles to the punctured lines $\lambda i \zeta^k$, $\lambda \in \mathbb{R} \setminus \{0\}$, $k = 0, 1, 2$. $s$ is the shape of an automedian triangle if and only if $-\bar{s} \in \{s, \zeta s, \zeta^2 s\}$. The normalized automedian triangle with vertex $z = \frac{1}{2} + \frac{\sqrt{3}}{2} e^{i\phi}$ has the shape $f(z) = \tan(\frac{\pi}{4} - \frac{\phi}{2}) e^{i\pi/6}$.

For $R \in [0, \infty]$, $f$ maps the circle $|z| = R$ to the Apollonius circle $| \frac{z - \zeta^2}{z - \zeta} | = R$,

i.e., to the circle $|s + \frac{1}{2} - i \frac{\sqrt{3} R^2 + 1}{R^2 - 1}| = \frac{\sqrt{3} R}{|R^2 - 1|}$ if $R \neq 1$ and to the extended real axis if $R = 1$. Conversely, $f^{-1}$ maps the circle $|s| = R$ to the Apollonius circle $C_R: \left| \frac{z + \zeta^2}{z + \zeta} \right| = \left| \frac{z - e^{i\pi/3}}{z - e^{-i\pi/3}} \right| = R$, (2) i.e., to the circle $|z - \frac{1}{2} + i \frac{\sqrt{3} R^2 + 1}{R^2 - 1}| = \frac{\sqrt{3} R}{|R^2 - 1|}$ if $R \neq 1$ and to the extended real axis if $R = 1$ (Figure 3). If $R \neq 1$, $C_R$ is a Neuberg circle [17, p. 287], [29],
i.e., the locus of the vertex \( z \) of the triangles \( \Delta' (z) \) with an appropriate constant Brocard angle \( \omega \) in the upper or lower half-plane according as \( R < 1 \) or \( R > 1 \): since the base side \( 1 \) of \( \Delta' (z) \) subtends the angle \( 2 \omega \) from the center of \( C_R \), one has (see also [10])

\[
\cot \omega = \sqrt{3} \left| \frac{R^2 + 1}{R^2 - 1} \right| \text{ for } R = |\sigma_{\Delta' (z)}|.
\]

If \( |\sigma_{\Delta' (z_0)}| = R \neq 0, 1, \infty \), the sides of \( \Delta' (z_0) \) issued from \( z_0 \in C_R \) cut \( C_R \) in one or two other points: these are vertices of normalized triangles inversely similar to \( \Delta' (z_0) \); the reflections of \( z_0 \) and of these vertices in the line \( \Re z = \frac{1}{2} \) give the other normalized triangles similar to \( \Delta' (z_0) \) with the same orientation [17, p. 289].

Figures 1, 2, and 3 show how many right-angled, isosceles, and automedian normalized triangles are equibrocardal with a shape of given modulus \( R \neq 0, 1, \infty \). There are two inversely similar automedian triangles with opposite shapes (and their companions with cyclically shifted vertices); there are two isosceles triangles with opposite shapes (and their companions), whose base angles are

\[
\theta_1 = \arctan \frac{\sqrt{3}(1 - R)}{1 + R}, \quad \theta_2 = \arctan \frac{\sqrt{3}(1 + R)}{1 - R}.
\]

Note that \( \tan \theta_1 \cdot \tan \theta_2 = 3 \) and that \( \theta_1, \theta_2 \) are \( < 0 \) when the orientation is negative. For \( R = 2 \pm \sqrt{3} \) or \( R \in ]2 - \sqrt{3}, 2 + \sqrt{3} [ \) there are one (isosceles) right-angled triangle or two inversely similar right-angled triangles with conjugate shapes (and their companions), respectively. Note that some of the above triangles may be simultaneously right-angled and isosceles or automedian.

We will prove later that two nontrivial triangles have opposite shapes (possibly after multiplying one of the shapes by \( \zeta^{\pm 1} \)) if and only if they are directly similar to the median triangle of each other.

\( f \) maps the circle \( C' \) through \( e^{i\pi/3}, e^{-i\pi/3} \), and \( z_0 \in C \setminus \{ e^{i\pi/3}, e^{-i\pi/3} \} \) to the line \( \lambda f(z_0), \lambda \in \mathbb{R} \cup \{ \infty \} \), and \( Z_0 = \frac{z_0 - 2}{z_0 - 1} \) lies on this circle since \( f \left( \frac{z - 2}{z - 1} \right) = -f(z) \). When \( z_0 \) is not real, \( \Delta'(z_0) \) is automedian if and only if the line through \( z_0 \) and \( Z_0 \) is horizontal or contains 0 or 1, because two equibrocardal triangles with a common angle are similar and because \( \Delta'(z_0) \) is directly similar to the median triangle of \( \Delta'(z_0) \). For each \( L \geq 0 \), \( f \) admits the values \( \pm L f(z_0) \) at the points \( z_\pm \) given by the intersections of \( C' \) with the Apollonius circle \( C_L(f(z_0)) \) (Figure 4).

\[
\tilde{\sigma}(z_0, z_1, z_2) = \frac{z - z_0}{z_1 - z_0} \cdot \frac{z - z_1}{z_2 - z_1} \quad [36], \quad \text{which is invariant under triangle translation and}
\]

under homothety and rotation, is a shape function that is more intuitive than \( \sigma \). One has \( \tilde{\sigma}(0, 1, z) = z, \quad \sigma_0(0, 1, z) = \zeta \tilde{\sigma}(z, e^{-i\pi/3}, e^{i\pi/3}), \quad \text{and} \quad \sigma(z_0, z_1, z_2) = \sigma(0, 1, \tilde{\sigma}(z_0, z_1, z_2)).\)

4. Convolution filters

We consider a filter \( T_\Gamma : \mathbb{C}^3 \rightarrow \mathbb{C}^3 \) given by the cyclic convolution \(*\) with a fixed triangle \( \Gamma = (c_0, c_1, c_2) \), i.e., by a circulant matrix:

\[
T_\Gamma : \Delta \mapsto \Delta \ast \Gamma = (z_0, z_1, z_2) \ast (c_0, c_1, c_2) = (z_0, z_1, z_2) \begin{pmatrix} c_0 & c_1 & c_2 \\ c_2 & c_0 & c_1 \\ c_1 & c_2 & c_0 \end{pmatrix}.
\]
The $k$th entry of $\Delta \ast \Gamma = \Gamma \ast \Delta$ is $\sum_{\ell=0}^{2} z_{\ell} c_{k-\ell \pmod{3}}, k = 0, 1, 2$.

Since the operator $\ast$ is bilinear and since $e_{k} \ast e_{\ell} = \begin{cases} 3 e_{k} & (k = \ell) \\ (0, 0, 0) & (k \neq \ell) \end{cases}$, one has

$$T_{\Gamma}(\Delta) = \Delta \ast \Gamma = \left( \sum_{k=0}^{2} \hat{z}_{k} e_{k} \right) \ast \left( \sum_{\ell=0}^{2} \hat{\ell}_{\ell} e_{\ell} \right) = \sum_{k=0}^{2} 3 \hat{c}_{k} \hat{z}_{k} e_{k},$$

i.e., $\Delta \ast \Gamma = 3 \hat{\Delta} \cdot \hat{\Gamma}$, where $\cdot$ is the entrywise product: the Fourier base is a base of eigenvectors of the convolution $T_{\Gamma}$ with eigenvalues $3 \hat{c}_{k}$ (and the ratio $\frac{\hat{c}_{0}}{\hat{c}_{2}}$ of the two “equilateral” eigenvalues is the shape of $\Gamma$). $T_{\Gamma}$ maps thus trivial, equilateral and nequilateral triangles to triangles of the same category or to trivial triangles. $T_{\Gamma}(\Delta)$ and $\Delta$ always have the same centroid if and only if $c_{0} + c_{1} + c_{2} = 1$, which means $\hat{c}_{0} = \frac{1}{3}$; the centroid is always translated to the origin if and only if $\hat{c}_{0} = 0$.

The image by $T_{\Gamma}$ of the Dirac triangle $(1, 0, 0) = \frac{1}{3}(e_{0} + e_{1} + e_{2})$ of shape 1 is $\Gamma$, the impulse response of the filter $T_{\Gamma}$, and the filter output is the convolution of the input with the impulse response, as for every linear time-invariant filter; the convolution with the Dirac triangle is the identity. It is immediate that

$$\sigma_{\Delta \ast \Gamma} = \sigma_{\Delta} \sigma_{\Gamma}$$

when $\Gamma$ and $\Delta$ are not trivial, except that $\Delta \ast \Gamma$ is trivial when $\Gamma$ and $\Delta$ are equilateral with opposite orientations (i.e., $0 \cdot \infty = \text{trivial}$). When $\Gamma$ and $\Delta$ are mixed, $\Delta \ast \Gamma$ is degenerate if and only if $|\sigma_{\Delta} \sigma_{\Gamma}| = 1$. When $\Delta$ is mixed, $T_{\Gamma}(\Delta)$ can have any prescribed shape $\sigma_{\Delta}$, and this is the case if and only if $\sigma_{\Gamma} = \frac{\sigma_{\Delta}}{\sigma_{\Delta}}$.

One has $T_{\Gamma_{1}} \circ T_{\Gamma_{2}} = T_{\Gamma_{2}} \circ T_{\Gamma_{1}} = T_{\Gamma_{1} \ast \Gamma_{2}}$. The iterates of $T_{\Gamma}$ are the convolution filters $T^{n}_{\Gamma} : (z_{0}, z_{1}, z_{2}) \rightarrow \sum_{k=0}^{2} (3 \hat{c}_{k})^{n} \hat{z}_{k} e_{k}, n \in \mathbb{N}$, with centroid $(3 \hat{c}_{0})^{n} \hat{z}_{0}$. The sum of the squared distances between the centroid and the vertices of $T^{n}_{\Gamma}(\Delta)$ is

Figure 4. $\Delta'(z_{0})$ and $\Delta'(Z_{0})$ have opposite shapes; $\Delta'(z_{\pm})$ have opposite shapes $\pm L_{\sigma_{\Delta}}$ for $L = \frac{1}{2}$.
Thus it is a right-hand ear if and only if \( K \) by the centroid-preserving convolution with
\[ z \]
with fixed \( a \neq 0 \) and fixed \( b \). A (generalized) Kiepert triangle consists of the
apices of ears that are erected on the sides of the initial triangle (opposite to the
vertices in order) and that all have the shape of the normalized
\[ \Delta \]
...
only shape-preserving Kiepert filter is the convolution with $K(\frac{1}{2}) = (0, \frac{1}{2}, \frac{1}{2}) = \frac{1}{6}(2e_0 - e_1 - e_2)$ of shape 1, which maps a triangle $\Delta$ to its medial triangle, i.e., to $\frac{1}{2}$ times the half-turned $\Delta$ (shrunk and rotated about the centroid). More generally, $T_{K(x)}(\Delta)$, $x \in \mathbb{R}$, is the $(1 - x)$-medial triangle of $\Delta$ [11]: this is the introducing Emmerich transformation! Since $K(x)$ has the shape of $\Delta'(\frac{x}{1-x})$, $\sigma_{K(x)}$ turns anticlockwise on the unit circle, starting and ending at $-1 = \sigma(0,1,-1)$ excluded, as $x$ grows on the real axis.

Note that $\sigma_{K(x)}$ is real if and only if $\text{Re } z = \frac{1}{2}$, i.e., if and only if the added ears are isosceles with equal angles at 0 and 1 (this corresponds to the classical Kiepert triangles). To get isosceles ears with base angles $|\theta| < \frac{\pi}{2}$, one has to convolve with the isosceles $K_{\text{iso}}(\theta) = K\left(\frac{1 + i \tan \theta}{2}\right)$ with apex angle $2|\theta|$ and shape $\frac{1 - \sqrt{3} \tan \theta}{1 + \sqrt{3} \tan \theta}$; the ears are right-hand or left-hand according as $\theta \geq 0$ or $\theta \leq 0$. Since $K_{\text{iso}}(\frac{\pi}{3}) = \frac{1}{3}(1, -2, 1)$, one can retrieve $\Delta$ from $\Delta * K_{\text{iso}}(\frac{\pi}{3})$ by constructing $\frac{1}{2}(T_{K_{\text{iso}}(\frac{\pi}{3})}(\Delta) + T_{K_{\text{iso}}(\frac{\pi}{3})}^2(\Delta)) = \Delta$: this is Lemoine’s problem [21]. With the same idea, one finds $\Delta = \frac{1}{2}(T_{K_{\text{iso}}(-\frac{\pi}{3})}(\Delta) + T_{K_{\text{iso}}(-\frac{\pi}{3})}^2(\Delta))$, $\Delta = 2T_{K(\frac{3}{2})}^2(\Delta) - T_{K(\frac{3}{2})}(\Delta)$, and $\Delta = T_{K(x)}(\Delta) + \frac{1}{3z-(\xi-\eta)}(T_{K(z)}^3(\Delta) - T_{K(\frac{\pi}{3})}(\Delta))$ for $z \neq \xi, \eta$. A triangle $\Delta$ and its classical Kiepert triangle $\Delta * K_{\text{iso}}(\theta)$ are always perspective [4, 20] and, if $\Delta$ is proper and nonisosceles, the perspectors form the equilateral Kiepert hyperbola of $\Delta$ (Figure 5) as $\theta$ runs from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, i.e., the hyperbola through the vertices of $\Delta$, the centroid $G$, and the orthocenter $H$ (which is the perspector in the limit case $|\theta| = \frac{\pi}{2}$). We now look at this limit case more closely.

The vertices of $\Delta * K_{\text{iso}}(\theta)$ tend for $|\theta| \to \frac{\pi}{2}$ to the infinite points of the altitudes of $\Delta$ on the line at infinity (when $\Delta$ has three different vertices). On the other hand, $\lim_{|\theta| \to \frac{\pi}{2}} - \sigma_{K_{\text{iso}}(\theta)} = -1 = \sigma(0,1,-1)$: the limit shape of $\Delta * K_{\text{iso}}(\theta)$ when $|\theta| \to \frac{\pi}{2}$ is thus the shape of $\Delta * (0,1,-1)$ for any nontrivial triangle $\Delta$. This is clear geometrically: with $\Delta = (z_0, z_1, z_2)$, the angles of $\Delta * K_{\text{iso}}(\theta)$ tend for $|\theta| \to \frac{\pi}{2}$ to the angles of the (quarter-turned) triangle

$$(z_2 - z_1, z_0 - z_2, z_1 - z_0) = \Delta * (0,1,-1),$$

whose vertices are the tips of the vectors $z_1 \to z_2$, $z_2 \to z_0$, $z_0 \to z_1$ starting from 0 (Figure 6). Convolving $\Delta$ with the normalized $(0,1,-1)$ gives a scaled down and quarter-turned “equally shaped” bounded copy of “$\Delta$ with infinite similar isosceles ears”. Here is another description of the filter $T_{(0,1,-1)}$: it translates $\Delta = (z_0, z_1, z_2)$ to $\Delta_1 = \Delta - z_0e_0 = (w_0, w_1, w_2)$ with centroid at the origin and then blows up the $\frac{1}{3}$-medial triangle (with cyclically shifted vertices) $\Delta_2 = \Delta_1 * (\frac{1}{3}, \frac{2}{3}, 0)$ to $3\Delta_2 = (W_0, W_1, W_2) = \Delta * (0,1,-1)$ (Figure 6). $T_{(0,1,-1)}^2(\Delta)$ is three times the half-turned $\Delta_1$ (enlarged and rotated about the origin). $\Delta * (0,1,-1)$ is thus directly similar to $\Delta * (\frac{1}{3}, \frac{2}{3}, \frac{1}{3}) = \Delta * K(\frac{3}{2})$, the $\frac{1}{3}$-medial triangle of $\Delta$, and two triangles have opposite shapes (possibly after multiplying one of the shapes by $e^{\pm 1}$) if and only if they are directly similar to the $\frac{1}{3}$-medial triangle of each other: since the $\frac{1}{3}$-medial triangle is directly similar to the $\frac{2}{3}$-medial triangle and to the median triangle (Figure 6), this proves that two triangles have
opposite shapes (possibly up to a factor $\zeta^{\pm 1}$) if and only if they are directly similar to the median triangle of each other. The well-known fact that the median triangle of the median triangle is directly similar to the start triangle is equivalent to the fact that the shape of $(0, 1, -1) * (0, 1, -1)$ is 1. If $\Delta'(z)$ is a normalized triangle, remember that $\Delta'(z) * (0, 1, -1)$ and $\Delta'(\frac{z-2}{2z-1})$ have the same shape $-\sigma_{\Delta'(z)}$ (Figure 4).

The following theorem is almost immediate.

**Theorem 2.** (1) The convolution filter $T_\Gamma$ is bijective if and only if $\Gamma$ has a full spectrum, i.e., if and only if $\Gamma = a(0, 1, -1) + b e_0$ with $a, b \neq 0$, or $\Gamma = a K(z) + b e_0$ with $z \notin \{\xi, \bar{\xi}\}$, $a \neq 0$, $a + 3b \neq 0$; if $\Gamma = (c_0, c_1, c_2)$, the inverse filter is then the convolution with $\sum_{k=0}^{2} \frac{1}{9e_k} e_k$. 
(2) A pequilateral filter (of shape 0), i.e., a convolution with \( aK(\xi) + b\epsilon_0, a \neq 0 \), maps nequilateral triangles to trivial triangles and all other nontrivial triangles to pequilateral triangles. A nequilateral filter (of shape \( \infty \)), i.e., a convolution with \( aK(\xi) + b\epsilon_0, a \neq 0 \), maps pequilateral triangles to trivial triangles and all other nontrivial triangles to nequilateral triangles.

(3) A proper nonequilateral filter \( \Gamma \), i.e., with \( |\sigma_\Gamma| \neq 0, 1, \infty \), is smoothing: its action on equilateral triangles is shape-preserving; and according to the filter is positively or negatively oriented, i.e., according as \( 0 < |\sigma_\Gamma| < 1 \) or \( 1 < |\sigma_\Gamma| < \infty \), the iterates \( T^n_\Gamma(\Delta) \) of every mixed \( \Delta \) are eventually positively or negatively oriented and have a (never reached) pequilateral or nequilateral limit in shape, respectively.

(4) A nontrivial filter \( \Gamma \) is degenerate (with \( |\sigma_\Gamma| = 1 \)) if and only if \( \Gamma = aK + b\epsilon_0 \) for some \( a \neq 0 \) and some degenerate \( K = K(x), x \in \mathbb{R} \), or \( K = (0, 1, -1) \).

(5) If \( \Gamma = a(0, 1, -1) + b\epsilon_0, a \neq 0 \), and if \( \Delta = (z_0, z_1, z_2) \), one has

\[
T^n_\Gamma(\Delta) = (-i\sqrt{3}a)^n(\hat{z}_1e_1 + (-1)^n\hat{z}_2e_2) + (3b)^n\hat{z}_0e_0.
\]

If \( \Delta \) is mixed, these iterates \( T^n_\Gamma(\Delta) \) are 2-periodic in shape with shape \((-1)^n\sigma_\Delta\) (Figures 6 and 4).

(6) If \( x \in \mathbb{R} \), one has

\[
\tilde{K}(x) = (\hat{c}_0, \hat{c}_1, \hat{c}_1) = \frac{1}{6} (2, -1 + i\sqrt{3}(2x - 1), -1 - i\sqrt{3}(2x - 1)) \quad \text{and}
\]

\[
\sigma_{K(x)} = -e^{i2\arg(\xi-x)} = e^{i2\kappa(x)} \quad \text{with } \kappa(x) = \arctan(\sqrt{3}(2x - 1)).
\]

If \( \Gamma = aK(x) + b\epsilon_0 \) for some \( a \neq 0 \), one has

\[
T^n_\Gamma(\Delta) = (-3|\hat{c}_1|e^{-i\kappa(x)}a)^n(\hat{z}_1e_1 + e^{i2\kappa(x)}\hat{z}_2e_2) + (a + 3b)^n\hat{z}_0e_0 \quad \text{(5)}
\]

for every \( \Delta = (z_0, z_1, z_2) \), where \( 3|\hat{c}_1| = \sqrt{3x^2 - 3x + 1} \). When \( \Delta \) is mixed, these iterates \( T^n_\Gamma(\Delta) \) are periodic or nonperiodic in shape (with chaotic behavior) according as \( \kappa(x) / \pi \) is rational or irrational, respectively (the period in similarity may be shorter than the period in shape). The period length is \( m = 1 \) if and only if \( \kappa(x) = 0 \), i.e., if and only if \( K(x) = K(\frac{1}{2}) \); \( m \geq 2 \) is the minimal period length (the same for all mixed \( \Delta \)) if and only if \( \sigma_\Gamma = \sigma_{K(x)} = e^{i2\pi\ell/m} \) for some integer \( \ell \in [1, m - 1] \) coprime to \( m \), i.e., if and only if \( m \geq 3 \) and

\[
x = \frac{1}{2} + \frac{1}{2\sqrt{3}}\tan\left(\frac{\ell\pi}{m}\right)
\]

for some integer \( \ell \in [1, m - 1] \) coprime to \( m \) (note that the period 2 corresponds to \( K = (0, 1, -1) \) and that \( x \in [0, 1] \) exactly when \( \ell / m \notin \left[\frac{1}{3}, \frac{2}{3}\right] \); \( T^n_{K(x)}(\Delta) \) is then given by the homothety of ratio \((-1)^{m+\min(\ell, m-\ell)}(3|\hat{c}_1|)^m\) about the centroid of \( \Delta \).

(7) When \( \Delta \) is a mixed triangle and when \( \Gamma \) is degenerate but not trivial, the shapes \((-1)^n\sigma_\Delta \) or \( e^{i2\kappa(x)}\sigma_\Delta \) of the iterates \( T^n_\Gamma(\Delta) \) lie on the circle \( |s| = |\sigma_\Delta| = R \in [0, \infty] \) and correspond to (equibrocardal) normalized triangles \((0, 1, z)\) with vertex \( z \) on the Neuberg circle \( \mathcal{C}_R \) given by (2) if \( R \neq 1 \), and to nontrivial degenerate triangles if \( R = 1 \) (Figures 3 and 4). If \( m \)-periodic, \( m \geq 1 \), these shapes are the vertices (in order) of a regular oriented \( \{m/\ell\}\)-gon with start at \( \sigma_\Delta \) for some
\( \ell \in [0, m - 1] \) coprime to \( m \) (note that this is the Fourier base polygon \( \tilde{e}_\ell \) of \( \mathbb{C}^m \) (Figure 7) multiplied by \( \sigma_\Delta \) and that the choice of another \( \ell \) coprime to \( m \) only changes the order of the shapes); if nonperiodic, i.e., if \( \kappa(x) \) is an irrational multiple of \( \pi \), these shapes \( T^n_1(\Delta) \) are dense on the circle \( |s| = |\sigma_\Delta| = R \), i.e., the accumulation triangles (in shape) of the sequence \( (T^n_1(\Delta))_{n \geq 0} \) are the (equiangular) normalized triangles \((0, 1, z)\) with vertex \( z \) on the Neuberg circle \( \mathcal{C}_R \) if \( R \neq 1 \) and the nontrivial degenerate triangles if \( R = 1 \).

(8) When \( \Delta \) is a proper positively oriented mixed triangle with the shape of \( \Delta'(z_0) \) and when \( x \) grows on the whole real axis, the shape of \( \Delta * K(x) \) travels counterclockwise over the whole circle \( |s| = |\sigma_\Delta| \) starting and ending at \(-\sigma_\Delta\) excluded, whereas the vertex \( z \) of the normalized triangle \( \Delta'(z) \) with the shape of \( \Delta * K(x) \) turns counterclockwise over the whole Neuberg circle \( \mathcal{C}_{|\sigma_\Delta'(z_0)|} \) of Figure 4 starting and ending at \( Z_0 \) excluded. \( \Delta * (0, 1, -1) \) fills the holes \(-\sigma_\Delta\) and \( Z_0 \). The rotation on the Neuberg circle is clockwise if \( \Delta \) is negatively oriented. In the degenerate case \( |\sigma_\Delta| = 1 \), \( z \) runs rightwards over the whole extended real axis starting and ending at \( Z_0 = \frac{z_0 - 2}{2m - 1} \) excluded.

Note that the result (6) or an equivalent one can be found in [22, 36, 16, 11] and that the last two parts of Theorem 2 probably furnish the solution that the quite incomprehensible paper [19] aimed at. Note also that the iterates \( T^n_1(\Delta) \) of a mixed \( \Delta \) are 3- periodic in shape if and only if \( \Gamma = aK(1) + b \) or \( aK(0) + b, a \neq 0, K(1) \) and \( K(0) \) causing the left and right shifts of \( \Delta \)’s vertices.

The proper nontrilaterable triangle \( \Delta \) is directly similar to its \((1 - t)\)-medial triangle \( \Delta * K(t), t \in \mathbb{R} \), if and only if \( \sigma_{K(t)} = \zeta^k \), \( k \in \{0, 1, -1\} \), i.e., if and only if \( t = \frac{1}{2}, 1, 0 \), \( \Delta * K(t) \) is then the medial triangle or a copy of \( \Delta \) with cyclically shifted vertices. The proper nontrilaterable triangle \( \Delta \) is inversely similar to \( \Delta * K(t) \) if and only if

\[
t = \frac{1}{2} + \frac{1}{2\sqrt{3}} \tan \left( \frac{k\pi}{3} - \arg \sigma_\Delta \right), \quad k \in \{0, 1, -1\}.
\]

(7)
the solutions \( t \) depend only on \( \arg \sigma_\Delta \pmod{\pi} \), i.e., the set of solutions for the nonequilateral normalized triangle \( \Delta'(z_0) = (0,1,z_0) \) remains the same for all nonequilateral \( \Delta'(z) \) with \( z \) on the circle \( C' \) through \( e^{i\pi/3}, -e^{i\pi/3} \), and \( z_0 \) (Figure 4). These solutions \( t \) are again \( 0, 1/3, 1 \) if \( \Delta \) is isosceles, they are \( 1/3, 2/3, 1 \) and the infinite point of the \( t \)-axis if \( \Delta \) is automedian (then \( \sigma_{K(t)} = \zeta^ke^{i\pi/3}, k = -1, 0, 1 \), and the infinite solution corresponds to \( \Delta* (0,1,-1) \)), and the solutions are three different real numbers \( t \neq 0, 1/3, 2/3, 1 \) in the other cases. It is clear that \( 1/3 \) and \( 2/3 \) appear only for automedian triangles, because the \( 2/3 \)- and \( 1/3 \)-medians \( \Delta* K(1/3) \) and \( \Delta* K(2/3) \), of shape \( e^{i\pi/3}\sigma_\Delta \), are directly similar to the median triangle (Figure 6). If \( \Delta \) is degenerate and nontrivial, \( \Delta \) and \( \Delta* K(t) \) are similar if and only if \( t = 0 \), \( 1/2 \), 1, or a solution of (7). Figure 8 shows the values of \( t \) for which \( \Delta'(z_0) \) or \( K(t) \) is similar to \( \Delta'(z_0) \) as functions of the midpoint \( M \) of \( C' \), whose radius is \( r = \sqrt{M^2 - M + 1} \): by considering a real point \( M \pm r \) of \( C' \) and by plugging \( \arg \sigma_{\Delta'(M \pm r)} \) given by (1) into (7), one obtains for \( k = 0, 1, -1 \) the solutions

\[
t_0 = \frac{M - 1}{M - 2}, \quad t_1 = \frac{1}{M + 1}, \quad t_{-1} = \frac{M}{2M - 1}
\]

(8)
given cyclically by \( t_{k+1}(M) = t_k\left(1 - \frac{1}{M}\right) \) (plain, dashed, and dotted hyperbolas of Figure 8, respectively); the values of \( M \) corresponding to isosceles or automedian triangles are the midpoints of the dotted circles of Figure 2. If \( \Delta \) is neither trivial, nor isosceles, nor automedian, exactly two of the \( (1 - t) \)-medial triangles \( \Delta* K(t) \) inversely similar to \( \Delta \) are inscribed in \( \Delta \), and the sum of these two values of \( t \), on each side of \( 1/2 \), is never 1. The cosine law in \( \Delta'(z_0) \) with \( z_0 = x_0 + iy_0 \) on \( C' \) and sides \( a = |z_0 - 1|, b = |z_0|, c = 1 \), and the equation \( M^2 - M + 1 = r^2 = (x_0 - M)^2 + (b^2 - x_0^2) \) give \( M = \frac{1 - b^2}{2a^2 - b^2 - c^2} \), and thus by (8)

\[
1 - t_0 = \frac{a^2 - b^2}{2a^2 - b^2 - c^2}, \quad 1 - t_1 = \frac{b^2 - c^2}{2b^2 - c^2 - a^2}, \quad 1 - t_{-1} = \frac{c^2 - a^2}{2c^2 - a^2 - b^2}
\]

(9)

which are correct by similarity for any nonequilateral and nontrivial triangle \( ABC \) with sides \( a, b, c \) opposite to the vertices (then \( M = \frac{c^2 - b^2}{a^2 - c^2}, t_0 = \frac{M - c}{M - 2c}, t_1 = \frac{M - b}{M - 2b} \) if \( A = 0, B = c > 0 \). Since the set \( \{t_0, t_1, t_{-1}\} \) is invariant under a cyclic shift of the vertices, each set of solutions \( t \) appears for three different \( M \) in Figure 8: for \( M = \frac{a^2 - b^2}{a^2 - c^2}, M = \frac{b^2 - c^2}{b^2 - a^2}, \) and \( M = \frac{c^2 - a^2}{c^2 - b^2} \) (whose product is \(-1\) in the nonisosceles case) given cyclically by \( M \mapsto 1 - \frac{1}{M} \) (once in \( \left[\frac{1}{2}, 1\right], \) once in \( \left[-1, \frac{1}{2}\right] \), and once in the rest of the extended real axis). Note that (9) already appeared in [18] with another proof.

Start for example from a proper nonequilateral isosceles \( \Delta \) with real shape \( \lambda \) (after a cyclic permutation of its vertices, if necessary); choose \( x = \frac{1}{2} + \frac{1}{2\sqrt{3}} \tan\left(\frac{\pi}{4m}\right) \) for some odd \( m \geq 3 \) and some integer \( \ell \in [1, m - 1] \) coprime to \( m \): the \( m \)-periodic shapes of the equibrocardal \( T^n_{K(x)}(\Delta) \) are the vertices in order of \( \lambda \ell \) in \( C_m \); the cycle contains no automedian triangle; \( T^k_{K(x)}(\Delta) \) and \( T^{m-k}_{K(x)}(\Delta) \) form a pair of inversely similar triangles for each integer \( k \in [1, \frac{m-1}{2}] \) since their shapes are
complex conjugate; if $m$ is moreover coprime to 3, the cycle contains no two directly similar triangles (hence no other isosceles triangle than $\Delta$) and the minimal period length in similarity is also $m$; if $m$ is divisible by 3, the first third of the cycle in shape contains no two directly similar triangles, the other thirds are obtained by multiplying the shapes of the first third by $\zeta^{\pm 1}$, and the minimal period length in similarity is $\frac{m}{3}$. (Since similarity does not depend on the order of the vertices, the condition that the isosceles $\Delta$ has a real shape can in fact be dropped.)

Start now from a proper nonequilateral automedian $\Delta$ with, say, purely imaginary shape $i\lambda$ (after a cyclic permutation of its vertices, if necessary); choose $x$ and $\ell$ as above, but with an even $m \geq 2$: the $m$-periodic shapes of the equibrocardal $T_{K(x)}^m(\Delta)$ are the vertices in order of $i\lambda \tilde{e}_\ell$ in $\mathbb{C}^m$; each triangle in the second half of the cycle is directly similar to the $\frac{1}{3}$-medial triangle of the triangle with the same rank in the first half (and conversely), since they have opposite shapes; $T_{K(x)}^{m/2}(\Delta)$ is thus inversely similar to the automedian $\Delta$; each pair $T_{K(x)}^k(\Delta)$ of the first half consists of inversely similar triangles (except $T_{K(x)}^{m/4}(\Delta)$ if $m$ is divisible by 4), and this property is inherited by the second half. If $m$ is moreover coprime to 3, the cycle contains no two directly similar triangles and no other automedian triangles than $\Delta$ and $T_{K(x)}^{m/2}(\Delta)$; if $m$ is divisible by 3 (hence by 6), there are no two directly similar triangles in the same third of the cycle in shape and the only automedian triangles are the $T_{K(x)}^{km/6}(\Delta)$, which are directly and inversely similar to $\Delta$ for $k = 0, 2, 4$ and $k = 1, 3, 5$, respectively. We consider for example the automedian triangle $\Delta_0$ with sides $1$, $\sqrt{2}$, $\sqrt{3}$ and $x = \frac{3+\sqrt{6}-\sqrt{3}}{6} \approx 0.61957$ corresponding to $m = 8$, $\ell = 1$ to construct the iterated $(1 - x)$-medial triangle
of \( \Delta_0 \), whose minimal period in shape has length 8: \( \sigma_{\Delta_0} = (\sqrt{3} - \sqrt{2})i \) when \( \Delta_0 = (0, 1, 1 + \sqrt{2}i) \), and \( \sigma_{\Delta_{k+1}} = e^{i\pi/4}\sigma_{\Delta_k} \) (Figure 9). \( \Delta_k \) and \( \Delta_{k+4} \) are similar to the \( \frac{1}{7} \)-medial triangle of each other and are thus nonsimilar if not automedian; \( \Delta_0 \) and \( \Delta_1 \) are right-angled, automedian and inversely similar; \( \Delta_1 \) and \( \Delta_3 \) are inversely similar, as are \( \Delta_5 \) and \( \Delta_7 \), and these four triangles are neither isosceles, nor automedian, nor right-angled (Figures 1 and 2); \( \Delta_2 \) and \( \Delta_6 \) are isosceles; by (5), \( \Delta_8 \) is obtained from \( \Delta_0 \) by a half-turn about the centroid of \( \Delta_0 \) followed by a homothety of ratio \( \frac{17}{4} - 3\sqrt{2} \approx 0.00736 \) about this centroid. Figure 10 shows the corresponding normalized triangles with their vertices \( f^{-1}(e^{ik\pi/4}\sigma_{\Delta_0}) \) on the Neuberg circle: note the position of the vertices of the inversely similar and of the isosceles triangles [17, p. 289].

**Remark.** In [22], the triangle \( S_{p,q}(\Delta) = (w_0, w_1, w_2) \) is constructed cyclically from the proper \( \Delta = (z_0, z_1, z_2) \) for real \( p \) and \( q \) with \( pq \neq 1 \): \( w_0 \) is the intersection of the cevian issued from \( z_1 \) dividing the side \( z_2 \rightarrow z_0 \) in the ratio \( p : (1 - p) \) and of the cevian issued from \( z_0 \) dividing the side \( z_1 \rightarrow z_2 \) in the ratio \( (1 - q) : q \). \( S_{p,1-p}(\Delta) \) is the \( p \)-Rooth triangle of [11].

The centroid-preserving transformation \( S_{p,q} \) amounts to the convolution of \( \Delta \) with the degenerate triangle \( \frac{1}{1-pq}(p(1-q), (1-p)(1-q), q(1-p)) \): by Theorem 1, \( S_{p,q}(\Delta) \) is thus obtained for \( p(2q-3) \neq -1 \) by submitting \( \Delta \ast K \left( \frac{q-p}{p(2q-3)+1} \right) \) to a homothety of ratio \( \frac{p(2q-3)+1}{1-pq} \) with respect to the centroid of \( \Delta \). Remember that \( \Delta \ast K(x) = S_{0,x}(\Delta) \) is the \( (1-x) \)-medial triangle of \( \Delta \) for \( x \in \mathbb{R} \). If
\[ p(2q - 3) = -1, \quad S_{p,q} \] is the convolution with \[ \frac{1-2q}{3}(0,1,-1) + \frac{1}{3}e_0 \] and \[ S_{p,q}(\Delta) \] is the \( \frac{1}{3} \)-medial triangle of \( \Delta \) transformed by a cyclic left shift of its vertices and by a homothety of ratio \( 1 - 2q \) with respect to the centroid of \( \Delta \); \( S_{\frac{1}{2},\frac{1}{2}} \) maps in particular every triangle to its centroid.

For every fixed real \( h \neq 0, -1, \pm 2 \), the different pairs \( (p_1, q_1) = \left( \frac{h-1}{h+2}, \frac{h+1}{h+2} \right) \) and \( (p_2, q_2) = \left( \frac{1}{h+2}, \frac{1-h}{2} \right) \) are such that \( S_{p_1,q_1}(\Delta) \) and \( S_{p_2,q_2}(\Delta) \) have the same vertices: \( S_{p_1,q_1}(\Delta) \) is the \( \frac{1}{3} \)-medial triangle transformed by a homothety of ratio \( h \) with respect to its centroid, and \( S_{p_2,q_2}(\Delta) \) is the same triangle after a cyclic left shift of the vertices; \( h = -3 \) corresponds in particular to \( S_{\frac{1}{2},\frac{1}{2}} \) and \( S_{-1,2} \).

For \( p \in \mathbb{R} \setminus \{ \frac{1}{2} \} \), the triangle \( \frac{1-p(1-p)}{1-2p} S_{p,1-p}(\Delta) = \Delta * \frac{1}{1-2p}(p^2, p(1-p), (1-p)^2) \) is a \( p \)-median triangle of \( \Delta \) [11], \( i.e.\), a triangle whose sides are parallel to and as long as the cevians connecting the corresponding vertices of \( \Delta \) and \( \Delta * K(1-p) \). The \( \frac{1}{2} \)-median or median triangle is for example \( \Delta * \left( -\frac{1}{2}, 0, \frac{1}{2} \right) \) with centroid 0.

5. Isosceles ears

The added ears are isosceles if and only if the shape of the convolving triangle is real or \( \infty \) and different from \( -1 \). The convolving triangle is then \( aK_{iso}(\theta) + be_0 \), \( a \neq 0 \), for some \( K_{iso}(\theta) = K(\frac{1+i\tan \theta}{2}, |\theta| < \frac{\pi}{2} \), with shape \( \frac{1-\sqrt{3}\tan \theta}{1+\sqrt{3}\tan \theta} = \frac{2}{1+\sqrt{3}\tan \theta} - 1 \). Since a product of real shapes is real, a composition of convolutions with \( a_1K_{iso}(\theta_1) + b_1e_0 \) and \( a_2K_{iso}(\theta_2) + b_2e_0 \) is again a convolution with some \( aK_{iso}(\theta) + be_0 \), or with \( a(0, 1, -1) + be_0 \), or with a trivial triangle. Since \( \sigma K_{iso}(\theta) \)
is real and nonzero for \( \theta \neq \pm \frac{\pi}{6} \), the triangles \( \Delta \) and \( \Delta \ast K_{\text{iso}}(\theta), \theta \neq \pm \frac{\pi}{6} \), are by Figure 2 always simultaneously isosceles or automedian, respectively.

If \( \Delta'(z_0) \) is a normalized triangle with finite \( z_0 \neq e^{\pm i\pi/3} \), the shapes of the classical Kiepert triangles \( \Delta'(z_0) \ast K_{\text{iso}}(\theta), |\theta| < \frac{\pi}{2}, \) and of \( \Delta'(z_0) \ast (0, 1, -1) \) form the extended line \( \lambda \sigma_{\Delta'(z_0)} \) and are the shapes of the normalized triangles \( \Delta'(z) \) with vertex \( z \) on the circle \( C' \) through \( e^{i\pi/3}, e^{-i\pi/3} \), and \( z_0 \) (Figure 4). As \( \theta \) grows on \( [0, \frac{\pi}{2}[ \), the vertex \( z \) of the triangle \( \Delta'(z) \) with the shape of \( \Delta'(z_0) \ast K_{\text{iso}}(\theta) \) moves from \( z_0 \) to \( Z_0 \) (excepted) on the arc of \( C' \) that contains \( e^{i\pi/3} \).

Each proper nonequilateral \( \Delta \) of shape \( s \) has exactly two degenerate classical Kiepert triangles \( \Delta \ast K_{\text{iso}}(\theta) \): for \( \theta = \arctan \frac{|s|-1}{\sqrt{3(1+|s|)}} \) and \( \theta = \arctan \frac{|s|-1}{\sqrt{3(|s|-1)}} \); these are inward Kiepert triangles, i.e., the ears intersect \( \Delta \)'s interior, their position does not depend on the vertices' order in \( \Delta \), they correspond to the real points of the circle \( C' \) and mark the transition between the positively and negatively oriented \( \Delta \ast K_{\text{iso}}(\theta) \). These two perpendicular [20] degenerate triangles intersect at the centroid of \( \Delta \) (which is their common centroid) and they are parallel to the asymptotes of \( \Delta \)'s Kiepert hyperbola (Figure 5). The degenerate classical Kiepert triangles of a nontrivial degenerate \( \Delta \) are the medial triangle and the almost Kiepert \( \Delta \ast (0, 1, -1) \).

We determine now the conditions under which the triangles \( \Delta \) and \( \Delta \ast K_{\text{iso}}(\theta) \) are similar. We suppose that \( \Delta \) is proper, not equilateral, and positively oriented, and we exclude the evident case \( K_{\text{iso}}(0) = K(\frac{1}{2}) \). We set \( \sigma_{\Delta} = s \) with \( 0 < |s| < 1 \) and denote the shape \( \frac{1-\sqrt{3}\tan\theta}{1+\sqrt{3}\tan\theta} \) of \( K_{\text{iso}}(\theta) \) by \( \mu \in \mathbb{R} \setminus \{-1, 1\} \). \( \mu \) is the shape of a triangle inversely similar to \( \Delta \) if and only if \( \mu s = \frac{1}{s} \), i.e., if and only if \( \mu = \frac{1}{|s|^2} (> 1) \): this corresponds to inward ears. \( \mu \) is the shape of a triangle directly similar to \( \Delta \) if and only if \( \mu s \in \left\{ \frac{1}{s}, \frac{1}{s^2}, \frac{1}{s^2}, \frac{1}{s}, \frac{1}{s^2}, \frac{1}{s^2}, \frac{1}{s}, \frac{1}{s^2}, \frac{1}{s^2}, \frac{1}{s}, \frac{1}{s^2}, \frac{1}{s^2} \right\} \), and this is possible in two cases:

1. \( s = \lambda \zeta^k, \lambda \in [-1, 1[, k = 0, 1, 2, \) and \( \mu = \frac{1}{|s|^2} (> 1) \): \( \Delta \) is then isosceles and \( K_{\text{iso}}(\theta) \) is the same as in the inversely similar case.
2. \( s = \lambda i\zeta^k, \lambda \in [-1, 1[, k = 0, 1, 2, \) and \( \mu = \frac{1}{|s|^2} (< -1) \): \( \Delta \) is then automedian.

Writing \( |s| = R \) and dropping now the condition that \( \Delta \) is positively oriented, one sees that the proper nonequilateral \( \Delta \) and \( \Delta \ast K_{\text{iso}}(\theta), \theta \neq 0 \), are inversely similar (and inversely oriented) exactly when \( \theta = \Theta_1 = \arctan \frac{R^2-1}{\sqrt{3(R^2+1)}} \), and directly similar (and inversely oriented) exactly when \( \Delta \) is automedian and \( \theta = \Theta_2 = \arctan \frac{R^2+1}{\sqrt{3(R^2-1)}} \). Note that \( 0 < |\Theta_1| < \frac{\pi}{6} < |\Theta_2| < \frac{\pi}{2} \) and that these are inward ears.

By (3), \( \Delta \ast K_{\text{iso}}(\Theta_1) = \Delta \ast K_{\text{iso}}(\mp \omega) \) is the first Brocard triangle (the base angles of the ears are the Brocard angle of \( \Delta \) ): it is thus immediate that the first Brocard triangle has the centroid of \( \Delta \). Figure 11 shows two equibrocardal isosceles triangles with the same base (the apex angle of the right triangle is \( 30^\circ \)), their first Brocard triangles and their Brocard points. Equibrocardal isosceles triangles \( \Delta_{1,2} \) with the same orientation have base angles \( \theta_1, \theta_2 \) given by (4), and
\( \tan \Theta_1 = \frac{-2 \tan \theta_{1,2}}{3 + \tan^2 \theta_{1,2}} \); if one neglects the order of the vertices, the first Brocard triangle is obtained from \( \Delta_{1,2} \) by a homothety of ratio \(-\frac{1}{2} + \frac{3}{3 + \tan^2 \theta_{1,2}} \in \left] -\frac{1}{2}, \frac{1}{2} \right[ \) with respect to the centroid of \( \Delta_{1,2} \) (the homothety ratio can be computed directly by considering the normalized \( \Delta = (0, 1, \frac{1 + i \tan \theta_{1,2}}{2}) \)); the sign of the homothety ratio changes at \( \theta = \frac{\pi}{3} \), and the homothety ratios corresponding to \( \theta_1 \) and to \( \theta_2 \) differ only by their sign (Figure 11).

Suppose now that \( \Delta \) is proper and automedian with Brocard angle \( \omega \). One has \( 3 \tan|\Theta_2| = \cot \omega \) by (3). On the other side, \( \cot \omega = 3 \cot \gamma \) when \( \gamma \) is the middle angle of any nontrivial automedian triangle [5, p. 17]: thus \( |\Theta_2| = \frac{\pi}{2} - \gamma \), the ears’ apex angle is \( 2\gamma \), the apex of the ear over the middle side is the circumcenter of \( \Delta \) by the inscribed angle theorem, and the sides of \( \Delta \star K_{\text{iso}}(\Theta_2) \) are perpendicular to the sides of \( \Delta \) with middle side opposite to the circumcenter. \( \Delta \star K_{\text{iso}}(\Theta_2) \) is thus obtained from \( \Delta \) by a quarter-turn about the centroid of \( \Delta \) followed by a homothety of ratio \( \frac{|\cot \varphi|}{2} \) about this centroid, where \( \varphi \) (positive or negative as the orientation of \( \Delta \)) is the angle between the middle side of \( \Delta \) and its median (the homothety ratio can be computed directly by considering the normalized automedian triangle \( (0, 1, \frac{1}{2} + \frac{\sqrt{3}}{2} e^{i \varphi}) \)). Notice that one has also \( \tan \Theta_2 = \frac{-1}{\sqrt{3} \sin \varphi} \), \( \tan \Theta_1 = \tan(\mp \omega) = \frac{-1}{\sqrt{3}} \sin \varphi \), and that the similarity ratio of the first Brocard triangle to \( \Delta \) is \( \frac{1}{4} \left| 1 + e^{i 2 \varphi} \right| \in [0, \frac{1}{2}] \) (0 when \( \Delta \) is equilateral).

The following theorem is proven.

**Theorem 3.** (1) A proper nonequilateral triangle \( \Delta \) is similar to exactly three or two of its classical Kiepert triangles according as it is automedian or not: it is directly similar to its medial triangle, inversely similar to its first Brocard triangle, and, if automedian, directly similar to the triangle constructed from inward isosceles ears with apex angle twice the middle angle of \( \Delta \) (the apex of the ear over the middle side is then the circumcenter of \( \Delta \) and the triangles are perpendicular to
each other. If $\Delta$ is automedian, the two cases $\lim_{\theta \to \frac{\pi}{2}} \Delta \ast K_{iso}(\pm \theta)$ are asymptotically inversely similar to $\Delta$.

(2) A proper nonequilateral triangle $\Delta$ is isosceles or automedian if and only if it is directly similar to one of its classical Kiepert triangles other than the medial triangle: the corresponding sides are then parallel in the isosceles and perpendicular in the automedian case.

![Figure 12. The (up to similarity) only nontrivial triangle with a congruent classical Kiepert triangle, together with its first Brocard triangle](image)

The automedian triangle with sides $1$, $\sqrt{1 - \frac{\sqrt{3}}{5}}$, and $\sqrt{1 + \frac{\sqrt{3}}{5}}$ (Figure 12) is up to similarity the only nontrivial triangle with a congruent classical Kiepert triangle: the cotangent of the angle formed by the side 1 and its median is 2. The base angles of the inward ears are $\arctan \frac{\sqrt{5}}{3}$ for the directly congruent classical Kiepert triangle and $\omega = \arctan \frac{1}{\sqrt{15}}$ for the first Brocard triangle, whose similarity ratio to the initial triangle is $\frac{1}{\sqrt{3}}$.

6. Sequences of outward and inward Kiepert triangles

If the triangle $\Delta = (w_0, w_1, w_2)$ is positively oriented or degenerate and if $\text{Im} \ z \geq 0$, the outward and inward Kiepert triangles of $\Delta$ corresponding to outward and inward ears directly similar to the triangle $(0, 1, z)$ are defined by $\Delta^{\text{out}}(z) = \Delta \ast K(z)$ and $\Delta^{\text{in}}(z) = \Delta \ast K(1 - z)$, respectively. If $\Delta$ is negatively oriented and proper, $\Delta^{\text{out}}(z) = \Delta \ast K(1 - z)$ and $\Delta^{\text{in}}(z) = \Delta \ast K(z)$. The outward ears added to
Δ do not intersect Δ’s interior. We set further ∆_out(∇) = ∆_in(∇) = ∆*(0, 1, −1).

Given a sequence \((z_n)_{n \in \mathbb{N}}\) with values in \(\{z \in \mathbb{C} \mid \text{Im } z \geq 0\} \cup \{\nabla\}\), and starting from \(\Delta = \Delta_0 = \Delta_0^0 = \Delta_0^0\), we define inductively \(\Delta_{n+1}^\text{out} = \Delta_{n}^\text{out}(z_n)\) and \(\Delta_{n+1}^\text{in} = \Delta_{n}^\text{in}(z_n)\). The centroid remains the centroid of \(\Delta_0\) until the first convolution with \((0, 1, -1)\), if any, moves it to the origin.

**Theorem 4.** Let \((z_n)_{n \geq 0}\) be a sequence of symbols \(\nabla\) and of complex numbers with nonnegative imaginary part; set \(\xi = \frac{1}{\sqrt{3}} e^{i\pi/6}\) and \(\frac{\nabla - \xi}{\nabla - 6} = -\sigma(0, 1, -1) = 1\).

If \(z_n \in \mathbb{C}\), denote by \(\tilde{z}_n = \frac{1 + i \tan \theta_n}{2}\), with

\[
\theta_n = \arctan \frac{|z_n - \xi| - |z_n - 6|}{\sqrt{3}(|z_n - \xi| + |z_n - 6|)} \in [0, \frac{\pi}{6}],
\]

the south pole of the Apollonius circle \(\frac{z - \xi}{z - 6}\) = \(R\) containing \(z_n\), and set \(\theta_n = 0\) if \(z_n = \nabla\). Let \(\Delta_0\) be a nontrivial and nonequilateral triangle.

The following properties are equivalent.

1. The sequence \((\Delta_{n+1}^\text{out})_{n \geq 0}\) constructed from \(\Delta_0\) and \((z_n)\) converges in shape to an equilateral limit.
2. \(\lim_{n \to \infty} \prod_{k=0}^{n} \frac{z_k - \xi}{z_k - 6} = 0\).
3. The sequence of classical Kiepert triangles \((\Delta_{n+1}^\text{out})_{n \geq 0}\) constructed from \(\Delta_0\) and \((\tilde{z}_n)\) converges in shape to an equilateral limit.
4. \(\lim_{n \to \infty} \prod_{k=0}^{n} \frac{1 - \sqrt{3}\tan \theta_k}{1 + \sqrt{3}\tan \theta_k} = 0\).
5. \(\theta_n = \frac{\pi}{6}\) for some \(n\) or \(\sum_{n=0}^{\infty} \theta_n = \infty\).

The existence of the equilateral limit does not depend on the choice of the nonequilateral \(\Delta_0\).

One can also allow to choose each \(\tilde{z}_n\) freely as the north or south pole of the Apollonius circle (in order to always leave \(\tilde{z}_n = z_n\) when \(\text{Re } z_n = \frac{1}{2}\) for example). One has then to take \(\theta_n = \arctan \frac{|z_n - \xi| + |z_n - 6|}{\sqrt{3}(|z_n - \xi| - |z_n - 6|)}\) when \(\theta_n \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]\), and the condition \(\sum_{n=0}^{\infty} \theta_n = \infty\) has to be replaced by \(\sum_{n=0}^{\infty} \min(\theta_n, \frac{\pi}{2} - \theta_n) = \infty\), as we showed in [23].

Theorem 4 generalizes [34], where the iterated convolution with a constant triangle \(K_{\text{iso}}(\theta)\) is analyzed, and [23], where only classical iterated Kiepert triangles are considered.

**Proof:** If \(\Delta_0\) is positively oriented or degenerate, so are all \(\Delta_{n+1}^\text{out}\), and \(\sigma_{\Delta_{n+1}^\text{out}} = \sigma_{\Delta_0} \prod_{k=0}^{n} \frac{\xi - z_k}{\xi - 6}\). \(\lim_{n \to \infty} \sigma_{\Delta_{n+1}^\text{out}} = 0\) means \(\lim_{n \to \infty} \prod_{k=0}^{n} \frac{z_k - \xi}{z_k - 6} = 0\) since \(\Delta_0\) is not equilateral, and each factor in this product is constant on the corresponding Apollonius circle. We proved the equivalence of (4) and (5) in [23].

If \(\Delta_0\) is negatively oriented and proper, so are all \(\Delta_{n+1}^\text{out}\). Since \(\sigma_{K(1-z)} = 1/\sigma_{K(z)}\), one has then \(\sigma_{\Delta_{n+1}^\text{out}} = \sigma_{\Delta_0} \prod_{k=0}^{n-1} \frac{z_k - \xi}{z_k - 6}\); every factor of this product has a modulus \(\geq 1\), and one has \(\lim_{n \to \infty} \sigma_{\Delta_{n+1}^\text{out}} = \infty\) under the same conditions as in the first case. \(\square\)
A nonequilateral limit shape for \((\Delta_n^{\text{out}})\) is only possible when \(\frac{z_n - \xi}{z_n - \zeta}\) converges to 1, i.e., when \(\lim_{n \to \infty} z_n = \frac{1}{2}\). But this condition is not sufficient: if \(z_n = \frac{1}{2} + \frac{1}{n}\), \(n \geq 1\), for example, \(\arg(\sigma_{\Delta_n^{\text{out}}})\) diverges like the harmonic series.

Suppose that the sequence \((\Delta_n^{\text{out}})\) has no equilateral limit shape: the infinite product \(\prod_{n=0}^{\infty} \left|\frac{z_n - \xi}{z_n - \zeta}\right|\), whose factors lie in \([0, 1]\), has then a limit \(L \in [0, 1]\) (and any such \(L\) can be obtained by an appropriate choice of the \(z_n\)'s). The accumulation points of \((\sigma_{\Delta_n^{\text{out}}})\) lie on the circle \(|s| = L|\sigma_{\Delta_0}|\) \((\leq |\sigma_{\Delta_0}| \leq 1)\) if \(\Delta_0\) is positively oriented or degenerate, and on the circle \(|s| = \frac{1}{L}|\sigma_{\Delta_0}|\) \((\geq |\sigma_{\Delta_0}| > 1)\) if \(\Delta_0\) is negatively oriented and proper: these accumulation shapes correspond to equibrocardal normalized triangles \((0, 1, z)\) with vertex \(z\) on the Neuberg circle \(\frac{z - e^{i\pi/3}}{z - e^{-i\pi/3}} = L^\pm 1|\sigma_{\Delta_0}|\), respectively (Figure 4). The sequence \((\sigma_{\Delta_n^{\text{out}}})\) can tend to the accumulation circle with any behavior since the argument of each factor \(\frac{z_n - \xi}{z_n - \zeta}\) can be changed arbitrarily by replacing \(z_n\) by an appropriate number of the Apollonius circle \(\left|\frac{z - \xi}{z - \zeta}\right| = R\) containing \(z_n\).

Suppose that a sequence of classical iterated Kiepert triangles is given by the successive convolutions with \(K_{\text{iso}}(\theta_n)\), \(0 \leq \theta_n < \frac{\pi}{2}\), \(n \geq 0\), and that this sequence \((\Delta_n)\) starts from a positively oriented nonequilateral \(\Delta_0\) and has no equilateral limit, i.e., \(\sum_{n=0}^{\infty} \min(\theta_n, \frac{\pi}{2} - \theta_n) < \infty\): the one or two accumulation points of the sequence \((\theta_n)\) belong to \(\{0, \frac{\pi}{2}\}\), and the corresponding subsequences converge rapidly to the accumulation points, since the sum of the corresponding \(\theta_n\) or \(\frac{\pi}{2} - \theta_n\) is finite; the convolution with \(K_{\text{iso}}(\theta_n)\) multiplies the shape by \(\lambda_n = \sigma_{K_{\text{iso}}(\theta_n)} = \frac{1 - \sqrt{3}\tan\theta_n}{1 + \sqrt{3}\tan\theta_n} \in [-1, 1]\), and \(\lambda_n\) is about 1 or -1 when \(\theta_n\) is near to 0 or to \(\frac{\pi}{2}\), respectively. As above, \(\prod_{n=0}^{\infty} |\lambda_n| = L \in [0, 1]\), but the \(\lambda_n\) are now real: \((\sigma_{\Delta_n})\) has thus the nonzero limit \(\sigma_{\Delta_0} \prod_{n=0}^{\infty} \frac{1 - \sqrt{3}\tan\theta_n}{1 + \sqrt{3}\tan\theta_n} \in \pm L|\sigma_{\Delta_0}|\) if all \(\lambda_n\) are eventually positive, i.e., if \(\lim_{n \to \infty} \theta_n = 0\). Otherwise, the infinite subsequences of the \(\sigma_{\Delta_n}\) with positive and negative \(\prod_{k=0}^{n-1} \lambda_k\) have nonzero limits \(L|\sigma_{\Delta_0}|\) and \(-L|\sigma_{\Delta_0}|\), respectively, and the sequence \((\sigma_{\Delta_n})\) has exactly two accumulation points given by \(\pm L|\sigma_{\Delta_0}|\) with \(0 < L < 1\). The limit or accumulation shapes and the \(\sigma_{\Delta_n}\) are shapes of classical Kiepert triangles of \(\Delta_0\) of the form \(\Delta_0 * K_{\text{iso}}(\theta)\), \(0 \leq \theta < \frac{\pi}{2}\). If existing, the two accumulation shapes \(\pm L|\sigma_{\Delta_0}|\) are also the shapes of two normalized equibrocardal triangles \(\Delta'(z_\pm)\) that are directly similar to the median triangle of each other. If the normalized \(\Delta'(z_0)\) has the shape of \(\Delta_0\), i.e., if \(z_0 = \frac{\sigma_{\Delta_0} - 1}{\xi - \sigma_{\Delta_0}}\), the vertices \(z_\pm\) are given by the intersections of the Neuberg circle \(\left|\frac{z - e^{i\pi/3}}{z - e^{-i\pi/3}}\right| = L|\sigma_{\Delta_0}|\) with (the upper half of) the circle \(C'\) through \(e^{i\pi/3}, e^{-i\pi/3}, \) and \(z_0\), on both sides of \(e^{i\pi/3}\) (Figure 4): \(z_+\) lies on the arc between \(e^{i\pi/3}\) and \(z_0\), \(z_-\) between \(e^{i\pi/3}\) and \(Z_0 = \frac{z_0}{z_0 - 2}\), which corresponds to the shape \(-\sigma_{\Delta_0}, z_0\) and \(Z_0\) have a strictly positive imaginary part if \(\Delta_0\) is proper and are real if \(\Delta_0\) is degenerate.

Since \(\Delta\) and an outward Kiepert triangle of \(\Delta\) are always simultaneously in the category “positively oriented or degenerate” or in the category “negatively oriented
and proper”, an iterated outward Kiepert triangle remains unchanged if one modifies the order of the successive convolutions. This is not the case for an iterated inward triangle, because the orientation may change after a convolution, leading to the next convolution with \( K(1 - z) \) instead of \( K(z) \) for example, and these orientation changes may depend on the order of the convolutions. Note that the inward Kiepert triangle of an outward Kiepert triangle given by the same ears is in general not even similar to \( \Delta_0 \): if \( \Delta_0 \) is proper, nonequilateral, and positively oriented with shape \( s_0 \), for example, and if \( \sigma_{K(z)} = s \) with \( 0 < |s| < |s_0| \), \( \Delta_0 \cdot K(1 - z) \) is negatively oriented and the end triangle \( \Delta_0 \cdot K(1 - z) \) \( \cdot K(1 - z) \) is also negatively oriented with shape \( \frac{\sigma_{s}}{\sigma_{s_0}} \) of modulus \( \frac{1}{|s_0|} \); this end triangle is never similar to \( \Delta_0 \).

Except when the sequence has been stopped before by \( 0 \cdot \infty \) or \( \infty \cdot 0 \), the shape of \( \Delta_{n+1}^m \), \( n \geq 0 \), is given recursively by \( \sigma_{\Delta_{n+1}^m} = \frac{z_n - z}{\xi - z_n} \cdot \sigma_{\Delta_n^m} \) if \( |\sigma_{\Delta_n^m}| \leq 1 \) (then \( |\sigma_{\Delta_{n+1}^m}| \geq |\sigma_{\Delta_n^m}| \)) and by \( \sigma_{\Delta_{n+1}^m} = \frac{z_n - z}{\xi - z_n} \cdot \sigma_{\Delta_n^m} \) if \( |\sigma_{\Delta_n^m}| > 1 \) (then \( |\sigma_{\Delta_{n+1}^m}| \leq |\sigma_{\Delta_n^m}| \)). The turning points between stretching factors \( \left( \frac{z_n - z}{\xi - z_n} \right)^{\pm 1} \) of modulus \( \leq 1 \) or \( \geq 1 \), respectively, depend also on \( \Delta_0 \). By choosing \( \Delta_0 \) and the \( z_n \) appropriately, the sequence \( \{\Delta_n^m\} \) can thus have any behavior in shape within these constraints.

The iterated first Brocard triangles of a proper nonequilateral \( \Delta_0 \) are for example alternately inversely and directly similar to \( \Delta_0 \) when \( \Delta_0 \) is not isosceles, and all directly similar to \( \Delta_0 \) when \( \Delta_0 \) is isosceles. If \( \Delta_0 \) is automedian and if the ears are isosceles with constant apex angle twice the middle angle of \( \Delta_0 \), the iterated inward Kiepert triangles are all directly similar to \( \Delta_0 \): the automedian triangle of Figure 12 initiates in particular a 4-periodic sequence given by quarter-turns about the centroid.

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On the Conic Through the Intercepts of the Three Lines Through the Centroid and the Intercepts of a Given Line

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Abstract. Let $L$ be a line intersecting the sidelines of triangle $ABC$ at $X$, $Y$, $Z$ respectively. The lines joining these intercepts to the centroid give rise to six more intercepts on the sidelines which lie on a conic $\mathcal{Q}(L, G)$. We show that this conic (i) degenerates in a pair of lines if $L$ is tangent to the Steiner inellipse, (ii) is a parabola if $L$ is tangent to the ellipse containing the trisection points of the sides, (iii) is a rectangular hyperbola if $L$ is tangent to a circle $\mathcal{C}_G$ with center $G$. We give a ruler and compass construction of the circle $\mathcal{C}_G$. Finally, we also construct the two lines each with the property that the conic $\mathcal{Q}(L, G)$ is a circle.

1. Introduction

In the plane of a triangle $ABC$, consider a line $\mathcal{L}$ intersecting the sidelines $BC$, $CA$, $AB$ respectively at $X$, $Y$, $Z$. Consider also three lines, $\mathcal{L}_a$ through $X$ intersecting $CA$, $AB$ at $Y_a$, $Z_a$, $\mathcal{L}_b$ through $Y$ intersecting $AB$, $BC$ at $Z_b$, $X_b$, and $\mathcal{L}_c$ through $Z$ intersecting $BC$, $CA$ at $X_c$, $Y_c$. The six points $X_b$, $X_c$, $Y_c$, $Y_a$, $Z_a$, $Z_b$ lie on a conic $\mathcal{Q}$ whose equation can be determined as follows.

Let $L = px + qy + rz = 0$ be the equation of $\mathcal{L}$ in homogeneous barycentric coordinates with respect to triangle $ABC$, so that

$X = (0 : r : -q), \quad Y = (-r : 0 : p), \quad Z = (q : -p : 0)$.

Suppose further that $\mathcal{L}_a$ joins $X$ to a point $P_1 = (f_1 : g_1 : h_1)$, $\mathcal{L}_b$ joins $Y$ to a point $P_2 = (f_2 : g_2 : h_2)$, and $\mathcal{L}_c$ joins $Z$ to a point $P_3 = (f_3 : g_3 : h_3)$.
so that these lines are represented by linear equations \( L_a = 0, L_b = 0, L_c = 0 \) respectively. These linear forms are

\[
L_a = \begin{vmatrix} x & y & z \\ 0 & r & -q \\ f_1 & g_1 & h_1 \end{vmatrix} = (qg_1 + rh_1)x - qf_1y - rf_1z, \\
L_b = -pg_2x + (rh_2 + pf_2)y - rg_2z, \\
L_c = -ph_3x - qh_3y + (pf_3 + qg_3)z.
\]

**Proposition 1.** The polynomial \( L_a L_b L_c - L(P_1) L(P_2) L(P_3) xyz \) is divisible by \( px + qy + rz \).

**Proof.** If we put \( x = -\frac{qy + rz}{p} \), then

\[
L_a = (qg_1 + rh_1)x - f_1(qy + rz) = (qg_1 + rh_1)x + f_1 \cdot px = L(P_1)x, \\
L_b = L(P_2)y, \\
L_c = L(P_3)z.
\]

It follows that, regarded as a polynomial in \( x \), \( L_a L_b L_c - L(P_1) L(P_2) L(P_3) xyz \), evaluated at \( x = -\frac{qy + rz}{p} \) is equal to 0. Thus, the cubic polynomial is divisible by \( px + qy + rz \). \( \square \)

Since \( L_a L_b L_c - L(P_1) L(P_2) L(P_3) xyz \) is divisible by \( px + qy + rz \), the remaining quadratic factor yields a conic \( \mathcal{Q} \) containing the six points \( X_b, X_c, Y_c, Y_a, Z_a, Z_b \). For example, if we regard the given line \( \mathcal{L} \) as the trilinear polar of \( P = (u : v : w) \), and take \( L_a, L_b, L_c \) to be the lines \( AX, BY, CZ \) respectively, then \( \mathcal{Q} \) is the circumconic \( uz + vzx + wxy = 0 \) with perspector \( P \).

### 2. Preliminaries on conics

We shall make use of the following basic results on conics associated with a triangle. Other preliminary results of triangle geometry can be found in [3]. Consider a conic with barycentric equation

\[ C : \alpha x^2 + \beta y^2 + \gamma z^2 + 2\lambda yz + 2\mu zx + 2\nu xy = 0. \]

Since this equation can be expressed in the form

\[
\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} \alpha & \nu & \mu \\ \mu & \lambda & \gamma \\ \nu & \beta & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0,
\]

we call \( M := \begin{pmatrix} \alpha & \nu & \mu \\ \nu & \beta & \lambda \\ \mu & \lambda & \gamma \end{pmatrix} \) the matrix of the conic \( C \). The adjoint matrix of \( M \), namely,

\[
M^\# = \begin{pmatrix} \beta \gamma - \lambda^2 & \lambda \mu - \gamma \nu & \nu \lambda - \beta \mu \\ \lambda \mu - \gamma \nu & \gamma \alpha - \mu^2 & \mu \nu - \alpha \lambda \\ \nu \lambda - \beta \mu & \mu \nu - \alpha \lambda & \alpha \beta - \nu^2 \end{pmatrix},
\]
defines the dual conic of $\mathcal{C}$. It is easy to verify that
\[ MM^\# = M^\# M = (\det M)I_3. \]

Let $G := \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$. The characteristic of $M$ is the number
\[ \chi(M) := GM^\#G^t \]
\[ = 2\mu\nu + 2\nu\lambda + 2\lambda\mu - \lambda^2 - \mu^2 - \nu^2 \]
\[ - 2(\alpha\lambda + \beta\mu + \gamma\nu) + \beta\gamma + \gamma\alpha + \alpha\beta. \]

**Proposition 2.** The conic defined by the symmetric matrix $M$ (a) degenerates into a pair of lines if $\det M = 0$,
(b) is a parabola if $\det M \neq 0$ and $\chi(M) = 0$,
(c) has center $Q = GM^\#$ if $(\det M)\chi(M) \neq 0$.

**Proof.** Let $Q = GM^\#$. For arbitrary $1 \times 3$ matrix $P$ and real number $t$,
\[ (Q + tP)M(Q + tP)^t = QMQ^t + (PMQ^t + QMP^t)t + (PMP^t)t^2 \]
\[ = QMQ^t + 2(QMP^t)t + (PMP^t)t^2 \]
\[ = (GM^\#)M(GM^\#)^t + 2((GM^\#)MP^t)t + (PMP^t)t^2 \]
\[ = (\det M)(GM^\#G^t) + 2(\det M)(GP^t)t + (PMP^t)t^2 \]
\[ = (\det M)\chi(M) + 2(\det M)(GP^t)t + (PMP^t)t^2. \]

Consider the following possibilities.

(a) If $\det M = 0$, this equation becomes $(Q + tP)M(Q + tP)^t = (PMP^t)t^2$. With $t = 0$, this shows that $Q$ is a point on the conic, and for every point $P$ not on the conic, the line $PQ$ intersects $\mathcal{C}$ only at $Q$ (corresponding to $t = 0$). On the other hand, if $P$ lies on the conic, then every point on the line $PQ$ also lies on the conic. It follows that the conic is a union of two lines, possibly identical. The two lines are parallel (possibly identical) if $\chi(M) = 0$. Otherwise, they intersect at a finite point $Q$.

(b) If $\det M \neq 0$ and $\chi(M) = 0$, then $Q$ is an infinite point on the conic $\mathcal{C}$, and for every finite point $P$, there is at most one nonzero $t$ for which $tQ + P$ lies on the conic. This shows that $\mathcal{C}$ is a parabola whose axis has infinite point $Q$.

(c) Suppose $\det M \neq 0$ and $\chi(M) \neq 0$. In this case, $Q = GM^\#$ is a finite point. For every infinite point $P$ (satisfying $PG^t = 0$), we have
\[ (Q + tP)M(Q + tP)^t = (\det M)\chi(M) + (PMP^t)t^2. \]

There are at most two infinite point $P$ satisfying $PMP^t = 0$ (which are the infinite points of the asymptote when $\mathcal{C}$ is a hyperbola). Apart from these infinite points, if the line through $Q$ with infinite point $P$ intersects $\mathcal{C}$ at two real points, these intersections are symmetric with respect to $Q$. This shows that $Q$ is the center of the conic. \hfill \square

\footnote{The symbol $G$ also denotes the centroid of triangle $ABC$, which has homogeneous barycentric coordinates $(1 : 1 : 1)$.}
The equation of the conic \( C \) can be rewritten in the form
\[
\theta yz + \varphi zx + \psi xy + (x + y + z)(\alpha x + \beta y + \gamma z) = 0,
\]
where
\[
\theta = 2\lambda - (\beta + \gamma), \quad \varphi = 2\mu - (\gamma + \alpha), \quad \psi = 2\nu - (\alpha + \beta).
\]

**Proposition 3.** Suppose \((\det M)\chi(M) \neq 0\) and \(\theta\varphi\psi \neq 0\). The conic \( C \) is homothetic to the circumconic
\[
C_0 : \quad \theta yz + \varphi zx + \psi xy = 0.
\]
The ratio of homothety is the square root of \(\frac{4\det M}{\theta\varphi\psi}\).

**Proof.** By Proposition 2(c), the center of the conic \( C \) is the point \(Q = GM^\#\). The circumconic \( C_0 \) (with \(\theta, \varphi, \psi\) given by (3)) has matrix
\[
M_0 = \frac{1}{2} \begin{pmatrix} 0 & \psi & \varphi \\ \psi & 0 & \theta \\ \varphi & \theta & 0 \end{pmatrix}.
\]
Its center is the point
\[
Q_0 = (\theta(\varphi + \psi - \theta) : \varphi(\psi + \theta - \varphi) : \psi(\theta + \varphi - \psi)).
\]

For the matrix \(M_0\) in (4), we have
(i) \(\det(M_0) = \frac{\theta\varphi\psi}{4}\),
(ii) \(\chi(M_0) = \frac{1}{3}(2\varphi\psi + 2\psi\theta + 2\theta\varphi - \theta^2 - \varphi^2 - \psi^2) = \chi(M)\) by substitutions using (3).

Note that
\[
M = M_0 + \frac{1}{2} \begin{pmatrix} \alpha & \beta & \gamma \\ \alpha & \beta & \gamma \\ \alpha & \beta & \gamma \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \alpha & \alpha & \alpha \\ \beta & \beta & \beta \\ \gamma & \gamma & \gamma \end{pmatrix} = M_0 + \frac{1}{2}(G^xL + L^xG),
\]
where \(L = (\alpha \beta \gamma)\).

Let \(P = (u : v : w)\) be an infinite point. By (1), the point \(Q + tP\) lies on the conic \( C \) if and only if
\[
(PMP^t)t^2 + (\det M)\chi(M) = 0.
\]

Now,
\[
PMP^t = P \left( M_0 + \frac{1}{2}(G^xL + L^xG) \right) P^t = PM_0P^t + \frac{1}{2}(PG^x)(LP^t) + \frac{1}{2}(PL^x)(PG^t)^t = PM_0P^t.
\]
Applying equation (6) to the circumconic \( C_0 \), by replacing \( M \) by \( M_0 \), we conclude that the intersection of \( C_0 \) with the line through \(Q_0\) with the same infinite
point $P$ is $Q_0 + TP$ with $T$ given by

$$(PM_0 P^t)T^2 + (\det M_0)\chi(M_0) = 0,$$

$$(PM P^t)T^2 + \frac{\theta \varphi \psi \chi(M)}{4} = 0. \tag{7}$$

Comparing equations (6) and (7), we conclude that parallel lines (with infinite point $P$) through $Q$ and $Q_0$ intersect the conics $\mathcal{C}$ and $\mathcal{C}_0$ respectively at points $Q + tP$ and $Q_0 + TP$ with

$$\frac{t^2}{T^2} = \frac{4 \det M}{\theta \varphi \psi},$$

which is independent of the infinite point $P$ (for which $PM P^t \neq 0$). Furthermore, the line joining $Q_0 + TP$ to $Q + tP$ intersects the line $Q_0Q$ at a fixed point $S$ such that $\frac{SQ}{SQ_0} = \frac{t}{T}$ (see Figure 2). This shows that the two conics are homothetic at $S$, with ratio of homothety equal to $\frac{t}{T}$, the square root of $\frac{4 \det M}{\theta \varphi \psi}$. \hfill \Box

Remark. The conic $\mathcal{C}$ contains an infinite point if and only if $\chi(M) \leq 0$. \footnote{Proof: Putting $x = -(y + z)$ into the equation of $\mathcal{C}$, we obtain a quadratic in $y$ and $z$ with discriminant $-4\chi(M)$.}

3. The conic $\mathcal{D}(\mathcal{L}, G)$ associated with the centroid

We shall study the special case when $P_1 = P_2 = P_3 = G$, the centroid of triangle $ABC$. Here,

$$L_a = (q + r)x - qy - rz,$$

$$L_b = -px + (r + p)y - rz,$$

$$L_c = -px - qy + (p + q)z,$$
and
\[ L(P_1) = L(P_2) = L(P_3) = p + q + r. \]

It follows that
\[ L_a L_b L_c - (p + q + r)^2 xyz = (px + qy + rz)Q(x, y, z) \]
where
\[
Q(x, y, z) = p(q + r)x^2 + q(r + p)y^2 + r(p + q)z^2 - (p(p + q + r) + 2qr)yz \\
- (q(p + q + r) + 2rp)zx - (r(p + q + r) + 2pq)xy.
\]

The conic \( \mathcal{D}(L, G) \) in question is defined by the equation \( Q(x, y, z) = 0 \). The matrix of the conic is
\[
M(L) = \frac{1}{2} \begin{pmatrix}
2p(q + r) & -r(p + q + r) - 2pq & -q(p + q + r) - 2rp \\
-r(p + q + r) - 2pq & 2q(r + p) & -p(p + q + r) - 2qr \\
-q(p + q + r) - 2rp & -p(p + q + r) - 2qr & 2r(p + q)
\end{pmatrix},
\]

We shall investigate the possibilities that \( \mathcal{D}(L, G) \) be (i) degenerate, (ii) a parabola, (iii) a rectangular hyperbola, (iv) a circle.

**Lemma 4.** Let \( L \) be the line \( px + qy + rz = 0 \). The matrix \( M(L) \) has
(a) determinant \( \det M(L) = -\frac{1}{4}(p + q + r)^4(qr + rp + pq) \),
(b) adjoint matrix
\[
M(L)^\# = \frac{(p + q + r)^2}{4} \begin{pmatrix}
-p^2 & 2qr + 2rp + pq & 2qr + rp + 2pq \\
2qr + 2rp + pq & -q^2 & qr + 2rp + 2pq \\
2qr + rp + 2pq & qr + 2rp + 2pq & -r^2
\end{pmatrix},
\]
and
(c) characteristic \( \chi(M(L)) = \frac{(p+q+r)^2}{4}(-p^2 - q^2 - r^2 + 10qr + 10rp + 10pq) \).

### 4. Degenerate \( \mathcal{D}(L, G) \)

From Proposition 2(a), the conic \( \mathcal{D}(L, G) \) is degenerate when
(i) \( p + q + r = 0 \), or
(ii) \( pq + qr + rp = 0 \).

(i) corresponds to the trivial case when the line \( L : px + qy + rz = 0 \) contains the centroid \( G = (1 : 1 : 1) \). The conic \( \mathcal{D}(L, G) \) is simply the line \( L \) counted twice.

In (ii), \( p : q : r \) being a point on the Steiner circum-ellipse \( xy + yz + zx = 0 \), the line \( px + qy + rz = 0 \) is tangent to the dual conic, which is the Steiner in-ellipse. This means that if \( L \) is tangent to the Steiner in-ellipse, then the conic \( \mathcal{D}(L, G) \) degenerates into two lines. Actually, these two lines complete with \( L \) a poristic triangle between the two ellipses (see Figure 3).

We shall henceforth assume \( p + q + r \neq 0 \) and \( qr + rp + pq \neq 0 \).
5. The lines $\mathcal{L}$ for which $\mathcal{Q}(\mathcal{L}, G)$ are parabolas

By Proposition 2(b) and Lemma 4(b), we conclude easily that $\mathcal{Q}(\mathcal{L}, G)$ is a parabola if and only if $(p : q : r)$ is a point on the ellipse

$$\mathcal{E}_0 : \quad -x^2 - y^2 - z^2 + 10yz + 10zx + 10xy = 0$$

with center $G$. Equivalently, the line $\mathcal{L}$ is tangent to the conic dual to $\mathcal{E}_0$. This is the ellipse

$$\mathcal{E}_0^* : \quad -2x^2 - 2y^2 - 2z^2 + 5yz + 5zx + 5xy = 0,$$

also with center $G$, and containing the trisection points of the sides of the triangle.

**Theorem 5** (Dergiades). The conic $\mathcal{Q}(\mathcal{L}, G)$ is a parabola if and only if the line $\mathcal{L}$ is tangent to the ellipse $\mathcal{E}_0^*$.

**Proposition 6.** If $\mathcal{L} : px + qy + rz = 0$ is the tangent to $\mathcal{E}_0^*$ at $P$, then the parabola $\mathcal{Q}(\mathcal{L}, G)$ and the ellipse $\mathcal{E}_0^*$ have

(i) a common tangent parallel to $\mathcal{L}$ at the antipode of $P$ on $\mathcal{E}_0^*$, and

(ii) two remaining common points on the parallel to $\mathcal{L}$ through the centroid $G$.

**Proof.** We take the matrices of the dual ellipses $\mathcal{E}_0$ and $\mathcal{E}_0^*$ to be

$$M = \begin{pmatrix} -1 & 5 & 5 \\ 5 & -1 & 5 \\ 5 & 5 & -1 \end{pmatrix} \quad \text{and} \quad M^* = \begin{pmatrix} -4 & 5 & 5 \\ 5 & -4 & 5 \\ 5 & 5 & -4 \end{pmatrix}$$

respectively. The line $\mathcal{L} : px + qy + rz = 0$ is tangent to the ellipse $\mathcal{E}_0^*$ at the point

$$P = (p \quad q \quad r) M = (-p + 5q + 5r \quad 5p - q + 5r \quad 5p + 5q - r).$$
With the matrix $M(\mathcal{L})$ given in (9), and $t = \frac{1}{18}(p + q + r)^2$, it is routine to verify that

$$(x \ y \ z) (M(\mathcal{L}) + tM^*) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{9} L_1(x, y, z) L_2(x, y, z),$$

where

$$L_1(x, y, z) = (-p + 2q + 2r)x + (2p - q + 2r)y + (2p + 2q - r)z,$$

$$L_2(x, y, z) = (-2p + q + r)x + (p - 2q + r)y + (p + q - 2r)z.$$ 

Note that the lines $\mathcal{L}_1 : L_1(x, y, z) = 0$ and $\mathcal{L}_2 : L_2(x, y, z) = 0$ are both parallel to $\mathcal{L}$ (with infinite point $(q - r : r - p : p - q)$). Since the point $(-p + 2q + 2r : 2p - q + 2r : 2p + 2q - r)$ lies on both conics dual to the ellipse $\mathcal{E}_0^*$ and the parabola $\mathcal{Q}(\mathcal{L}, G)$, the line $\mathcal{L}_1$ is a common tangent of the two conics. The point of tangency is

$$(-p + 2q + 2r \ 2p - q + 2r \ 2p + 2q - r) M$$

$$= 3 \begin{pmatrix} 7p + q + r & p + 7q + r & p + q + 7r \end{pmatrix}.$$ 

This is the antipode of $P$ on the ellipse $\mathcal{E}_0^*$.

The other line $\mathcal{L}_2$ clearly contains the centroid $G$, and therefore a pair of antipodal points on the ellipse $\mathcal{E}_0^*$. These two points also lies on the parabola $\mathcal{Q}(\mathcal{L}, G)$. \hfill \Box
6. The lines $\mathcal{L}$ for which $\mathcal{Q}(\mathcal{L}, G)$ are rectangular hyperbolas

Apart from the cases of pairs of lines and parabolas, the type of the conic $\mathcal{Q}(\mathcal{L}, G)$ can be easily determined by an application of Proposition 2(c) and Proposition 3.

**Proposition 7.** If $p + q + r \neq 0$ and $qr + rp + pq \neq 0$, the conic $\mathcal{Q}(\mathcal{L}, G)$ has center

$$Q = (-p^2 + 3p(q + r) + 4qr : -q^2 + 3q(r + p) + 4rp : -r^2 + 3r(p + q) + 4pq).$$

It is homothetic to the circumconic

$$(p + 2q)(p + 2r)yz + (q + 2r)(q + 2p)zx + (r + 2p)(r + 2q)xy = 0,$$  \hspace{1cm} (11)

with ratio of homothety $\tau$ given by

$$\tau^2 = \frac{(p + q + r)^4(pq + qr + rp)}{(p + 2q)(p + 2r)(q + 2r)(q + 2p)(r + 2p)(r + 2q)},$$  \hspace{1cm} (12)

provided $(p + 2q)(p + 2r)(q + 2r)(q + 2p)(r + 2p)(r + 2q) \neq 0$.

Since the conic $\mathcal{Q}(\mathcal{L}, G)$ is homothetic to the circumconic defined by (11), it is a rectangular hyperbola if and only if this circumconic contains the orthocenter $H = \left(\frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C}\right)$, i.e.,

$$S_A(p + 2q)(p + 2r) + S_B(q + 2r)(q + 2p) + S_C(r + 2p)(r + 2q) = 0.$$ 

Equivalently, the line $\mathcal{L} : px + qy + rz = 0$ is tangent to the conic dual to

$$\mathcal{E}_1 : S_A(x + 2y)(x + 2z) + S_B(y + 2z)(y + 2x) + S_C(z + 2x)(z + 2y) = 0.$$ 

![Figure 5](image-url)
Proposition 8. The conic \( \mathcal{E}_1 \) is an ellipse with center \( G \). Its dual conic is a circle also with center \( G \).

Proof. The matrix of the conic \( \mathcal{E}_1 \) being

\[
M_1 = \begin{pmatrix}
S_A & S_A + S_B + 2S_C & S_A + 2S_B + S_C \\
S_A + S_B + 2S_C & S_B & 2S_A + S_B + S_C \\
S_A + 2S_B + S_C & 2S_A + S_B + S_C & S_C
\end{pmatrix},
\]

with

\[
\det M_1 = 9(S_A + S_B + S_C)(S_{BC} + S_{CA} + S_{AB}) = \frac{9}{2}(a^2 + b^2 + c^2)S^2 \neq 0,
\]

the conic is nondegenerate. The adjoint matrix is

\[
M_1^\# = \begin{pmatrix}
m_{aa} & m_{ab} & m_{ac} \\
m_{ab} & m_{bb} & m_{bc} \\
m_{ac} & m_{bc} & m_{cc}
\end{pmatrix}
\]

where

\[
m_{aa} = -4S_A(S_A + S_B + S_C) - (S_{BB} + S_{BC} + S_{CC}),
m_{ab} = -S_{CC} + 2S_{AB} + 2S_{BC} + 2S_{AC} + 5S_{AA} + 2S_{BB},
m_{ac} = -S_{BB} + 2S_{AB} + 2S_{BC} + 2S_{CC} + 5S_{AC} + 2S_{AA},
m_{bc} = -S_{AA} + 2S_{AB} + 2S_{AC} + 2S_{BB} + 5S_{BC} + 2S_{CC},
\]

and \( m_{bb}, m_{cc} \) are analogously defined. It is easy to check that

\[
m_{aa} + m_{ab} + m_{ac} = m_{ab} + m_{bb} + m_{bc} = m_{ac} + m_{bc} + m_{cc} = 3(S_{BC} + S_{CA} + S_{AB}) = 3S^2.
\]

From this we conclude that the conic \( \mathcal{E}_1 \) has center \( G \). Also, \( \chi(M_1) = 9S^2 > 0 \). It follows that \( \mathcal{E}_1 \) is an ellipse. \[3\]

The equation of the dual conic can be written in the form

\[
9(S_A + S_B + S_C)((S_B + S_C)yz + (S_C + S_A)zx + (S_A + S_B)xy) + (x + y + z)(m_{aa}x + m_{bb}y + m_{cc}z) = 0.
\]

From this it is clear that this dual conic is a circle. The center has coordinates given by \( G(M_1^\#)^\# \), which we may simply take as \( GM_1 = 3(S_A + S_B + S_C)G \). This is the centroid \( G \). \[\square\]

We denote this circle by \( \mathcal{C}_G \).

Theorem 9. The conic \( \mathcal{Q}(\mathcal{L}, G) \) is a rectangular hyperbola if and only if the line \( \mathcal{L} \) is tangent to the circle \( \mathcal{C}_G \).

\[3\] See Remark at the end of §2.
6.1. **Construction of the centroidal circle \( \mathcal{C}_G \).** Since

\[
\det M_1 = 9(S_A + S_B + S_C)(S_{BC} + S_{CA} + S_{AB}) = 9S^2(S_A + S_B + S_C),
\]

we have

\[
\det M_1^\# = (\det M_1)^2 = 81S^4(S_A + S_B + S_C)^2. \quad (13)
\]

Making use of Proposition 3 and (13), we determine the ratio of homothety of \( \mathcal{C}_G \) and the circumcircle as the square root of

\[
\frac{4 \det M_1^\#}{(9(S_A + S_B + S_C))^3(S_{BC} + S_{CA} + S_{AB})(S_A + S_B)} = \frac{8S^4}{9a^2b^2c^2(a^2 + b^2 + c^2)}.
\]

This means that the square of the radius of the circle \( \mathcal{C}_G \) is

\[
\frac{8S^4}{9a^2b^2c^2(a^2 + b^2 + c^2)} \cdot \frac{a^2b^2c^2}{4S^2} = \frac{2S^2}{9(a^2 + b^2 + c^2)} = \frac{S}{3a} \cdot \frac{2}{3} \cdot \frac{a^2}{a^2 + b^2 + c^2} \cdot S.
\]

In the last expression, the first factor is the height of the centroid \( G \) above the line \( BC \). The second factor is \( \frac{2}{3} \) of the height of the symmedian point \( K \) above the same line. This leads to the following construction of the circle.

**Construction 10.** Given triangle \( ABC \) with centroid \( G \) and symmedian point \( K \), construct

1. the pedals \( X \) and \( Y \) of \( G \) and \( K \) on the line \( BC \),
2. a point \( X' \) on the extension of \( XG \) such that \( GX' = \frac{2}{3} YK \),
3. the circle with diameter \( XX' \),
4. the perpendicular through \( G \) to the line \( XX' \) to intersect the circle in (3) at \( Z \) and \( Z' \).

The circle with center \( G \) and diameter \( ZZ' \) is the circle \( \mathcal{C}_G \).
7. The lines \( \mathcal{L} \) for which \( \mathcal{D}(\mathcal{L}, G) \) are circles

The conic \( \mathcal{D}(\mathcal{L}, G) \) is a circle if and only if

\[
(p + 2q)(p + 2r) : (q + 2r)(q + 2p) : (r + 2p)(r + 2q) = a^2 : b^2 : c^2. \tag{14}
\]

We may regard \((p : q : r)\) as a common point of conics defined by

\[
\frac{(x + 2y)(x + 2z)}{a^2} = \frac{(y + 2z)(y + 2x)}{b^2} = \frac{(z + 2x)(z + 2y)}{c^2},
\]
or equivalently,

\[
\frac{(x + y + z)^2 - (y - z)^2}{a^2} = \frac{(x + y + z)^2 - (z - x)^2}{b^2} = \frac{(x + y + z)^2 - (x - y)^2}{c^2}.
\]

The equation of the second and third expression can be rewritten as

\[
(b^2 - c^2)(x + y + z)^2 = b^2(x - y)^2 - c^2(z - x)^2; \tag{15}
\]
similarly for the other two equations. Thus, \((p : q : r)\) is a common point of the conics

\[
F_a := c^2y^2 - b^2z^2 - 2(b^2 - c^2)yz - 2(b^2 - 2c^2)zx - 2(2b^2 - c^2)xy = 0,
F_b := a^2z^2 - c^2x^2 - 2(c^2 - a^2)zx - 2(c^2 - 2a^2)xy - 2(2c^2 - a^2)yz = 0,
F_c := b^2x^2 - a^2y^2 - 2(a^2 - b^2)xy - 2(2a^2 - b^2)yz - 2(a^2 - b^2)zx = 0.
\]

We easily determine, by Proposition 2(c), that these conics all have center \(G\). Since

\[
F_a + F_b + F_c = 3((b^2 - c^2)yz + (c^2 - a^2)zx + (a^2 - b^2)xy) + (x + y + z)((b^2 - c^2)x + (c^2 - a^2)y + (a^2 - b^2)z) = 0
\]
is degenerate, this represents the two asymptotes of a hyperbola with center \(G\), homothetic to the Kiepert hyperbola. These asymptotes, as is well known, are parallel to the axes of the Steiner circum-ellipse. We therefore conclude that the common points of the conics \(F_a = F_b = F_c = 0\) are on an axis of the Steiner ellipse, which turns out to be the minor axis. Each of these two points leads to a conic \(\mathcal{D}(\mathcal{L}, G)\) which is a circle.

We shall make use of the following notations:

\[
P := a^2 + b^2 + c^2;
P_a := b^2 + c^2 - 2a^2, \quad P_b := c^2 + a^2 - 2b^2, \quad P_c := a^2 + b^2 - 2c^2;
Q := a^4 + b^4 + c^4 - 2b^2c^2 - 2c^2a^2 - 2a^2b^2.
\]

Lemma 11. The infinite points of the Kiepert hyperbola are

\[
((b^2 - c^2)(P_a + \varepsilon \sqrt{Q}) : (c^2 - a^2)(P_b + \varepsilon \sqrt{Q}) : (a^2 - b^2)(P_c + \varepsilon \sqrt{Q})
\]

for \(\varepsilon = \pm 1\).

The point for \(\varepsilon = -1\) is the infinite point of the minor axis of the Steiner ellipse. It is the point \(X_{3414}\) of [2]. The infinite point corresponding to \(\varepsilon = +1\) is \(X_{3413}\).
Proposition 12. The two real common points of the conics $F_a = 0$, $F_b = 0$, $F_c = 0$ are

$$P_\varepsilon = (\varepsilon t + (b^2 - c^2)(P_a - \sqrt{Q}) : \cdots : \cdots)$$

(16)

for $\varepsilon = \pm 1$, where $t$ is the square root of

$$\frac{1}{9}(P + 2\sqrt{Q})(P_a - \sqrt{Q})(P_b - \sqrt{Q})(P_c - \sqrt{Q}).$$

(17)

Proof. A point on the minor axis of the Steiner ellipse is of the form (16) for some $t$. Substituting into the equation $F_a = 0$, or equivalently (15), and solving, we obtain

$$t^2 = \frac{1}{9}(\lambda + \mu \sqrt{Q}),$$

where

$$\lambda = 4Q^2 + PP_aP_bP_c,$$

(18)

$$\mu = 2(P_aP_bP_c + PQ).$$

(19)

Since $P_a + P_b + P_c = 0$ and $P_aP_b + P_bP_c + P_cP_a = -3Q$, we have

$$\lambda + \mu \sqrt{Q} = 2Q \sqrt{Q}(P + 2\sqrt{Q}) + P_aP_bP_c(P + 2\sqrt{Q})$$

$$= (P + 2\sqrt{Q})(2Q \sqrt{Q} + P_aP_bP_c)$$

$$= (P + 2\sqrt{Q})(-\sqrt{Q}^3 + (P_a + P_b + P_c)Q$$

$$- (P_aP_b + P_bP_c + P_cP_a) \sqrt{Q} + P_aP_bP_c)$$

$$= (P + 2\sqrt{Q})(P_a - \sqrt{Q})(P_b - \sqrt{Q})(P_c - \sqrt{Q})$$

leading to the factorization of $t^2$ given in (17) above. \qed
The points \( P_\varepsilon, \varepsilon = \pm 1 \), correspond to two lines \( L_\varepsilon \) which are common tangents to the conics dual to \( F_a = 0 \), \( F_b = 0 \), and \( F_c = 0 \) (see Figure 8). For these two lines, the conics \( \mathcal{P}(L_\varepsilon, G) \) are circles, which we simply denote by \( \mathcal{C}_\varepsilon \) (see Figure 9).
Proposition 13. The two circles \( C_{\varepsilon}, \varepsilon = \pm 1 \), have centers

\[
Q_\varepsilon = \left( \frac{f + g\sqrt{Q}}{\varepsilon \cdot 3t} + (b^2 - c^2)(P_a - \sqrt{Q}) : \ldots : \ldots \right),
\]

where

\[
f = 4Q^2 - PP_aP_bP_c = 8Q^2 - \lambda, \tag{21}
g = 2(P_aP_bP_c - PQ) = 4P_aP_bP_c - \mu. \tag{22}
\]

They are congruent and have radius

\[
\rho = \frac{P + 2\sqrt{Q}}{18S} \cdot \sqrt{P - \sqrt{Q}}.
\]

Proof. With \((p, q, r)\) given by (14), and \(\lambda, \mu\) by (18), (19), we have

\[
(p + 2q)(p + 2r) = a^2W, \quad (q + 2r)(q + 2p) = b^2W, \quad (r + 2p)(r + 2q) = c^2W
\]

for

\[
W = \frac{(p - q)^2 - (r - p)^2}{b^2 - c^2} = 3(P_a - \sqrt{Q})(P_a^2 - P_a \sqrt{Q} - 2Q)
\]

\[
= 3(P_a - \sqrt{Q})^2(P_a + 2\sqrt{Q})
\]

\[
= 3(P_b - \sqrt{Q})^2(P_b + 2\sqrt{Q})
\]

\[
= 3(P_c - \sqrt{Q})^2(P_c + 2\sqrt{Q}).
\]

Note that

\[
p + q + r = \varepsilon \cdot 3t,
\]

\[
pq + qr + rp = 3t^2 - 2Q^2 - P_aP_bP_c\sqrt{Q}
\]

\[
= \frac{1}{3} \left( \lambda + \mu \sqrt{Q} - 6Q^2 - 3P_aP_bP_c \sqrt{Q} \right)
\]

\[
= \frac{1}{3} \left( 2PQ \sqrt{Q} - 2Q^2 + PP_aP_bP_c - P_aP_bP_c \sqrt{Q} \right)
\]

\[
= \frac{1}{3} \left( 2Q \sqrt{Q}(P - \sqrt{Q}) + P_aP_bP_c(P - \sqrt{Q}) \right)
\]

\[
= \frac{1}{3} \left( P - \sqrt{Q} \right) \left( 8Q \sqrt{Q} + 4(P_a + P_b + P_c)Q \\
+ 2(P_aP_b + P_bP_c + P_cP_a) \sqrt{Q} + P_aP_bP_c \right)
\]

\[
= \frac{1}{3} \left( P - \sqrt{Q} \right) (P_a + 2\sqrt{Q})(P_b + 2\sqrt{Q})(P_c + 2\sqrt{Q}).
\]
From these,
\[-p^2 + 3p(q + r) + 4qr
= 4(pq + qr + rp) - p(p + q + r)
= 4(3t^2 - 2Q^2 - P_aP_bP_c\sqrt{Q}) - (\varepsilon t + (b^2 - c^2)(P_a - \sqrt{Q}))\varepsilon \cdot 3t
= 9t^2 - 8Q^2 - 4P_aP_bP_c\sqrt{Q} - 3\varepsilon(b^2 - c^2)(P_a - \sqrt{Q})t
= \lambda - 8Q^2 + (\mu - 4P_aP_bP_c)\sqrt{Q} - 3\varepsilon(b^2 - c^2)(P_a - \sqrt{Q})t
= -f - g\sqrt{Q} - 3\varepsilon(b^2 - c^2)(P_a - \sqrt{Q})t
= -\varepsilon \cdot 3t \left( \frac{f + g\sqrt{Q}}{\varepsilon \cdot 3t} + (b^2 - c^2)(P_a - \sqrt{Q}) \right),
\]
where \(f\) and \(g\) are given in (21) and (22) above. Similarly,
\[-q^2 + 3q(r + p) + 4rp
= -\varepsilon \cdot 3t \left( \frac{f + g\sqrt{Q}}{\varepsilon \cdot 3t} + (c^2 - a^2)(P_b - \sqrt{Q}) \right),
\]
\[-r^2 + 3r(p + q) + 4pq
= -\varepsilon \cdot 3t \left( \frac{f + g\sqrt{Q}}{\varepsilon \cdot 3t} + (a^2 - b^2)(P_c - \sqrt{Q}) \right).
\]
By Proposition 2(c), the center of the circle \(C_\varepsilon\) is the point \(Q_\varepsilon\) given by (20) above. Furthermore, the homothetic ratio of \(C_\varepsilon\) and the circumcircle is the square root of
\[
\frac{(p + q + r)^4(pq + qr + rp)}{(p + 2q)(p + 2r)(q + 2p)(q + 2p)(r + 2p)(r + 2q)} = \frac{(P + 2\sqrt{Q})^2(P - \sqrt{Q})}{3 \cdot 27a^2b^2c^2}.
\]
From these, it follows that the common radius of the circles \(C_\varepsilon\) is
\[
\frac{P + 2\sqrt{Q}}{9abc} \sqrt{P - \sqrt{Q}} \cdot \frac{abc}{2S} = \frac{P + 2\sqrt{Q}}{18S} \sqrt{P - \sqrt{Q}}.
\]

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The *f*-belos

Antonio M. Oller-Marcén

**Abstract.** The arbelos is the shape bounded by three mutually tangent semicircles with collinear diameters. Recently, Sondow introduced the parabolic analog, the parbelos and proved several properties of the parbelos similar to properties of the arbelos. In this paper we give one step further and generalize the situation considering the figure bounded by (quite) arbitrary similar curves, the *f*-belos. We prove analog properties to those of the arbelos and parbelos and, moreover, we characterize the parbelos and the arbelos as the *f*-beloses satisfying certain conditions.

**1. Introduction**

The arbelos (ἀρβηλός, literally “shoemaker’s knife”) was introduced in Proposition 4 of Archimedes’ Book of Lemmas [1, p. 304]. It is the plane figure bounded by three pairwise tangent semicircles with diameters lying on the same line (see the left-hand side of Figure 1). In addition to the properties proved by Archimedes himself, there is a long list of properties satisfied by this figure. Boas’s paper [3] presents some of them and is a good source for references.

It is quite surprising to discover that for 23 centuries no generalizations of this figure were introduced. Recently, Sondow [4] has extended the original construction considering latus rectum arcs of parabolas instead of semicircles (see right-hand side of Figure 1). In his paper, Sondows proves several interesting properties of his construction (named parbelos) that are, in some sense, counterparts of properties of the arbelos.

![Figure 1. An arbelos (left) and a parbelos (right).](image_url)

The motivation for considering latus rectum arcs of parabolas instead of semicircles is clear. Just like all circles are similar, so too all parabolas are similar. Of course this is a very special property of these curves which is not shared even by other conics. Nevertheless, it gives the clue for a further generalization of both...
the arbelos and the parbelos. In a suitable generalization, the three considered arcs must be similar curves.

In Section 2 we present our construction and the resulting family of figures that we shall call \( f \)-belos. Subsequent sections are mainly devoted to extend and give analogs of some of the properties found in [4]. In passing we will see how arbelos and parbelos appear as particular cases of our construction imposing certain seemingly unrelated conditions.

2. The \( f \)-belos

Let \( f : [0, 1] \rightarrow \mathbb{R} \) be a function such that \( f(x) > 0 \) except for \( f(0) = f(1) = 0 \). We will assume that \( f \) is continuous in \([0, 1]\) and differentiable in \((0, 1)\). Given \( p \in (0, 1) \) we define functions \( g : [0, p] \rightarrow \mathbb{R} \) and \( h : [p, 1] \rightarrow \mathbb{R} \) given by:

\[
\begin{align*}
g(x) &= pf(x/p), \\
h(x) &= (1-p)f\left( \frac{x-p}{1-p} \right).
\end{align*}
\]

Observe that both \( g \) and \( h \) are similar to \( f \) (\( g \) is obtained by a homothety centered at the origin and \( h \) is obtained by a homothety followed by a translation). In what follows we will consider the case when the graphs of \( g \) and \( h \) are below the graph of \( f \) so that a situation like the one in Figure 2 makes sense.

![Figure 2. The \( f \)-belos.](image)

Given the function \( f \) we will call the figure obtained in the previous construction an \( f \)-belos. The point \( P = (p, 0) \) will be called the cusp of the \( f \)-belos. We will denote \( O = (0, 0) \) and \( I = (1, 0) \) (see Figure 2).

Observe that if \( f(x) = \sqrt{x-x^2} \) we recover the original arbelos, while if \( f(x) = x-x^2 \) we obtain Sondow’s parbelos.

3. Elementary properties of the \( f \)-belos

In spite of the generality of the latter construction, the \( f \)-belos satisfy several interesting properties which, in some sense, extend those of the arbelos and the parbelos. These properties are analogs of Properties 1 and 2 in [4].
**Proposition 1.** The upper and lower boundaries of an $f$-belos have the same length.

**Proof.** If $L_f$ is the length of the upper boundary, $L_g$ is the length of the lower arc corresponding to the graph of $g$ and $L_h$ in the length of the lower arc corresponding to the graph of $h$ it follows by their similarity that $L_g = pL_f$ and $L_h = (1 - p)L_f$. Hence, the result. □

The following lemma is easy to prove and it is closely related to Plato’s analogy of the line [2].

**Lemma 2.** Consider a segment $\overline{AB}$ and choose any point $C \in \overline{AB}$ except the endpoints. Now, let $D \in \overline{AC}$ and $E \in \overline{CB}$ be points such that $|AC| = |AD| = |CE| = |EB|$. Then,

$$|DC| = |CE| = \frac{|AC| \cdot |CB|}{|AB|}.$$

As a consequence we obtain the following property.

**Proposition 3.** Under each lower arc of an $f$-belos, construct a new $f$-belos similar to the original. Of the four lower arcs, the middle two are congruent, and their common length equals one half the harmonic mean of the lengths of the original lower arcs.

**Proof.** It is enough to apply the previous lemma noting that the lengths of the considered arcs are proportional to the length of the horizontal segment that they determine. □

4. The parallelogram associated to a point

Let $x_0 \in (0, 1)$ and consider the point $P_1 = (x_0, f(x_0))$. This point lies on the graph of $f$ and hence, by similarity, it has corresponding points $P_2$ and $P_3$ in the graphs of $g$ and $h$, respectively. Namely $P_2 = (px_0, pf(x_0))$ and $P_3 = ((1 - p)x_0 + p, (1 - p)f(x_0))$ (see Figure 3). Observe that $P_2P_3 = P_1P_2 = (1 - p)(x_0, f(x_0))$. Hence, $P_1P_2PP_3$ is a parallelogram. Since it depends on the choice of $x_0$, we will denote this parallelogram $P(x_0)$.

It is interesting to study when $P(x_0)$ is a rectangle. This leads to a surprising characterization of the arbelos.

**Proposition 4.** Given an $f$-belos, the parallelogram $P(x_0)$ is a rectangle if and only if $f(x_0)^2 = x_0 - x_0^2$. Consequently, $P(x_0)$ is a rectangle for every $x_0 \in (0, 1)$ if and only if $f$ describes a semicircle (i.e., the figure is an arbelos).

**Proof.** We have that $\overrightarrow{P_2P} = (p - px_0, -pf(x_0))$. Then, $P(x_0)$ is a rectangle if and only if $P_2P_3 \perp P_2P_1$; i.e., if and only if

$$0 = P_2P \cdot P_2P_1 = p(1 - p)[x_0(1 - x_0) - f(x_0)^2].$$
The area of the parallelogram $\mathcal{P}(x_0)$ can be easily computed by:

$$\text{area}(\mathcal{P}(x_0)) = \|\overrightarrow{P_2P_1} \times \overrightarrow{P_2P_3}\| = \|(0, 0, -p(1-p)f(x_0))\| = p(1-p)f(x_0).$$

This fact leads to the following property.

**Proposition 5.** Given an $f$-belos, let $c \in (0, 1)$ be such that $f(c)$ is the mean value of $f$ on $[0, 1]$ and let $\mathcal{P}(c)$ be the parallelogram associated to $c$ (see Figure 4). Then:

$$\text{area}(f\text{-belos}) = 2\text{area}(\mathcal{P}(c)).$$

**Proof.** Let us denote by $A$ the area below the upper arc of the $f$-belos; i.e.,

$$A = \int_0^1 f(x) \, dx.$$

By similarity, the area below the lower arc corresponding to the graph of $g$ is $p^2A$, while the area below the lower arc corresponding to the arc of $h$ is $(1-p)^2A$. Hence the area of the $f$-belos is:

$$A - p^2A - (1-p)^2A = 2p(1-p)A.$$

Moreover, the mean value theorem for integration states that there exists $c \in (0, 1)$ such that $A = \int_0^1 f(x) \, dx = f(c)$. Consequently, the area of the $f$-belos is

$$2p(1-p)f(c).$$
where $f(c)$ is the mean value of $f$ on $[0, 1]$.

Since, by the above consideration, $\text{area}(\mathcal{P}(c)) = p(1 - p)f(c)$, the result follows. \hfill \Box

We will see how this property in fact generalizes Property 3 in [4].

**Remark.** Consider a parbelos; i.e., an $f$-belos with $f(x) = x - x^2$. In this context Property 3 in [4] states that

$$\text{area}(\text{parbelos}) = \frac{4}{3}\text{area}(\mathcal{P}(1/2)).$$

Let us see how this follows from Property 4 above.

The mean value of $f$ in $(0, 1)$ is $1/6$. Consequently, $\text{area}(\mathcal{P}(c)) = \frac{p(1 - p)}{6}$.

On the other hand, while $f(1/2) = 1/4$, we have that $\text{area}(\mathcal{P}(1/2)) = \frac{p(1 - p)}{4}$.

Hence:

$$\text{area}(\text{parbelos}) = 2\text{area}(\mathcal{P}(c)) = \frac{4}{3}\text{area}(\mathcal{P}(1/2))$$

as claimed.

### 5. The tangent parallelogram

Throughout this section we will consider an $f$-belos such that $f$ is also differentiable in $x = 0$ and $x = 1$. Hence, we consider the tangents to $f$ in $x = 0$ and in $x = 1$ and the tangents in $x = p$ to $g$ and $h$ and a situation like Figure 5 makes sense.

![Tangent Parallelogram](Figure 5. The tangent parallelogram of an $f$-belos.)

The fact that $T_1T_2T_3P$ is a parallelogram (that we will denote by $\mathcal{T}$) follows readily from the similarity of $f$, $g$ and $h$ (observe that $g'(p) = h'(0)$ and that $g'(1) = f'(1)$). In the following property we compute its area.
Proposition 6. The area of $\mathcal{T}$ is given by:

$$\text{area}(\mathcal{T}) = p(1 - p) \left| \frac{f'(0)f'(1)}{f'(0) - f'(1)} \right|.$$ 

Proof. Since the sides of the parallelogram are given by the equations

- $T_1T_2 : y = f'(0)x,$
- $PT_3 : y = f'(0)(x - p),$  
- $T_1P : y = f'(1)(x - p),$  
- $T_2T_3 : y = f'(1)(x - 1),$

it is easy to find the coordinates of $T_i$ for $i = 1, 2, 3$ and obtain

$$\overrightarrow{T_1T_2} = \frac{(1 - p)f'(1)}{f'(1) - f'(0)}(1, f'(0)) \quad \text{and} \quad \overrightarrow{T_1P} = \frac{-pf'(0)}{f'(1) - f'(0)}(1, f'(1)).$$

With these, we have that

$$\text{area}(\mathcal{T}) = \left\| \overrightarrow{T_1T_2} \times \overrightarrow{T_1P} \right\| = \left\| \frac{p(1 - p)f'(0)f'(1)}{(f'(1) - f'(0))^2} (0, 0, f'(0) - f'(1)) \right\|$$

and the result follows. \(\square\)

In the parbelos, $\mathcal{T}$ is in fact a rectangle [4, Property 4]. The general situation is as follows.

Proposition 7. The tangent parallelogram $\mathcal{T}$ is a rectangle if and only if $f'(0)f'(1) = -1$. In such case we have that

$$\text{area}(\mathcal{T}) = p(1 - p) \frac{f'(0)}{1 + f'(0)^2}.$$ 

Proof. The first part is straightforward recalling that two lines of slopes $m_1$ and $m_2$ are perpendicular if and only if $m_1m_2 = -1$. For the second statement it is enough to put $f'(1) = -1/f'(0)$ in the previous property. \(\square\)

Since we have that $2f'(0) \leq 1 + f'(0)^2$, the following property follows (recall the notation from the previous section).

Proposition 8. Given an $f$-belos, let $f(c)$ be the mean value of $f$ on $[0, 1]$. Let also $\mathcal{P}(c)$ be the parallelogram associated to $c$ and $\mathcal{T}$ the tangent parallelogram. If $f'(0)f'(1) = -1$; i.e., if $\mathcal{T}$ is a rectangle, then:

$$\text{area}(\mathcal{P}(c)) \geq 2f(c)\text{area}(\mathcal{T});$$

and equality holds if and only if $f'(0) = -f'(1) = 1$.

Proof. Properties 4 and 6 imply that

$$\text{area}(\mathcal{P}(c)) = f(c)p(1 - p) = f(c)\frac{\text{area}(\mathcal{T})}{f'(0)} \geq 2f(c)\text{area}(\mathcal{T}).$$

And equality holds if and only if $2f'(0) = 1 + f'(0)^2$; i.e., if and only if $f'(0) = 1$. \(\square\)
Remark. In the parbelos $f(x) = x - x^2$, so we have $f'(0) = 1 = -f'(1)$ and equality holds in Property 7. Moreover (recall the remark after Property 4), $f(c) = 1/6$. Hence:

$$\text{area}(\mathcal{T}) = \frac{\text{area}(\mathcal{P}(c))}{2f(c)} = \frac{3}{2}\text{area(parbelos)},$$

as was already proved in [4, Property 4].

Now, if $f'(0)f'(1) = -1$, $\mathcal{T}$ is a rectangle, so it makes sense to consider its circumcircle $\Gamma$. In [4, Property 6] it was proved that, in the parbelos case (i.e., when $f(x) = x - x^2$) the circumcircle of the tangent rectangle passes through the focus of the upper parabola (i.e., through the point $(1/2, 0)$). This property can be generalized in the following way.

**Proposition 9.** Given an $f$-belos such that its tangent parallelogram $\mathcal{T}$ is a rectangle; i.e., such that $f'(0)f'(1) = -1$, the circumcircle $\Gamma$ of $\mathcal{T}$ intersects the axis $OX$ at the point $(p, 0)$ and $(1 + f'(0)^2, 0)$ (see Figure 6). Consequently:

- $\Gamma$ is tangent to $OX$ if and only if $p = \frac{1}{1 + f'(0)^2}$.
- $\Gamma$ intersects $OX$ at $x = 1/2$ if and only if $f'(0) = 1$.

**Proof.** The center of this circle is the midpoint of $T_1 T_3$; i.e.,

$$C_\Gamma = \left( \frac{p + 1 + p f'(0)^2}{2(1 + f'(0)^2)}, \frac{f'(0)}{2(1 + f'(0)^2)} \right).$$

This circle clearly intersects $OX$ at $P$ but, of course, it will intersect the axis $OX$ in another point (unless $\Gamma$ is tangent to $OX$). Hence we are looking for $p \neq \alpha \in (0, 1)$ such that $(\alpha, 0) \in \Gamma$. This condition leads to:

$$(\frac{p + 1 + p f'(0)^2}{2(1 + f'(0)^2)} - \alpha)^2 + \left( \frac{f'(0)}{2(1 + f'(0)^2)} \right)^2 = \left( \frac{p + 1 + p f'(0)^2}{2(1 + f'(0)^2)} - p \right)^2 + \left( \frac{f'(0)}{2(1 + f'(0)^2)} \right)^2.$$

Consequently

$$\alpha^2 - p^2 - (\alpha - p) \frac{p + 1 + p f'(0)^2}{2(1 + f'(0)^2)} = 0$$

and, since $p \neq \alpha$,

$$\alpha = \frac{p + 1 + p f'(0)^2}{2(1 + f'(0)^2)} - p = \frac{1}{1 + f'(0)^2}$$

as claimed. \qed

Now, we turn again to the general case, when $\mathcal{T}$ is “only” a parallelogram. The equation of the line passing through $T_1$ and $T_3$ is:

$$\overline{T_1T_3} : \left[ (1 - 2p) f'(0) + f'(1) \right] x + [pf'(0) - (1 - p)f'(1)] y + p^2 f'(0) f'(1) = 0.$$

We are interested in studying when this line is tangent to the graph of $f$ at $(p, f(p))$. This interest is motivated by [4, Property 5], where it was proved that, in a parbelos, the diagonal of the tangent rectangle opposite to the cusp is tangent to the upper parabola at the point $(p, f(p))$. In fact we will see how this property characterizes parbeloses.
Proposition 10. The point \((p, f(p))\) lies on \(T_1T_3\) (see Figure 7) if and only if
\[
f(p) = \frac{p(p - 1)f'(0)f'(1)}{pf'(0) - (1 - p)f'(1)}.
\]
In addition, the line \(T_1T_3\) is tangent to the graph of \(f\) at the point \((p, f(p))\) if and only if
\[
f'(p) = \frac{(2p - 1)f'(0)f'(1)}{pf'(0) - (1 - p)f'(1)}.
\]
Proof. For the first part it is enough to substitute \((p, f(p))\) in the equation of \(T_1T_3\). For the second part, the slope of \(T_1T_3\) must coincide with \(f'(p)\). \(\square\)
We will close the paper with a property that surprisingly characterizes the parbelos among all possible $f$-beloses. Parbeloses are the only $f$-beloses such that the diagonal of $T$ opposite to the cusp is tangent to $f$ at $(p, f(p))$ for every $p \in (0, 1)$.

**Proposition 11.** Given an $f$-belos, let $T$ be its tangent parallelogram. Then the diagonal of $T$ opposite to the cusp is tangent to $f$ at $(p, f(p))$ for every $p \in (0, 1)$ if and only if $f$ is a parabola $f(x) = k(x - x^2)$. Moreover, in this case $T$ is a rectangle if and only if $k = 1$.

**Proof.** The first part of the previous property leads to

$$f(x) = \frac{x(x - 1)f'(0)f'(1)}{xf'(0) - (1 - x)f'(1)}$$

for every $x \in [0, 1]$. In this case we have that

$$f'(x) = \frac{f'(0)f'(1)[x^2f'(0) + (1 - x)^2f'(1)]}{[xf'(0) - (1 - x)f'(1)]^2}.$$

Hence, if $f$ is tangent to $T_1T_3$ at $(p, f(p))$ for every $p \in (0, 1)$, the second part of the previous property implies that:

$$f'(x) = \frac{(2x - 1)f'(0)f'(1)}{xf'(0) - (1 - x)f'(1)}$$

for every $x \in (0, 1)$. Some computations lead to

$$(f'(0) + f'(1))(x - x^2) = 0$$

for every $x \in [0, 1]$ and hence $f'(0) = -f'(1)$ so

$$f(x) = \frac{-x(x - 1)f'(0)^2}{xf'(0) + (1 - x)f'(0)} = f'(0)(x - x^2),$$

and the result follows. \qed

**References**


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Why Are the Side Lengths of the Squares Inscribed in a Triangle so Close to Each Other?

Victor Oxman and Moshe Stupel

Abstract. We compare the side lengths of the squares inscribed in a non-obtuse triangle.

Given non-obtuse triangle $ABC$ we consider an inscribed square. The construction is well known (see, for instance, [1] and [2, Problem 9, pp.16,64]). One can easily note that the side lengths of the squares based on the various sides of the triangle are almost equal to each other (Figure 1). Why does it happen? In this note we will give the answer to this question.

Let $BC = a$, $AC = b$, $AB = c$, and $h_a, h_b, h_c$ the corresponding altitudes of triangle $ABC$. Denote the side lengths of the squares based on $BC$ and $AC$ by $x_a$ and $x_b$ respectively. One can easily verify that $x_a = \frac{ah_a}{a + h_a}$. Given $a$ and $h_a > 0$, consider a non-obtuse triangle $ABC$ with $BC = a$ and altitude $h_a$ on the side $BC$. The vertex $A$ lies on the side $B' C'$ of the rectangle $BB' C'C$ with $BB' = h_a$ (see Figure 2).

Let $AC' = t$. We have $b^2 = t^2 + h_a^2$ and $c^2 = (a - t)^2 + h_a^2$. Note that

$$x_b = \frac{bh_b}{b + h_b} = \frac{ah_a}{\sqrt{t^2 + h_a^2} + \frac{ah_a}{\sqrt{t^2 + h_a^2}}} = \frac{ah_a\sqrt{t^2 + h_a^2}}{t^2 + h_a(a + h_a)}.
$$

We shall assume $a \geq b$. This requires $h_a \leq a$ and $t \in [0, t_0]$, where $t_0 = \sqrt{a^2 - h_a^2}$. Since we require triangle $ABC$ to be non-obtuse, $b^2 + c^2 \geq a^2$. This implies $t^2 - at + h_a^2 \geq 0$. This is always the case when $h_a \geq \frac{a}{2}$. When $h_a < \frac{a}{2}$,
Comparing boundary values, we have
\[ t_1 = \frac{a - \sqrt{a^2 - 4h_a^2}}{2}, \quad t_2 = \frac{a + \sqrt{a^2 - 4h_a^2}}{2}. \]

Note that \( t_0^2 - t_2^2 = \frac{a(a - \sqrt{a^2 - 4h_a^2})}{2} \geq 0. \)

Now consider the function
\[ f(t) := \frac{x_a}{x_b} = \frac{t^2 + h_a(a + h_a)}{(a + h_a)\sqrt{t^2 + h_a^2}} \]

defined on
\[ D := \begin{cases} [0, t_0], & \text{if } \frac{a}{2} \leq h_a \leq a, \\ [0, t_1] \cup [t_2, t_0], & \text{if } 0 < h_a < \frac{a}{2}. \end{cases} \]

This is a restriction of the function
\[ F(t) = \frac{t^2 + h_a(a + h_a)}{(a + h_a)\sqrt{t^2 + h_a^2}} \]

defined on \([0, a]\). It has derivative
\[ F'(t) = \frac{1}{a + h_a} \frac{t(t^2 - h_a(a - h_a))}{(t^2 + h_a^2)^{3/2}}. \]

From this it is clear that the only interior critical point is \( t^* = \sqrt{h_a(a - h_a)} \), and that \( F(t) \) is decreasing on \([0, t^*]\) and increasing on \([t^*, a]\). Therefore, \( F(t) \geq F(t^*) = \frac{2\sqrt{ah_a}}{a + h_a} \) for every \( t \in [0, a] \).

Note that \( F(0) = F(t_0) = 1 \). This means \( f(t) \leq 1 \) for \( t \in D \), and \( x_a \leq x_b \).

1. For \( h_a \geq \frac{a}{2} \), comparing \( f(t^*) = \frac{2\sqrt{ah_a}}{a + h_a} \) with the boundary values \( f(0) = 1 \) and \( f(t_0) = 1 \), we conclude that
\[ \min\{f(t) : t \in D\} = \frac{2\sqrt{ah_a}}{a + h_a}. \]

As a function of \( h_a \in \left[ \frac{a}{2}, a \right] \), \( \frac{2\sqrt{ah_a}}{a + h_a} \) is increasing. Therefore, for \( t \in D = [0, t_0] \),
\[ \frac{x_a}{x_b} = f(t) \geq \frac{2\sqrt{ah_a}}{a + h_a} \geq \frac{2\sqrt{a \cdot \frac{a}{2}}}{a + \frac{a}{2}} = \frac{2\sqrt{2}}{3}. \]

2. If \( h_a < \frac{a}{2} \), then
\[ t_{*2} - t_{12}^2 = \frac{a(\sqrt{a^2 - 4h_a^2} - (a - 2h_a))}{2} = \frac{a\sqrt{a - 2h_a}(\sqrt{a + 2h_a} - \sqrt{a - 2h_a})}{2} > 0, \]
\[ t_{2-}^2 - t_{*2}^2 = \frac{a(\sqrt{a^2 - 4h_a^2} + (a - 2h_a))}{2} > 0. \]

It follows that the critical point \( t^* \) is not in the interior of \( D = [0, t_1] \cup [t_2, t_0] \).

Comparing boundary values, we have
\[ \min\{f(t) : t \in D\} = \min\{f(t_1), f(t_2)\}. \]
Note that $f(t_1) = f(t_2)$ because when $t = t_1$ or $t_2$, $\angle BAC = 90^\circ$. In fact, for $j = 1, 2$,
\[
f(t_j) = \frac{a(t_j + h_a)}{(a + h_a)\sqrt{at_j}} = \frac{\sqrt{a}}{a + h_a} \cdot \sqrt{\frac{(t_j + h_a)^2}{t_j}} = \frac{\sqrt{a(a + 2h_a)}}{a + h_a}.
\]
As a function of $h_a \in (0, \frac{a}{2}]$, $\frac{\sqrt{a(a + 2h_a)}}{a + h_a}$ is decreasing. Therefore, for $t \in D = [0, t_1] \cup [t_2, t_0]$,
\[
\frac{x_a}{x_b} = f(t) \geq \frac{\sqrt{a(a + 2h_a)}}{a + h_a} \geq \frac{\sqrt{a(a + 2 \cdot \frac{a}{2})}}{a + \frac{a}{2}} = \frac{2\sqrt{2}}{3}.
\]
We conclude that in all cases,
\[
1 \geq \frac{x_a}{x_b} \geq \frac{2\sqrt{2}}{3} = 0.94 \ldots.
\]
The difference between $x_a$ and $x_b$ is less than 6% of $x_b$. This explains why the lengths of the sides of the inscribed squares are very close to each other.

Note that from the above reasoning the smallest inscribed square in a non-obtuse triangle is based on the longest side.

References


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Pairings of Circles and Sawayama’s Theorem

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Abstract. In this article we study pairs of projectively related circles, tangent to a fixed circle, and point out properties, which in a class of particular cases specialize to Sawayama’s theorem.

1. Introduction

The well known Sawayama-Thebault theorem ([6]) states that the centers $O_1, O_2$ of the two circles $k_1, k_2$ defined by a cevian $AW$ of the triangle $ABC$ are collinear with the incenter $I$ of $ABC$ (See Figure 1). The circles $k_1, k_2$ are, by definition, tangent to $AW$, $BC$ and the circumcircle $c$ of the triangle. For the history of the problem and a synthetic proof I refer to Ayme’s article [1].

In this article we study pairs of projectively related circles $(k_1, k_2)$ called pairings of circles and reveal properties, that in some particular cases, called Sawayama pairings, lead to Sawayama’s theorem.

The general pairings of circles, considered here, result by fixing a triple $(c, h, P)$ of a circle, a line $h$ intersecting $c$ at two non-diametral points $B, C$ and a point inside the circle $c$ but $P \neq O$ and $P \notin h$.

Then we consider all lines through the point $P$. The intersection points $(X, X')$ of a line through $P$ with the circle, define a pair of circles $(k_1, k_2)$, which are correspondingly tangent to $c$ at $X$ and $X'$ and also tangent to $h$ at points $(Z, Z')$ (See Figure 2). These circles have their centers $Y, Y'$ correspondingly on lines
Figure 2. Circle pairing defined by the triple \((c, h, P)\)

\(OX, OX',\) and it is easy to see that their respective chords \(XZ, X'Z'\) pass always through the middle \(D\) of the smaller arc \(BC,\) defined on \(c\) by \(h.\) In fact, using this property, and considering a variable line through \(P\) intersecting the circle \(c\) at \(X, X',\) one can define \(Z, Z'\) by correspondingly intersecting \(DX, DX'\) with \(h\) and defining the two isosceles triangles \(XYZ, X'Y'Z'.\) The resulting triangles are line perspective, since corresponding sides intersect along line \(OD.\) Hence they are also point perspective and define a point \(H_X\) on line \(h,\) which is collinear with the centers \(Y, Y',\) as well as with the contact points \(X, X'\) of the circles \(k_1, k_2.\) This construct is called in the sequel the basic configuration.

In §2 we discuss a fundamental property of the basic configuration coming from the projective geometry aspect. In §3 we deduce that the line of centers \(YY'\) always passes through a fixed point \(I\) and locate its position. In addition we show a property of the map \(Z' = f(Z),\) defined on line \(h\) (Theorem 5). In §4 we investigate two pencils of circles, naturally connected to the basic configuration and giving the geometric meaning and consequences of the calculations of the preceding section. In §5 we establish the existence of a certain variable line, associated to pairing and pivoting about a fixed point (Theorem 13). This line, called pivoting line, specializes in Sawayama’s theorem to the cevian \(AW\) from the vertex of the triangle of reference \(ABC.\) In §6 we characterize the Sawayama pairings and note a resulting proof of Sawayama’s theorem. Finally in §7 we supply a few details concerning the case of Sawayama’s theorem.

2. The projective aspect

In the sequel \((AB, CD)\) denotes the intersection of the lines \(AB\) and \(CD,\) \(D = C(A, B)\) denotes the harmonic conjugate of \(C\) with respect to \((A, B),\) and \((ABCD) = \frac{CA}{CB} : \frac{DA}{DB}\) denotes the cross ratio of the four points. Next construction is a typical one of the definition of a homology of the projective plane [3, p.9]. Given two
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Figure 3. A typical homology \( Y = f(X) \)

lines \( e, h \), intersecting at point \( E \), and three points \( D, J, O \) on \( e \), define the transformation \( f \) as follows: For each point \( X \) of the plane, let \( Z \) be the intersection \( Z = (XD, h) \) and set \( Y = f(X) = (XO, ZJ) \). Defining \( H = (h, XO) \), the equality of cross ratios \( (OHXY) = (OEDJ) \) proves next lemma.

**Lemma 1.** The map \( f \) is a homology with center \( O \), axis \( h \) and homology coefficient equal to the cross ratio \( k = (OEDJ) \).

Of particular interest for our subject is the case in which point \( J \) goes to infinity,

then the configuration becomes like the one of figure 4(I) and the cross ratio has the value \( k = \frac{DO}{DE} \). A further specialization of the previous figure to 4(II), in which points \( D, E, O \) are defined by a circle \( c(O, r) \) and a line \( h \), intersecting the circle at two non-diametral points \( B, C \), leads to the projective aspect of our basic configuration. In this case \( E \) is the projection of the center \( O \) on \( h \) and \( D \) is the intersection of the half-line \( OE \) with \( c \). Using these notations and conventions the following lemma is valid.

**Lemma 2.** For each point \( X \) of the circle \( c \) the triangle \( XYZ \) is isosceles and the circle \( k_1(Y, |XY|) \) is tangent to the circle \( c \) at \( X \) and to line \( h \) at \( Z \). As \( X \) varies on the circle \( c \) the center \( Y \) of \( k_1 \) describes a parabola.

In fact, the first claim follows from an easy angle chasing argument (See Figure 5). From this, or considering the locus as the image \( f(c) \) of the circle via the
homology, follows also easily that the geometric locus of the centers $Y$ of the circles $k_1$ is a parabola tangent to the triangle $A_0BC$ at $B,C$ and point $A_0$ being the diametral of $D$. The parabola passes through the middle $N$ of $A_0E$ and is also tangent to the circle $c'(D, |DB|)$.

3. Pairs of circles

The circle pairing of our basic configuration results by considering all lines through the fixed point $P$. Joining each point $X$ of the circle $c$ with $P$ and considering the second intersection point $X'$ with $c$ one defines an involution ([7, I p.221]) of the points of $c$ denoted by $g_P$. Point $X$ on the circle and the corresponding point $X' = g_P(X)$ define, by the recipe of the previous section, two triangles $XYZ, X'Y'Z'$ and their corresponding circumcircles $k_1,k_2$ (See Figure 6).
Lemma 3. Circle $c'$ is orthogonal to circles $k_1$ and $k_2$ and points $X, Z, X'Z'$ are concyclic.

This follows by considering the inversion with respect to circle $c'$. It is easily seen that this interchanges circle $c$ and line $h$ and fixes, as a whole, circles $k_1$ and $k_2$. Hence these two circles are orthogonal to $c'$. The second claim is a consequence of the first one.

As noticed in the introduction, the two triangles are perspective and their perspectivity center $H_X$ is on line $h$. With the notations adopted so far the following is true.

Lemma 4. The lines $YY'$, joining the centers of the circles $k_1, k_2$, pass through a fixed point $I$ lying on line $OP$. The points $P, I$ are related by the homology $f$ defined by the triple $(c, h, P)$:

$$\frac{PO}{PS} = \frac{IO}{IS} \cdot \frac{DO}{DE}.$$

In fact, lines $YY'$ are images $YY' = f(XX')$ of lines $XX'$ under the homology $f$, hence they pass all through $I = f(P)$. As a result, $I$ lies on $OP$ and taking the intersection $S = (OP, h)$ we have $(OSPI) = \frac{DO}{DE}$. The formula results by expanding the cross ratio $(OSPI) = \frac{PO}{PS} : \frac{IO}{IS}$.

Theorem 5. With the previous notations the map $Z' = f(Z)$ on line $h$ is an involution, which in line coordinates with origin at the projection $Q$ of $I$ on $h$ obtains the form, for a constant $w$:

$$Z' = -\frac{w^2}{z}.$$

![Figure 7. The involution $Z' = g(Z)$](image)
on line \( h \). This, taking coordinates \( z, z' \) with origin at \( Q \), is described by a Moebius transformation ([7, I,p.157]) of the kind (See Figure 7).

\[
z' = \frac{az + b}{cz - a}.
\]

The rest of the theorem, on the form of this involution (vanishing of \( a \)), follows immediately from the collinearity of \( D, P, Q \), which is a consequence of Lemma 4. In fact, if the line \( XX' \) passing through \( P \), obtains the position of \( PD \), then points \( X', D \) become coincident and the corresponding circle \( k_2 \) becomes the tangent to \( c \) at \( D \), the line of centers \( YY' \) becomes orthogonal to \( h \) and passes through \( I \) (See Figure 7). For this particular position of \( XX' \) we see also that as point \( X' \) converges to \( D \), the corresponding \( Z' \) goes to infinity. Thus for \( z = 0 \) the value of \( f(z) \) must be infinity, hence \( a = 0 \), thereby proving the theorem by setting the constant \( w^2 = -b/c \). The value of the constant can be easily calculated by

![Figure 8. The value of the constant](image)

letting \( X \) take the place of the intersection point of the line \( OP \) with \( c \). Then \( XX' \) becomes a diameter of \( c \) and \( DZZ', DX'X \) become similar right angled triangles. Setting for \( D \) the coordinates \( D(e, d) \) and denoting the coordinates of all other points by corresponding small letters, we obtain

\[
(z - e)(z' - e) = -d^2 \quad \text{and} \quad (z - s)(z' - s) = \delta,
\]

where \( \delta = -|SX||SX'| \) is the power of \( S \) with respect to the circle \( c \). The second equation follows from the concyclicity of points \( Z, X, Z', X' \) (Lemma 3). From these we obtain the value

\[
-w^2 = zz' = \frac{s(d^2 + e^2) + e(\delta - s^2)}{e - s}.
\]
Interpreting geometrically the theorem, we deduce that there is a fixed point \( W(0, w) \) on the axis \( QI \), such that the involution \( Z' = g(Z) \) is defined by means of the intersections of line \( h \) by two orthogonal lines rotating about \( W \) ([5, Involution, p.3]), this is formulated in the following.

**Corollary 6.** The circle pairing induces an involution \( Z' = g(Z) \) on line \( h \), which coincides with the one defined by means of pairs of orthogonal lines rotating about a fixed point \( W \) lying on line \( IQ \).

4. Two auxiliary pencils of circles

Here we continue the exploration of the basic configuration using the notation and conventions of the previous sections.

**Theorem 7.** The following properties of the basic configuration are true.
1. The radical axis \( DN \) of circles \( k_1 \) and \( k_2 \) passes through \( D \) and the middle \( N \) of \( ZZ' \).
2. If \( M = (DN, IQ) \), then the circle \( m_X \) centered at \( M \) and orthogonal to \( k_1 \) and \( k_2 \), intersects the parallel \( h' \) to \( h \) from \( I \) at two fixed points \( V_1 \) and \( V_2 \).
3. The circles \( m_X \) are members of a pencil with base points \( V_1 \) and \( V_2 \).

Property (1) immediately follows from Lemma 3, since then \( D \) is on the radical axis of the two circles, and because of the tangency to line \( h \), the middle \( N \) of \( ZZ' \) is also on the radical axis of \( k_1 \) and \( k_2 \) (See Figure 10).

Property (2) follows from the fact that both circles \( c' \) and \( m_X \) being orthogonal to \( k_1 \) and \( k_2 \), their radical axis coincides with \( YY' \), which passes through \( I \). The power of \( I \) with respect to \( c' \) and \( m_X \) is constant and equal in absolute value to \( |IV_1|^2 \), as claimed.

Property (3) is an immediate consequence of (2).
Corollary 8. The radical axis of circle $m_X$ with base points $V_1, V_2$ passes through a fixed point $R$ on $V_1V_2$ and passes also through $H_X$.

The first claim is a well known property for the radical axis of members of a pencil and a fixed circle ([2, p.210]). The second claim follows from the fact that $H_X$ is the radical center of circles $c'$, $c$ and $m_X$.

Next theorem deals with the circles having diameter $ZZ'$. Since by Theorem 5 the map $Z' = g(Z)$ is an involution, we know that this is a pencil of circles ([4, vol.II,p.27]). Here we make precise the kind of the pencil and the location of its base points. Note that this pencil and the one of the preceding theorem have a common member $m_0$ carrying the base points of both pencils.

Theorem 9. The following properties are valid.

1. The circle $n_X$ with diameter $ZZ'$ is orthogonal to $k_1$ and $k_2$.
2. The circle $n_X$ intersects the circle $m_0$ centered at $Q$ and passing through $V_1$ and $V_2$, at its diametral points $W$ and $W'$.
3. The circles $n_X$ are members of a pencil with base points $W$ and $W'$.

Property (1) follows from the fact that $n_X$ has its center at the middle $N$ of $ZZ'$, from which the tangents to $k_1, k_2$ are equal (See Figure 11).

Property (2) results from Theorem 5. Another proof results from the fact that circle $n_X$ belongs to the pencil of orthogonal circles to $k_1$ and $k_2$. From this it follows that circles $m_X, c'$ and $n_X$ pass through the two base points $U_1, U_2$ lying on line $YY'$, which passes through $I$. Let $W$ and $W'$ be the diametral points of circle $m_0$ on the diameter passing through $I$. Then $|IW||IW'| = |IV_1||IV_2|$ but also, since $V_1V_2, U_1U_2$ are chords of the circle $m_X$ it is $|IV_1||IV_2| = |IU_1||IU_2|$. This implies that $I$ is on the radical axis of $m_0, n_X$, which, since both circles have their center on $h$, is orthogonal to $h$. Thus it coincides with $IQ$ or equivalently with $WW'$. 

Figure 10. The pencil of circles $m_X$ with base points $V_1, V_2$
Property (3) is an immediate consequence of (2). Note that point \( W \) coincides with the one defined in Corollary 6.

**Corollary 10.** The radical axis of circle \( n_X \) and \( c \) passes through a fixed point \( T \) on \( WW' \), and passes also through point \( H_X \).

The first claim is a well known property for the radical axis of members of a pencil and a fixed circle ([2, p.210]). The second claim follows from the fact that \( H_X \) is the radical center of circles \( c, m_X \) and \( n_X \).

5. The pivoting line of the pairing of circles

Next figure results by drawing orthogonals respectively to lines \( WZ \) and \( WZ' \) from the centers \( Y \) and \( Y' \) of the circles \( k_1 \) and \( k_2 \). These intersect at right angles at point \( W_X \) and intersect also the parallel \( h' \) to \( h \) at points \( W_1, W_2 \).

**Theorem 11.** Lines \( ZW_1 \) and \( Z'W_2 \) are parallel and \( W_X \) varies on a fixed line \( h'' \) parallel to \( h \) at distance equal to \( |WI| \).

To prove the theorem a short calculation seems unavoidable. For this, we use coordinates along line \( h \) and its orthogonal through \( Q \), as in §3. Denoting coordinates with respective small letters and by \( r, r', r_1, r_2 \) respectively the radii of circles \( c, c', k_1, k_2 \), we find easily the relations

\[
\begin{align*}
z z' &= -w^2, \\
r^2 &= -2rd, \\
r_1 &= \frac{1}{2d}(2rd + (z - e)^2 + d^2), \\
r_2 &= \frac{1}{2d}(2rd + (z' - e)^2 + d^2), \\
w_1 &= \frac{wz + (r_1 - i)z'}{w}, \\
w_2 &= \frac{wz' + (r_2 - i)z}{w}. 
\end{align*}
\]

From these, by a short calculation, we see that

\[w_2 - w_1 = (z' - z).\]
This means that $W_1W_2$ is equal in length to $ZZ'$, thus lines $ZW_1$ and $Z'W_2$ are parallel, as claimed. The other claims are consequences of this property, since then $W_1W_2WX$ and $Z'ZW$ are equal right angled triangles and projecting $WX$ on $IW$ to $T$, we get $|QT| = |WI|$ (See Figure 12).

Next theorem states a property of the tangents to $k_1, k_2$ at points $K, K'$, resulting by intersecting these circles respectively with lines $WZ$ and $WZ'$.

Theorem 13. The tangents to circles $k_1, k_2$, which are reflections of $h$, respectively, on $WX_1$, $WX_2$ are parallel and equidistant to the median $WXM$ of the right angled triangle $WX_1W_2$. This median passes through a fixed point $A$, independent of the position of $X$ on the circle $c$.

Lemma 12. If a variable line intersects the $x-$axis $e$ and a parallel to it $e'$ at distance $q$, so that the coordinates $x, x'$ of the intersection points satisfy an equation of the form

$$ax + bx' + c = 0,$$

then the variable line passes through the fixed point with coordinates

$$A = \frac{1}{a + b}(-c, qb).$$
That the tangents are parallel follows easily by measuring the angles at their intersection points with $h$. Similarly follows their parallelity to the median $W_X M$ (See Figure 14).

To show that line $W_X M$ passes through a fixed point it suffices, according to the previous lemma, to show that the $x$-coordinates of points $M$ and $W_X$ satisfy a linear relation. In fact, a somewhat extended calculation shows that $(W_X)_x$ and $M_x$ satisfy the equation

$$(w - d)(W_X)_x + (2d - w)M_x - ew = 0.$$

From this and the previous lemma follows that line $W_X M$ passes through the fixed point

$$A = \frac{1}{d}(ew, i(2d - w)).$$

In the sequel line $W_X M$ is called the pivoting line of the circle pairing.

6. Sawayama pairings

I call the pairing of circles $k_1, k_2$ defined by the basic configuration $(c, h, P)$ a Sawayama pairing, when the corresponding point $I$ of the configuration is on the arc of circle $c'$ lying inside $c$ (See Figure 15). The formulas of the preceding section imply then that points $W$ and $I$ coincide ($i = w$) and consequently that $W_X$ lies on line $h$. This implies in turn that the three parallels coincide with the pivoting line of the pairing, which is also the median of triangle $W_X W_1 W_2$ and which becomes simultaneously tangent to the two circles $k_1, k_2$ while passing also through the fixed point $A$. A short calculation shows also that in these circumstances point $A$ lies on circle $c$, thus producing the figure of Sawayama’s theorem. In fact, the last ingredient of the figure is the collinearity of points $A, I$ and $D$, which, in view of the previous conventions for the coordinates reduces to an easy check of the vanishing of a determinant. By a well known property of the incenter ([2, p.76]),
point $I$ coincides with the incenter of triangle $ABC$, thereby proving the next theorem.

**Theorem 14.** Each Sawayama pairing $k_1, k_2$ corresponds to a triangle $ABC$ with incenter $I$ and cevians $AW_X$, such that the two circles are tangent to the circum-circle $c$ of the triangle and also tangent to the cevian $AW_X$ and side $BC$. The centers $Y, Y'$ of the circles and the incenter are collinear.

Fixing the basic configuration $(c, h, P)$, so that the corresponding point $I$ lies on the arc of $c'$ contained in $c$, we obtain another proof of Sawayama’s theorem, once we can show, that point $I$ can obtain every possible position on the aforementioned arc of $c'$ and, for each position of $I$, the corresponding pivoting lines $AW_X$ can obtain all positions of the cevians through $A$ of triangle $ABC$. That $I$ can take every position on the claimed arc of $c'$, is a simple consequence of the homographic relation of $I$ to $P$ described in §3. Next lemma makes this point clear and shows that the whole arc $BC$ on $c$ is obtained from a proper arc of an ellipse via the homography $f$, defined in §2. The somewhat more interesting verification that the pivoting line $AW_X$ obtains all positions of cevians from $A$ is handled in the next section.

**Lemma 15.** For $I$ varying on circle $c'$ the corresponding point $P$ varies on an ellipse, which is tangent to circle $c$ at points $B$ and $C$.

In fact, since $I = f(P)$ is a homology, the claimed relation is given through the inverse homology $P = f^{-1}(I)$, which has the same center and axis and ratio $k' = \frac{DE}{DO}$. Hence the curve $c'' = f^{-1}(c')$ is a conic and the other claims result from simple geometric considerations.
7. The cevians

Using the notation of the previous sections, we deduce here various properties of the Sawayama pairings, appearing to be of interest and leading to the proof that, for a fixed point $I$ on the arc of $c'$ lying inside $c$, the corresponding pivoting lines $AW_X$, for variable $X$, obtain all positions of the cevians through $A$.

**Lemma 16.** The circle $c''$ with diameter $Z_1Z_2$ is orthogonal to circles $k_1, k_2$ and the diametral point $I'$ defines line $II'$ which is parallel to $BC$. Also the triples of points $(Z_1, I', V)$ and $(Z_2, I', V')$ are collinear.

The proof of parallelity follows from the equality of angles $W_XZ'Z_2, W_XZ_2Z', TIZ_2$ (See Figure 17). The collinearity of $Z_1, I', V$ follows from the equality of angles $W_XZ'I$, $ZW_XY$ and the parallelity of $VZ_1$ to $YW_X$. 

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**Figure 16.** The locus $c'$ of $P$ as $Q$ varies on $c'$

**Figure 17.** The parallel to $h$ line $II'$ and the radical axis of $c_x, c_X$. 

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Pairings of circles and Sawayama’s theorem
Corollary 17. Line $YY'$ passes through the intersection points $I, I''$ of the circles $c'$ and $c''$.

![Figure 18. Line $I'W_X$](image)

Lemma 18. Line $I'I''$ is orthogonal to $II''$ and passes through point $W_X$ and also through points $G, G'$, which are correspondingly diametrals of $I$ with respect to the circle with diameter $ZZ'$ and the circle $c'$.

In fact, by the previous lemma, angle $I'I''I$ is a right one (See Figure 18). Also $I''$ is on the circle with diameter $YW_X$, which passes through $Z, Z_1$. Thus $I', I'', W_X$ are collinear. The center $N$ of the circle with diameter $ZZ'$ is on line $RD$, which is the radical axis of circles $k_1, k_2$ and is orthogonal to $II'$ at its middle. Hence $I'W_X$ is parallel to $RD$. This implies the two other statements of the lemma.

Corollary 19. The cross ratio $\kappa = (BCW_XH_X)$ is constant and equal to $\frac{UB}{UC}$. Here $U$ is the projection of $G'$ on $BC$.

This follows by considering the pencil $I''(B, C, W, H_X)$ of lines through $I''$. The lines of the pencil pass through fixed points of the circle $c'$. Thus their cross ratio is independent of the position of $I''$ on this circle. Its value is easily calculated
Pairings of circles and Sawayama’s theorem

by letting \( I'' \) take the place of \( U' \), which is the projection of \( G' \) on \( II' \). Then \( H_X \) goes to infinity and \( \kappa \) becomes equal to the stated value.

**Corollary 20.** Fixing point \( I \), which becomes the incenter of triangle \( ABC \), the pivoting lines \( AW_X \) of the Sawayama pairing take the positions of all cevians through \( A \) and Sawayama’s theorem is true.

In fact, by the previous corollary, the constancy of cross ratio implies that for variable \( X \) on \( c \) points \( H_X \) obtain all positions on line \( BC \) and consequently \( W_X \), being related to \( H_X \) by a line-homography, obtains also all possible positions on this line.

**References**

Derivation of the Law of Cosines via the Incircle

Larry Hoehn

Abstract. The law of cosines can be derived without using special cases and without using the Pythagorean theorem.

Most trigonometry textbooks treat the derivation of the law of cosines as three separate cases depending on whether the triangle is acute, obtuse, or right. In this note we derive a proof using an arbitrary general triangle and we do not use the Pythagorean theorem.

We let \(ABC\) be any triangle whose incircle is tangent at \(D, E,\) and \(F\) as shown in Figure 1. Let \(a = BC, b = CA, c = AB,\) and \(r\) be the radius of the incircle with incenter \(I.\) We derive our formula by equating independent expressions for \(r.\)

\[
\text{If } x = CD = CE, \text{ then } c = (a - x) + (b - x) \text{ so that } x = \frac{1}{2}(a + b - c). \text{ In the right triangle } CEI, x = r \cot \frac{C}{2}. \text{ Solving } \frac{1}{2}(a + b - c) = r \cot \frac{C}{2} \text{ for } r \text{ yields}
\]

\[
r = \frac{a + b - c}{2 \cot \frac{C}{2}}. \quad (1)
\]

By computing the area of triangle \(ABC\) in two ways we obtain

\[
\frac{1}{2}ab \sin C = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr,
\]

and

\[
r = \frac{ab \sin C}{a + b + c}. \quad (2)
\]
From (1) and (2), we have \( \frac{a+b-c}{2 \cot \frac{C}{2}} = \frac{ab \sin C}{a+b+c} \). Cross multiplying and making use of some well known trigonometric identities we obtain the following.

\[
(a + b)^2 - c^2 = 2ab \sin C \cot \frac{C}{2} \\
= 2ab \left( 2 \sin \frac{C}{2} \cos \frac{C}{2} \right) \cdot \frac{\cos \frac{C}{2}}{\sin \frac{C}{2}} \\
= 4ab \cos^2 \frac{C}{2}.
\]

Therefore,

\[
c^2 = (a + b)^2 - 4ab \cos^2 \frac{C}{2} \\
= a^2 + b^2 - 2ab \left( 2 \cos^2 \frac{C}{2} - 1 \right) \\
= a^2 + b^2 - 2ab \cos C.
\]

Since the identity \( \sin^2 x + \cos^2 x = 1 \) (and hence \( \cos 2x = 2 \cos^2 x - 1 \)) can be derived independently [2] without the Pythagorean theorem, we have derived the law of cosines without the use of the Pythagorean theorem. This also answers a question in a footnote in [1, p. 135] whether this is possible.

References


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On the Fermat Geometric Problem

Zvonko Čerin

Abstract. We consider the Fermat configuration generated by a point on a circle over the side of a rectangle. When the ratio of the rectangle’s sides is $\sqrt{2}$, then many properties do not depend on a position of the point. Some properties hold for all ratios and other ratios can also have interesting geometric consequences. Our proofs use analytic geometry but some parts include also synthetic arguments.

1. Introduction

Among the numerous questions that Pierre Fermat formulated, the following geometric problem is our main concern (see Figure 1).

\textbf{Fermat Problem.} Let $P$ be a point on the semicircle that has the top side $AB$ of the rectangle $ABB'A'$ as a diameter. Let $\frac{|AB|}{|AA'|} = \sqrt{2}$. Let the segments $PA'$ and $PB'$ intersect the side $AB$ in the points $C$ and $D$, respectively. Then $|AD|^2 + |BC|^2 = |AB|^2$.

The great Leonard Euler in [6] has provided the first rather long proof, which is old fashioned (for his time), and avoids the analytic geometry (which offers rather simple proofs as we shall see later). Several more concise synthetic proofs are now known (see [10], [7, pp. 602, 603], [1, pp. 168, 169] and [8, pp. 181, 264]). A very nice description of Euler’s proof is available on the Internet (see [11]).
The analytic proof was recently recalled in [9] where it was observed that the above relation holds for all points on the circle with the segment $AB$ as a diameter. For a circle, we shall consider a slightly more general situation where the quotient $|\frac{AB}{AA'}|$ is a positive real number $m$ (see Figure 2.).

For given different points $A$ and $B$ and any points $P_1, P_2, P_3, P_4$ in the plane, let

$$\varphi(P_1P_2, P_3P_4) := \frac{|P_1P_2|^2 + |P_3P_4|^2}{|AB|^2}.$$

In this notation, the above Fermat Problem for the circle is the implication $(a) \Rightarrow (b)$ in Theorem 1 below.

**Theorem 1.** Let $U_b = \varphi(AD, BC)$. The following statements are equivalent.

(a) $m = \sqrt{2}$, and
(b) $U_b = 1$.

**Proof.** We shall use analytic geometry, which offers a simple proof. Let the origin of the rectangular coordinate system be the midpoint $O$ of the side $AB$ so that the points $A$ and $B$ have coordinates $(-r, 0)$ and $(r, 0)$ for some positive real number $r$ (the radius of the circle). The equation of the circle is a standard $x^2 + y^2 = r^2$.

The coordinates of the points $A'$ and $B'$ are $(-r, -\frac{2r}{m})$ and $(r, -\frac{2r}{m})$. For any real number $t$, let $u = 1 - t^2, v = 1 + t^2, z = mt, \eta = v - z$ and $\vartheta = v + z$. An arbitrary point $P$ on the circle has coordinates $\left(\frac{ru}{\sqrt{\eta}}, \frac{2rt}{\sqrt{\eta}}\right)$. From similar right-angled triangles $PVC$ and $PQA'$ and $s = \frac{2r}{m}$, we easily find that $C\left(\frac{ru - z}{\sqrt{\vartheta}}, 0\right)$...
and $D \left( \frac{r(u+z)}{y}, 0 \right)$. The equivalence of the statements (a) and (b) follows from the identity $U_b - 1 = \frac{t^2(m^2-2)}{y^2}$.

The fact that $(a) \implies (b)$ can be proved more simply by synthetic methods. Here is an adaptation of Lionnet’s proof from [7, p. 602], which works for any point on the circle (see Figure 2).

Let the directed lengths $\overrightarrow{AC}, \overrightarrow{CD}, \overrightarrow{DB}$ be $a, b, c$. Then

$$AD^2 + BC^2 - AB^2 = (a + b)^2 + (b + c)^2 - (a + b + c)^2 = b^2 - 2ac,$$

so that

$$AD^2 + BC^2 = AB^2 \iff b^2 = 2ac. \quad (1)$$

Now draw $CY$ and $DZ$ perpendicular to $AB$, with $Y$ on $PA$ and $Z$ on $PB$. Using pairs of similar triangles, we have

$$\frac{YC}{AA'} = \frac{PC}{PA'} = \frac{CD}{A'B'} = \frac{PD}{PB'} = \frac{ZD}{BB'}.$$

Hence $YCDZ$ is a rectangle similar to $AA'B'B$. The triangles $YCA$, $BDZ$ are equiangular, so $\frac{YC}{a} = \frac{CD}{DZ}$. But $YC = DZ = \frac{CD}{\sqrt{2}} = \frac{b}{\sqrt{2}}$. Thus $b^2 - 2ac$ vanishes, hence $AD^2 + BC^2 = AB^2$.

![Figure 3. Simple analytic proof.](image)

We can also use the equivalence (1) to obtain a simple analytic proof (see Figure 3). There is no loss of generality in taking the radius of the circle as the unit of
length. By similar triangles,
\[
\frac{a}{m} = \frac{1 + x}{2 + m + y}, \quad \frac{b}{m} = \frac{2}{2 + m + y} \quad \text{and} \quad \frac{c}{m} = \frac{1 - x}{2 + m + y}.
\]
Simplifying \(b^2 - 2ac\) and putting \(x^2 = 1 - y^2\) we find that this difference is
\[4y^2(m^2 - 2)(2 + my)^2.\]

Let \(A''\), \(B''\), \(P''\) be the reflections of the points \(A', B', P\) in the line \(AB\). We close this introduction with a remark that most of our results come in related pairs. The second version, which requires no extra proof, comes (for example in Theorem 1) by replacing the points \(C\) and \(D\) with the points \(C'\) and \(D'\), which are the intersections of the line \(AB\) with the lines \(PA''\) (or \(P'A'\)) and \(PB''\) (or \(P'B'\)). In other words, if \(V_b = \varphi(AD', BC')\), then \((a)\) and
\[(b')\ V_b = 1\are also equivalent. Moreover, \((a)\) and
\[(b'')\ U_b = V_b\are equivalent as well.

2. Invariants of the Fermat configuration

Our primary goal is to present several statements similar to \((b)\), \((b')\) and \((b'')\) that could replace it in Theorem 1. In other words, we explore what other relationships in the Fermat configuration remain invariant as the point \(P\) changes position on the circle.

We begin with the diagonals of the trapezium \(A'B'DC\) (see Figure 2).

**Theorem 2.** Let \(U_c = \varphi(A'D, B'C)\) and \(V_c = \varphi(A'D', B'C')\).

Consider the statements
\[(c)\ U_c = 2, \quad (c')\ V_c = 2, \quad (c'')\ U_c = V_c.

The following are true.
\[(a) \iff (c''),\]
\[(a)\ implies each of the statements \((c)\) and \((c')\).

**Proof.** With straightforward computations one can easily check that
\[
V_c - U_c = \frac{2(m^2 - 2)v t^3}{\eta^2 \theta^2},
\]
\[
2 - U_c = \frac{(m^2 - 2)v(2z + v)}{m^2 \eta^2 \theta^2},
\]
\[
V_c - 2 = \frac{(m^2 - 2)v(2z - v)}{m^2 \eta^2}.
\]

Of course, Pythagoras’ theorem might be useful in computing the function \(\varphi\). For instance, for \(U_c\) we have
\[A'D^2 + B'C^2 = AA'^2 + BB'^2 + AD^2 + BC^2.\]
But \(AA'^2 + BB'^2 = AB^2\) and \(AD^2 + BC^2 = AB^2\). Hence \(U_c = 2.\)
This method will also yield generalizations. Take points \( A_s, \ B_s, \ C_s, \ D_s \) with \( \text{AA}_s = \lambda \text{AA}' \), \( \text{BB}_s = \lambda \text{BB}' \), \( \text{BC}_s = \mu \text{BC} \), \( \text{AD}_s = \mu \text{AD} \). Then it follows easily that \( \varphi(A_sD_s, B_sC_s) = \lambda^2 + \mu^2 \).

For points \( X \) and \( Y \), let \( X \oplus Y \) be the center of the square built on the segment \( XY \) such that the triangle \( X(X \oplus Y)Y' \) has the positive orientation (counterclockwise). When the point \( X \oplus Y \) is shortened to \( M \), then \( M^* \) denotes \( Y \oplus X \).

The midpoints \( \text{G}, \ \text{H}, \ \text{G}', \ \text{H}' \) of the segments \( \text{AC}, \ \text{BD}, \ \text{AC}', \ \text{BD}' \) and the top \( \text{N} \) of the semicircle over \( \text{AB} \) are used in the next theorem. In other words, \( N = B \oplus A \). The center \( \text{O} \) of the circle (i.e., the midpoint of the segment \( AB \); the origin of the rectangular coordinate system) appears also.

**Theorem 3.** Let \( U_d = \varphi(NG, NH), \ V_d = \varphi(NG', NH') \), \( U_e = \varphi(OG, OH) \) and \( V_e = \varphi(OG', OH') \) (see Figure 4).

The following statements are equivalent.

(a) \( m = \sqrt{2} \),
(b) \( U_d = \frac{3}{4}, \ (d') \ V_d = \frac{3}{4}, \ (d'') \ U_d = V_d, \ (d'') \ U_d = 3U_e, \ V_d = 3V_e \).
(c) \( U_e = \frac{1}{4}, \ (e') \ V_e = \frac{1}{4}, \ (e'') \ U_e = V_e, \ (e'') \ U_e = 3U_e, \ V_e = 3V_e \).

**Proof:** This time the differences \( U_d - \frac{3}{4} \) and \( U_e - \frac{1}{4} \) both simplify to \( \frac{\sqrt{2} (m^2 - 2)}{4 \sqrt{s}} \), which has the factor \( m^2 - 2 \) again. Similarly, \( V_d - U_d \) and \( V_e - U_e \) both are equal \( \frac{m \sqrt{4 (m^2 - 2)}}{2 \sqrt{s}} \). Finally, \( 3U_e - U_d \) is \( \frac{\sqrt{2} (m^2 - 2)}{4 \sqrt{s}} \). The other differences are analogous quotients that have \( \eta \) instead of \( \vartheta \).

The following synthetic proof shows that the statements (a) and (b) together imply (d) and (c).

A dilatation with center \( A \) and scale factor 2 maps \( GO \) on to \( CB \), thus \( GO = \frac{CB}{2} \); similarly \( OH = \frac{AB}{2} \). Consequently for \( U_e \) we have

\[
OG^2 + OH^2 = \frac{BC^2}{4} + \frac{AD^2}{4} = \frac{1}{4} AB^2.
\]

Similarly, for \( U_d \) we have \( NG^2 = NO^2 + OG^2 \) and \( NH^2 = NO^2 + OH^2 \) so that

\[
NG^2 + NH^2 = 2 NO^2 + \frac{BC^2 + AD^2}{4} = \frac{AB^2}{2} + \frac{AB^2}{4} = \frac{3}{4} AB^2.
\]

\[\square\]

Let \( G_s, \ H_s, \ G'_s, \ H'_s \) be the points that divide the segments \( NG, \ NH, \ NG', \ NH' \) in the same ratio \( s \neq -1 \) (i.e., \( NG_s : G_sG = s : 1 \), etc.).

**Theorem 4.** Let \( m_s = \frac{s^2 + 2}{4(s+1)^2}, \ n_s = \frac{3s^2}{4(s+1)^2} \).

\( U_f = \varphi(OG_s, OH_s), \ V_f = \varphi(OG'_s, OH'_s), \)
\( U_g = \varphi(NG_s, NH_s), \ V_g = \varphi(NG'_s, NH'_s). \)

If \( s \neq 0 \), then the following statements are equivalent.

(a) \( m = \sqrt{2} \),
(b) \( U_f = m_s, \ (f') \ V_f = m_s, \ (f'') \ U_f = V_f, \)
(c) \( U_g = n_s, \ (g') \ V_g = n_s, \ (g'') \ U_g = V_g. \)
Proof. Since $G_s = \left( -\frac{rs(t+m+t)}{(s+1)^2}, \frac{r}{s+1} \right)$ and $H_s = \left( \frac{rs(1+z)}{(s+1)^2}, \frac{r}{s+1} \right)$, the difference $U_f - m_s$ is $\frac{s^2t^2(m^2-2)}{4(s+1)^2}$. The other parts are proved with analogous arguments. □

Theorem 5. Let $\lambda = \frac{3s^2}{s^2+2}$. If $s \neq 0, 1$, then the following statements are equivalent.

(a) $m = \sqrt{2}$,

(f) $U_g = \lambda U_f$,

(g) $V_g = \lambda V_f$.

Proof. For $s \neq 0, 1$, the equivalence of (a) and (f) is a consequence of the equality $\lambda U_f - U_g = \frac{s^2t^2(s-1)(m^2-2)}{2(s+1)(s^2+2)}$. □

Let $N_1$, $N_2$, $N_3$, $N_4$ denote the highest points on the semicircles built on the segments $AC$, $BD$, $AC'$, $BD'$ above the line $AB$. In other words, $N_1 = C \oplus A$, $N_2 = B \oplus D$, $N_3 = C' \oplus A$, $N_4 = B \oplus D'$.

Theorem 6. Let $U_h = \varphi(BN_1, AN_2)$, $V_h = \varphi(BN_3, AN_4)$,

$U_i = \varphi(NN_1, NN_2)$, $V_i = \varphi(NN_3, NN_4)$ (see Figure 4). The following statements are equivalent.

(a) $m = \sqrt{2}$,

(h) $U_h = \frac{3}{2}$, (h') $V_h = \frac{3}{2}$, (h'') $U_h = V_h$, (h'') $3U_i = U_h$,

(i) $U_i = \frac{1}{2}$, (i') $V_i = \frac{1}{2}$, (i'') $U_i = V_i$, (i'') $3V_i = V_h$.
Proof. Since $N_1 = \left( \frac{-r(1+z)}{\eta}, \frac{r}{\eta} \right)$ and $N_2 = \left( \frac{r(1+z)}{\eta}, \frac{r^2}{\eta} \right)$, we get

$$U_h - \frac{3}{2} = U_i - \frac{1}{2} = \frac{r^2(m^2-2)}{2\eta^2}.$$  

This proves the equivalence of $(a)$ with $(h)$ and $(i)$. The proofs of the other equivalences are similar.

The synthetic proof of the implication $(a) \Rightarrow (h)$ uses the right-angled triangles $AHN_2$ and $BGN_1$ to get $AN_2^2 = (a+b+\frac{c}{2})^2 + \frac{c^2}{4}$ and $BN_1^2 = (\frac{a}{2} + b + c)^2 + \frac{a^2}{4}$. But $b^2 = 2ac$ implies $AN_2^2 + BN_1^2 = \frac{3}{2}AB^2$.

Let $a' = \frac{a}{\sqrt{2}}$, etc. For the implication $(a) \Rightarrow (i)$, from the isosceles right-angled triangles $ABN$, $ACN_1$, $BDN_2$ we get $NN_1^2 = (b' + c')^2$ and $NN_2^2 = (a' + b')^2$. But $b^2 = 2ac$ implies $NN_1^2 + NN_2^2 = \frac{1}{2}AB^2$. \hfill $\square$

In the following theorem we also use the points $N_1$, $N_2$, $N_3$ and $N_4$. However, we do not use the function $\varphi$.

**Theorem 7.** The following statements are equivalent.

1. $m = \sqrt{2}$.
2. $|N_3N_2| = |AN|$, (j') $|N_3N_4| = |AN|$, (j'') $|N_1N_2| = |N_3N_4|$.
3. $|N_1N_2|^2 + |N_2N_3|^2 + |N_3N_4|^2 + |N_4N_1|^2 = 2|AB|^2$.

**Proof.** For $(a) \Leftrightarrow (j)$, we easily get $|N_1N_2|^2 - |AN|^2 = \frac{2r^2t^2(m^2-2)}{\eta^2}$.

Since the coordinates of $N_3$ and $N_4$ are $\left( \frac{r(tm-t)}{\eta}, \frac{r}{\eta} \right)$ and $\left( \frac{r(t-1-z)}{\eta}, \frac{r^2}{\eta} \right)$,

$$|N_1N_2|^2 + |N_2N_3|^2 + |N_3N_4|^2 + |N_4N_1|^2 - 2|AB|^2 = \frac{8r^2t^2(m^2-2)(v^2+z^2)}{\eta^2\eta^2}.$$  

This shows that $(a) \Leftrightarrow (k)$.

Note that by projecting $BC$ orthogonally on to $AN$, we obtain $NN_1$, and its length is $\frac{BC}{\sqrt{2}}$. Similarly, $|NN_2| = \frac{\overline{AD}}{\sqrt{2}}$. The fourth vertex of the rectangle with sides $NN_1$, $NN_2$ is the point $M_1$ used later in the paper. Hence $|N_1M_1| = |N_1N_2| = |AN|$, proving $(a) \Rightarrow (o)$ in Theorem 9 below.

Notice that $|N_1N_2|^2 + |N_2N_3|^2 + |N_3N_4|^2 + |N_4N_1|^2 = 2|AB|^2$ if and only if $m = 1$.

Let $N_5 = A \oplus D$, $N_6 = C \oplus B$, $N_7 = A \oplus D'$ and $N_8 = C' \oplus B$.

**Theorem 8.** Let $U_\ell = \varphi(AN_5, BN_6)$, $V_\ell = \varphi(AN_7, BN_8)$, $U_m = \varphi(GN_6, HN_5)$, $V_m = \varphi(G'N_5, H'N_5)$ and $U_n = \varphi(NN_5, NN_6)$, $V_n = \varphi(NN_7, NN_8)$.

The following ten statements are equivalent:

1. $m = \sqrt{2}$.
2. $U_\ell = \frac{1}{2}$, (\ell') $V_\ell = \frac{1}{2}$, (\ell'') $U_\ell = V_\ell$.
3. $U_m = \frac{3}{4}$, (m') $V_m = \frac{3}{4}$, (m'') $U_m = V_m$.
4. $U_n = \frac{3}{2}$, (n') $V_n = \frac{3}{2}$, (n'') $U_n = V_n$. 

Proof. Since \( N_5 = \left( \frac{-r t^2}{q}, \frac{-r (1+z)}{q} \right) \) and \( N_6 = \left( \frac{r}{q}, \frac{-r t (m+t)}{q} \right) \), we obtain \( U_\ell = -\frac{1}{2} = \frac{t^2 (m^2 - 2)}{2 q^2} \). This shows the equivalence of (a) and (\ell). The other equivalences have similar proofs.

In order to prove the implication (a) \( \Rightarrow \) (\ell) in synthetic fashion, from the isosceles right-angled triangles \( ADN_5 \) and \( BCN_6 \), we get

\[
AN_5^2 + BN_6^2 = AD^2 + BC^2 = \frac{1}{2} AB^2.
\]

\( \square \)

Let \( \lambda_s = \frac{3((s+1)^2 + 1)}{4(s+1)^2} \). The equality \( \varphi(G_s N_6, H_s N_5) = \lambda_s \) is true if and only if \( m = \sqrt{2} \). In fact, this is the first equivalence from another similar group that involve points \( G_s \) and \( H_s \).

![Figure 5. Points \( M_1 \) and \( M_2 \) in Theorem 9.](image)

The next theorem uses the centers of squares on the segments \( CD \) and \( C'D' \).

Let \( M_1 = C \oplus D \) and \( M_2 = C' \oplus D' \).

**Theorem 9.** Let \( U_o = |N M_1| - |A N|, \ V_o = |N M_2| - |A N|, \ U_p = \varphi(M_1 N_1, M_1 N_2) \) and \( V_p = \varphi(M_2 N_3, M_2 N_4) \) (see Figure 5).

The following statements are equivalent:

(a) \( m = \sqrt{2} \),

(o) \( U_o = 0 \), \( V_o = 0 \), \( (o') \ U_o = V_o \),

(p) \( U_p = \frac{1}{2} \), \( V_p = \frac{1}{2} \), \( (p') \ U_p = V_p \).

**Proof.** Since \( M_1 = \left( \frac{u}{q}, \frac{-z}{q} \right) \), the difference \( |M_1 N|^2 - |A N|^2 \) is \( \frac{2 r^2 t^2 z (m^2 - 2)}{q^2 z^2} \). Similarly, \( |M_2 N|^2 - |M_1 N|^2 = \frac{8 r^2 t^2 z (m^2 - 2)}{q^2 z^2} \). This shows that (a) \( \Leftrightarrow \) (o) and (a) \( \Leftrightarrow \) (o'). The (a) \( \Rightarrow \) (o) is proved also as follows.
From the right-angled triangle $NN_1M_1$, since \( \frac{b^2}{2} = ac \), we get $NM_1^2 = (a' + b')^2 + (b' + c')^2 = a'^2 + b'^2 + c'^2 + ab + bc = (a' + b' + c')^2 = NA^2$. \(\square\)

For any point $X$ in the plane, let $G_1, G_2, G_3, G_4, G_5$ and $G_6$ denote the centroids of the triangles $ACX, CDX, DBX, AC'X, C'D'X$ and $BD'X$.

**Theorem 10.** Let $U_q = \varphi(G_2G_1, G_2G_3)$ and $V_q = \varphi(G_5G_4, G_5G_6)$.

The following statements are equivalent:

(a) $m = \sqrt{2}$,

(b) $U_q = \frac{1}{3}$,  \( (q) \) $V_q = \frac{1}{9}$,  \( (q') \) $U_q = V_q$.

**Proof.** If $X = (x, y)$, then the points $G_1, G_2$ and $G_3$ have the same ordinate \( \frac{y}{3} \) while their abscissae are \( \frac{2}{3} - \frac{2r(t(m+t)}{9\theta}, \frac{2}{3} + \frac{2r(t(m+1)}{9\theta} \) and \( \frac{2}{3} + \frac{2r(z+1)}{9\theta} \). It follows that

\[
U_q - \frac{1}{3} = \frac{t^2(m^2-2)}{9\theta^2} \quad \text{that proves} \quad (a) \Leftrightarrow (q).
\]

The implication \( (b) \Rightarrow (q) \) could be proved as follows. Let $I$ denote the midpoint of the segment $CD$. A dilatation with center $C$ and scale factor $2$ maps $GI$ on to $AD$, thus $GI = \frac{AB}{2}$; similarly $HI = \frac{BC}{2}$. Hence $\varphi(GI, HI) = \frac{1}{4}$. On the other hand, a dilatation with center $X$ and scale factor $\frac{3}{2}$ maps $G_1G_2$ on to $GI$, thus $G_1G_2 = \frac{2}{3}GI$; similarly $G_3G_2 = \frac{2}{3}HI$. Hence $U_q = \frac{1}{9}$. \(\square\)

Let $U$ and $V$ be the midpoints of the segments $CC'$ and $DD'$.

**Theorem 11.** Let $U_s = \varphi(NU, NV)$, $U_t = \varphi(OU, OV)$ and $V_t = \varphi(N_6U, N_5V)$.

The following statements are equivalent:

(a) $m = \sqrt{2}$,

(b) $U_s = 1$,

(c) $U_t = \frac{1}{3}$,  \( (t') \) $V_t = \frac{1}{2}$,  \( (t'') \) $U_t = V_t$.

**Proof.** Since abscissae of $U$ and $V$ are \( \frac{r(u^2v)}{\eta \sigma} \) and \( \frac{r(u^2v)}{\mu \sigma} \), we get

\[
U_s - 1 = U_t - \frac{1}{2} = \frac{t^2(m^2-2)}{\eta \sigma^2} \quad \text{that proves} \quad (a) \Leftrightarrow (s) \quad \text{and} \quad (a) \Leftrightarrow (t). \quad \square
\]

**Theorem 12.** Let $W = U \oplus V$, $U_a = |WO| - \frac{|AB|}{2}$,

$U_{v_{i,j}} = \varphi(WN_i, WN_j)$ for $i \in \{1, 3\}$ and $j \in \{2, 4\}$ and $U_w = \varphi(W^*N, WN)$.

The following statements are equivalent:

(a) $m = \sqrt{2}$,

(b) $U_a = 0$,

(c) $U_w = 1$,

(x) the lines $WN_1$ and $WN_2$ are perpendicular,

(y) the lines $WN_3$ and $WN_4$ are perpendicular,

(v) \( U_{v_{i,j}} = \frac{1}{2} \) for $i \in \{1, 3\}$ and $j \in \{2, 4\}$.

**Proof.** Since the coordinates of the point $W$ is the pair \( \left( \frac{ru}{\eta \sigma}, \frac{rz^2}{\eta \sigma} \right) \), we get that

\[
|WO|^2 - r^2 = \frac{2r^2 t^2 v^2 (m^2-2)}{\eta^2 \sigma^2} . \quad \text{This proves} \quad (a) \Leftrightarrow (u).
\]

For the equivalence \( (a) \Leftrightarrow (x) \), note that the lines $WN_1$ and $WN_2$ have equations

\[
(z^2 - \eta)x + (z^2 - \theta)y = \lambda \quad \text{and} \quad (m^2 - \eta)x - (m^2 - \theta)y = \mu,
\]
where \( \lambda \) and \( \mu \) are real numbers. These lines are perpendicular if and only if \( 2v z (m^2 - 2) \) is zero.

Let \( K_1 = B \oplus N_1 \), \( K_2 = N_2 \oplus A \), \( K_3 = B \oplus N_3 \), \( K_4 = N_4 \oplus A \). These points can be defined more simply. They all are at the same height as \( N \) and vertically above the points \( N_6, N_5, N_8, N_7 \), respectively.

**Theorem 13.** Let \( \lambda = \frac{1}{2} + \sqrt{2} \), \( U_z = \varphi(A'K_2, B'K_1) \) and \( V_z = \varphi(A'K_4, B'K_3) \).

For the statements
(a) \( m = \sqrt{2} \),
(b) \( U_z = \lambda, \ (z') \ V_z = \lambda \),

we have (a) \( \Rightarrow \) (z) and (a) \( \Rightarrow \) (z').

**Proof.** Since \( K_2 \) and \( K_1 \) have abscissae \( -\frac{z^2}{\delta} \) and \( \frac{z^2}{\delta} \), we conclude that \( \lambda - U_z = \frac{1}{(4(m+1)\delta^2-z^2)\sqrt{2}+m(z+2v)(3z+2v)(m-\sqrt{2})}. \) When (a) is true, then this difference is zero. The proof of (a) \( \Rightarrow \) (z') is similar.

Let \( K = P \oplus A', L = P \oplus B', S = P \oplus A'' \), \( T = P \oplus B'' \).

**Theorem 14.** Let \( \lambda = 1 + \sqrt{2} \), \( U_\alpha = \varphi(AS, BL) \) and \( V_\alpha = \varphi(AS, BT^*) \).

The following statements are equivalent:
(a) \( m = \sqrt{2} \),
(b) \( U_\alpha = \lambda, \ (\alpha') \ V_\alpha = \lambda \).

**Proof.** Since \( \left( \frac{r(\delta-tz)}{m v}, -\frac{r(m+\eta)}{m v} \right) \) and \( \left( \frac{r(m-\eta)}{m v}, -\frac{r(tz+\eta)}{m v} \right) \) are the coordinates of the points \( K^* \) and \( L \), we get that \( \lambda - U_\alpha \) is equal to \( \frac{1+\sqrt{2}}{2} \left( 1 + \sqrt{2} \right) \left( m - 2 \right) \). It follows from Theorem 18 that the same holds for \( V_\alpha \).

Let \( S_1, T_1, S_2 \) and \( T_2 \) denote the midpoints of the segments \( A'C, B'D, A'C' \) and \( B'D' \). Note that (a) implies that
\[
\varphi(G_sS_1, H_sT_1) = \varphi(G_s' S_2, H_s'T_2) = \frac{(s+1+\sqrt{2})^2+1}{4(s+1)^2}.
\]

**Theorem 15.** Let \( \lambda_\pm = 1 \pm \sqrt{2} \),
\( U_\beta = \varphi(NS_1, NT_1) \), \( V_\beta = \varphi(NS_2, NT_2) \),
\( U_\gamma = \varphi(N S_1, N T_1) \), \( V_\gamma = \varphi(N S_2, N T_2) \),
\( U_\delta = \varphi(OS_1, OT_1) \) and \( V_\delta = \varphi(OS_2, OT_2) \).

Consider the statements
(a) \( m = \sqrt{2} \),
(b) \( U_\beta = \lambda_+, \ (\beta') \ V_\beta = \lambda_+, \ (\beta'') \ U_\beta = V_\beta \),
(\gamma) \( U_\gamma = \lambda_-, \ (\gamma') \ V_\gamma = \lambda_- \), \( (\gamma'') \ U_\gamma = V_\gamma \),
(\delta) \( U_\delta = \frac{1}{2} \), \( (\delta') \ V_\delta = \frac{1}{2} \), \( (\delta'') \ U_\delta = V_\delta \).

The following are true:
(i) The statements (a), \( (\beta'') \), \( (\gamma'') \) and \( (\delta'') \) are equivalent.
(ii) (a) implies each of the statements \( (\beta), (\beta'), (\gamma), (\gamma'), (\delta) \) and \( (\delta') \).
Proof. For the implication \((a) \Rightarrow (\delta)\), from the right-angled triangles \(OGS_1\) and \(OHT_1\) and Theorem 3, we get that the sum \(OS_1^2 + OT_1^2\) is equal to \((OG^2 + GS_1^2) + (OH^2 + HT_1^2)\), i.e., to \((OG^2 + OH^2) + \frac{AB^2}{4} = \frac{1}{2}AB^2\). \(\square\)

For points \(X\) and \(Y\), let \(\mathcal{g}_X^Y\) be the reflection of the point \(X\) in the point \(Y\). Let \(Q = \mathcal{g}_A^D, R = \mathcal{g}_B^C, Q' = \mathcal{g}'_A^D, R' = \mathcal{g}'_B^C\).

Theorem 16. Let \(U_\varepsilon = \varphi(A'Q, B'R)\) and \(V_\varepsilon = \varphi(A'Q', B'R')\).

Consider the statements
(a) \(m = \sqrt{2}\),
(\(\varepsilon\)) \(U_\varepsilon = 5\), \((\varepsilon')\) \(V_\varepsilon = 5\), \((\varepsilon'')\) \(U_\varepsilon = V_\varepsilon\).

The following are true:
(i) The statements (a), (\(\varepsilon''\)) are equivalent.
(ii) (a) implies each of the statements (\(\varepsilon\)) and (\(\varepsilon'\)).

Proof. Since \(Q = \left(\frac{r(u+3z+2)}{\sqrt{a}}, 0\right)\) and \(R = \left(\frac{r(3u-3z-2)}{\sqrt{a}}, 0\right)\), we get that \(5 - U_\varepsilon\) is equal to \(\frac{\eta(m^2-2)(\varphi + 2z)}{m^2}\). From this we conclude that \((a) \Rightarrow (\varepsilon)\).

From the right-angled triangles \(AA'Q\) and \(BB'R\), we get that \(A'Q^2\) and \(B'R^2\) are \(4(a + b)^2 + A'A'^2\) and \(4(b + c)^2 + B'B'^2\). By adding we conclude from (1) that \(U_\varepsilon = 5\). Also, we have \(\varphi(N_5Q, N_6R) = \frac{5}{2}\) and \(\varphi(AQ, BR) = 4\). \(\square\)

3. Common properties for all ratios

Of course, there are many properties that hold for all ratios \(m\). The following are two examples of such properties.

![Figure 6. Common orthocenters in Theorem 17.](image-url)
Theorem 17. (i) The triangles $ADP$ and $BCP$ have the same orthocenter that lies on a circle with the segment $CD$ as a diameter.

(ii) The triangles $ADP'$ and $BCP'$ have a common orthocenter. It lies on a circle with the segment $C'D'$ as a diameter.

Proof. The orthocenters of the triangles $ADP$ and $BCP$ both have coordinates $\left( \frac{ru}{v}, \frac{2mrt^2}{dv} \right)$.

If $K_0$ is the midpoint of the segment $CD$ and $L_0$ is the orthocenter of the triangle $ADP$, then $|CK_0|^2 - |K_0L_0|^2 = 0$ so that $L_0$ lies on the circle with the segment $CD$ as a diameter. □

Theorem 18. The equalities

$$\varphi(AK, BL^*) = \varphi(AS^*, BT)$$ and $$\varphi(AK^*, BL) = \varphi(AS, BT^*)$$ hold for all points $P$ on the circle and every ratio $m$.

Proof. Both $\varphi(AK, BL^*)$ and $\varphi(AS^*, BT)$ have the value $\frac{(m-1)^2+1}{2m^2}$ while both $\varphi(AK^*, BL)$ and $\varphi(AS, BT^*)$ have the value $\frac{(m+1)^2+1}{2m^2}$. □

4. Some other interesting ratios

Our last result is different because in it some other ratios besides $\sqrt{2}$ appear. This has happened already in a comment following the proof of Theorem 9. Let $\lambda = 1 - \frac{\sqrt{2}}{2}$.

Theorem 19. The ratio $m$ is either $\sqrt{2}$ or $2 + \sqrt{2}$ if and only if for every point $P$ on the circle the equalities $\varphi(AK, BL^*) = \lambda$ and/or $\varphi(AS^*, BT) = \lambda$ hold.

Proof. Since $\left( -\frac{r(\vartheta+tz)}{mv}, \frac{r(m-\eta)}{mv} \right)$ and $\left( \frac{r(m+\vartheta)}{mv}, \frac{r(tz-\eta)}{mv} \right)$ are the coordinates of the points $K$ and $L^*$, we get that $\lambda - \varphi(AK, BL^*)$ will factor out as the quotient $\frac{(1-\sqrt{2})(m-\sqrt{2})}{2m^2}$. The same is true for $\varphi(AS^*, BT)$ by Theorem 18. □

Conclusion. Fermat conjectured that $(a) \Rightarrow (b)$ and Euler proved this. We added $(b')$, $(b'')$, $(c)$, . . . , $(e'')$ and discovered here that many are equivalent. Also, in related papers, we consider how to replace the circle with an arbitrary conic and propose the space versions for the sphere and some other surfaces (see [2], [3], [4] and [5]).

References


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**Jigsawing a Quadrangle from a Triangle**

Floor van Lamoen

**Abstract.** For an acute-angled triangle $ABC$, we construct two isotomic points $P$ and $Q$ on $BC$ and $P'$ and $Q'$ on $AB$ and $AC$ respectively, such that $BP'P$ and $CQQ'$ are right triangles which, when rotated about $P'$ and $Q'$ respectively through appropriate angles fit with $AP'Q'$ to a quadrangle with a new fourth vertex $A'$. We show that $AA'$ passes through the circumcenter $O$.

Given an acute-angled triangle $ABC$ consider the construction of a pair of isotomic points $P$ and $Q$ on the side $BC$ (with $BP = QC < \frac{1}{2} \cdot BC$) such that the perpendiculars to $BC$ at $P$ and $Q$ intersect $AB$ and $AC$ at $P'$ and $Q'$ satisfying $P'Q' = PP' + QQ'$.

If we put $BP = QC = t$, then $PP' = t \cdot \tan B$, $QQ' = t \cdot \tan C$, and $P'Q' = t(\tan B + \tan C) = \frac{t \sin A}{\cos B \cos C}$. Applying the law of cosines to triangle $AP'Q'$, we have

$$(c - \frac{t}{\cos B})^2 + (b - \frac{t}{\cos C})^2 - 2 \left( c - \frac{t}{\cos B} \right) \left( b - \frac{t}{\cos C} \right) \cos A = t^2 \cdot \frac{1 - \cos^2 A}{\cos^2 B \cos^2 C}.$$

Clearing denominators and rearranging terms, we have

$$(1 - \cos^2 A - \cos^2 B - \cos^2 C + 2 \cos A \cos B \cos C) t^2$$

$$+ 2 \cos B \cos C(b \cos B + c \cos C - a \cos A)t - a^2 \cos^2 B \cos^2 C = 0.$$

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Making use of the following identities
\[ b \cos B + c \cos C - a \cos A = 2a \cos B \cos C, \]
\[ 1 - \cos^2 A - \cos^2 B - \cos^2 C = 2 \cos A \cos B \cos C, \]
we rewrite (1) as
\[ 4t^2 \cos A + 4at \cos B \cos C - a^2 \cos B \cos C = 0 \] (2)
after cancelling a common factor \( \cos B \cos C \). Writing \( x = \frac{a}{2} - t \), or \( 2t = a - 2x \), we have
\[ (a - 2x)^2 \cos A + 2a(a - 2x) \cos B \cos C - a^2 \cos B \cos C = 0 \]
\[ 4x^2 \cos A - 4ax(\cos A + \cos B \cos C) + a^2(\cos A + \cos B \cos C) = 0 \]
\[ 4x^2 \cos A - 4ax \sin B \sin C + a^2 \sin B \sin C = 0. \] (3)

From this,
\[ x = \frac{a \sin B \sin C - a \sqrt{\sin B \sin C} \sin B \sin C - \cos A}{2 \cos A} \]
\[ = \frac{a \sin B \sin C - a \sqrt{\sin B \cos B \cdot \sin C} \cos C}{2 \cos A} \] (4)

Applying the law of sines to triangle \( ABC \), with \( b = 2R \sin B \) and \( c = 2R \sin C \) for the circumradius \( R \), we have
\[ \frac{2x}{a} = \frac{2R \sin B \sin C - 2R \sqrt{\sin B \sin C} \cdot \cos B \cos C}{2R \cos A} \]
\[ = \frac{b \sin C - \sqrt{b \cos B \cdot c \cos C}}{2R \cos A} \]
\[ = \frac{AX - \sqrt{BX \cdot XC}}{2R \cos A} = \frac{AX - X_1X}{AH} = \frac{AX_1}{AH}, \]
where \( H \) is the orthocenter of triangle \( ABC \), and the altitude \( AX \) intersects the semicircle with diameter \( BC \) at \( X_1 \) (see Figure 2). This leads to the following construction of the trapezoid \( PQQP' \).

Let \( O \) and \( G \) be the circumcenter and centroid of triangle \( ABC \), and \( D \) the midpoint of \( BC \).

1. Construct the semicircle with diameter \( BC \) (on the same side of \( A \)) to intersect the \( A \)-altitude at \( X_1 \).
2. Construct the line \( X_1G \) to intersect \( OD \) at \( X_2 \).
3. Construct the parallels to \( OB \) and \( OC \) through \( X_2 \) to meet \( AC \) in \( P \) and \( Q \) respectively.
4. Construct the perpendiculars to \( BC \) at \( P \) and \( Q \) to intersect \( AB \) and \( AC \) at \( P' \) and \( Q' \) respectively.

These points satisfy \( PP' + QQ' = P'Q' \).

**Proof.** In Figure 2,
\[ \frac{X_2D}{OD} = \frac{PD}{BD} = \frac{2x}{a} = \frac{AX_1}{AH}. \]
Figure 2.

Since $AH = 2 \cdot OD$, we have $AX_1 = 2 \cdot X_2D$, and $HX_1 = AH - AX_1 = 2 \cdot OD - 2 \cdot X_2D = 2 \cdot OX_2$. From this it is clear that $X_1X_2$ passes through the centroid $G$ which divides $HO$ in the ratio $HG : GO = 2 : 1$. □

We rotate triangle $BP P'$ about $P'$ and triangle $CQQ'$ about $C$ so that the images of $P$ and $Q$ coincide at a point on $P'Q'$. Then the images of $B$ and $C$ coincide at a point $A'$.

Consider the quadrilateral $AP' A'Q'$. Note that $\angle Q' A' P' = \angle B + \angle C$, so that the quadrilateral is cyclic. Hence,

$$\angle Q' A' A = \angle Q' P' A' = \angle P P' B = \frac{\pi}{2} - B.$$

This means that $AA'$ passes through the circumcenter $O$ of triangle $ABC$.

By symmetry, if we perform similar constructions on the sides $CA$ and $AB$, and obtain points $B', C'$ corresponding to $A'$, the lines $BB'$ and $CC'$ also pass through the circumcenter $O$, as does $AA'$. 
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Pedal Polygons

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Abstract. We study the pedal polygon $H_1H_2\cdots H_n$ of a point $P$ with respect to a polygon $P$, where the points $H_i$ are the feet of the perpendiculars drawn from $P$ to the sides of $P$. In particular we prove that if $P$ is a quadrilateral which is not a parallelogram, there exists one and only one point $P$ for which the points $H_i$ are collinear.

1. Introduction

Consider a polygon $A_1A_2\cdots A_n$ and call it $P$. Let $P$ be a point and let $H_i$ be the foot of the perpendicular from $P$ to the line $A_iA_{i+1}$, $i = 1, 2, \ldots, n$ (with indices $i$ taken modulo $n$). The points $H_i$ usually form a polygon $H_1H_2\cdots H_n$, which we call the pedal polygon of $P$ with respect to $P$, and denote by $H$ (see Figure 1). We call $P$ the pedal point. See ([2, p.22]) for the notion of pedal triangle.

In this article we find some properties of the pedal polygon $H$ of a point $P$ with respect to $P$. In particular, when $P$ is a triangle we find the points $P$ such that the pedal triangle $H$ is a right, obtuse or acute triangle. When $P$ is a quadrilateral which is not a parallelogram, we prove that there exists one and only one point $P$ for which the points $H_i$ are collinear. Moreover, we find the points $P$ for which
the pedal quadrilateral $H$ of $P$ has at least one pair of parallel sides. We also prove that, in general, there exists one and only one pedal point with respect to which $H$ is a parallelogram. In the last part of the paper, we find some properties of the pedal polygon $H$ in the general case of a polygon $P$ with $n$ sides.

2. Properties of the pedal triangle

Let $P$ be a triangle. The pedal triangle of the circumcenter of $P$ is the medial triangle of $P$; the one of the orthocenter is the orthic triangle of $P$; the one of the incenter is the Gergonne triangle of $P$ (i.e., the triangle whose vertices are the points in which the incircle of $P$ touches the sides of $P$).

**Theorem 1.** [2, p.41] If $P$ is a triangle, the points $H_i$ are collinear if and only if $P$ lies on the circumcircle of $P$.

The line containing the points $H_i$ is called *Simson line* of the point $P$ with respect to $P$ (see Figure 2).

![Figure 2.](image)

**Theorem 2.** [1, p.108] The points $P$ for which the pedal triangle $H$ is isosceles are all and only the points that lie on at least one of the Apollonius circles associated to the vertices of $P$.

The Apollonius circle associated to the vertex $A_i$ is the locus of points $P$ such that $PA_{i+1} : PA_{i+2} = A_iA_{i+2} : A_iA_{i+1}$. The three Apollonius circles are coaxial and they intersect in the two isodynamic points of the triangle $P$, $I_1$ and $I_2$. Therefore, the isodynamic points of $P$ are the only points whose pedal triangles with respect to $P$ are equilateral (see Figure 3).
We will find now the points $P$ whose pedal triangle is a right, acute, obtuse triangle. Let $P$ be a point (see Figure 4) and let $A_iA_{i+1} = a_{i+2}$, $PA_i = x_i$, $H_iH_{i+1} = h_{i+2}$. Since the quadrilateral $A_iH_iPH_{i+2}$ is cyclic, $h_i = x_i \sin A_i$ ([2, p.2]).
By the Pythagorean theorem and its converse, the pedal triangle of $P$ is right in $H_i$ if and only if
\[ x_i^2 \sin^2 A_i = x_{i+1}^2 \sin^2 A_{i+1} + x_{i+2}^2 \sin^2 A_{i+2}. \]

By the law of sines, this is equivalent to
\[ a_i^2 x_i^2 = a_{i+1}^2 x_{i+1}^2 + a_{i+2}^2 x_{i+2}^2. \]

(1)

This relation represents the locus $\gamma_i$ of points $P$ for which the triangle $H_i$ is right in $H_i$. Therefore, the locus of points $P$ whose pedal triangle is a right triangle is $\gamma_1 \cup \gamma_2 \cup \gamma_3$. Observe that $\gamma_i$ contains the points $A_{i+1}$ and $A_{i+2}$; moreover, $\gamma_i$ and $\gamma_{i+1}$ intersect only in the point $A_{i+2}$.

We verify now that $\gamma_1$ is a circle. Set up an orthogonal coordinate system such that $A_2 \equiv (1,0)$ and $A_3 \equiv (-1,0)$; let $A_1 \equiv (a,b)$ and $P \equiv (x,y)$. The relation (1) becomes:
\[ 4((x-a)^2+(y-b)^2) = ((a+1)^2+b^2)((x-1)^2+y^2)+((a-1)^2+b^2)((x+1)^2+y^2). \]

Simplifying, we obtain the equation of a circle:
\[ (a^2 + b^2 - 1)(x^2 + y^2) + 4by - (a^2 + b^2 - 1) = 0. \]

Moreover, it is not hard to verify that the tangents to the circumcircle of $P$ in the points $A_2$ and $A_3$ pass through the center of $\gamma_1$.

Analogously the same holds for $\gamma_2$ and $\gamma_3$. We can then state that $\gamma_i$ is a circle passing through the points $A_{i+1}$ and $A_{i+2}$; the tangents to the circumcircle of $P$ in the points $A_{i+1}$ and $A_{i+2}$ pass through the center $C_i$ of $\gamma_i$; moreover, $\gamma_i$ and $\gamma_{i+1}$ are tangent in $A_{i+2}$. Then, if $C_1C_2C_3$ is the tangential triangle of $A_1A_2A_3$, $\gamma_i$ is the circle with center $C_i$ passing through $A_{i-1}$ and $A_{i+1}$ (see Figure 5).

![Figure 5](image_url)

Observe that, by the law of cosines, the angle in $H_i$ of the pedal triangle of $P$ is obtuse if and only if:
\[ a_i^2 x_i^2 > a_{i+1}^2 x_{i+1}^2 + a_{i+2}^2 x_{i+2}^2. \]
i.e., the point $P$ lies inside the circle $\gamma_i$. Thus, we have established the following theorem.

**Theorem 3.** The pedal triangle of a point $P$ is
(a) a right triangle if and only if $P$ lies on one of the circles $\gamma_i$,
(b) an obtuse triangle if and only if $P$ is inside one of the circles $\gamma_i$,
(c) an acute triangle if and only if $P$ is external to all the circles $\gamma_i$.

3. Properties of the pedal quadrilateral

Let $P$ be a cyclic quadrilateral. The pedal quadrilateral of the circumcenter of $P$ is the Varignon parallelogram of $P$, and the one of the anticenter ([6, p.152]) is the principal orthic quadrilateral of $P$ ([5, p.80]).

Let $P$ be a tangential quadrilateral. The pedal quadrilateral of the incenter of $P$ is the contact quadrilateral of $P$, i.e., the quadrilateral whose vertices are the points in which the incircle of $P$ touches the sides of $P$.

For a generic quadrilateral, we consider the problem of finding the pedal points for which the points $H_i$ are collinear.

It is easy to verify that if $P$ has only one pair of parallel sides, there is only one pedal point $P$ for which the points $H_i$ are collinear. $P$ is the common point to the lines containing opposite and non parallel sides of $P$, and the points $H_i$ lie on the perpendicular from $P$ to the parallel sides of $P$. On the other hand, if $P$ is a parallelogram, there is no point with respect to which the points $H_i$ are collinear.

Suppose now that $P$ is a quadrilateral without parallel sides (see Figure 6). Let $A_5$ be the common point to the lines $A_1A_2$ and $A_3A_4$, and $A_6$ the common point

![Figure 6](image-url)
to $A_2A_3$ and $A_1A_4$. Consider the four triangles $A_1A_2A_6$, $A_2A_3A_5$, $A_3A_4A_6$, $A_1A_4A_5$, and let $C_1, C_2, C_3, C_4$ be their circumcircles, respectively.

If the pedal point $P$ lies on one of the circles $C_i$, then, by Theorem 1, at least three of the points $H_i$ are collinear. It follows that the four points $H_i$ are collinear if and only if $P$ lies in every $C_i$. The four circles are concurrent in the Miquel point of the quartet of lines containing the sides of $P$ ([3, p.82]). Thus, we have established the following theorem.

**Theorem 4.** If $P$ is a quadrilateral, that is not a parallelogram, there exists one and only one pedal point with respect to which the points $H_i$ are collinear.

We call this point the Simson point of the quadrilateral $P$, and denote it by $S$. We call the Simson line of $P$ the line containing the points $H_i$. Observe that the points $H_i$ determine a quadrilateral if and only if $P \not\in C_1 \cup C_2 \cup C_3 \cup C_4$.

**Theorem 5.** If $P$ is a quadrilateral which is not a parallelogram, the reflections of the Simson point with respect to the lines containing the sides of $P$ are collinear and the line $\ell$ containing them is parallel to the Simson line.

**Proof.** The theorem is trivially true if $P$ has one pair of parallel sides. Suppose that $P$ is without parallel sides (see Figure 7). Let $K_i, i = 1, 2, 3, 4$, be the reflection of $S$ with respect to the line $A_iA_{i+1}$. The points $S, H_i$ and $K_i$ are collinear and $SH_i = H_iK_i$, then $K_i$ is the image of $H_i$ under the homothety $h(S, 2)$. Then, since the points $H_i$ are collinear (Theorem 4), the points $K_i$ are also collinear. Moreover, the line $\ell$ containing the points $K_i$ is parallel to the Simson line of $P$. \qed
**Conjecture.** If $\mathbf{P}$ is a cyclic quadrilateral without parallel sides, the line $\ell$ passes through the anticenter $H$ of $\mathbf{P}$, and the Simson line bisects the segment $SH$.

The conjecture was suggested by using a dynamic geometry software (see Figure 8). However, we have been unable to prove it.

If $\mathbf{P}$ is a cyclic quadrilateral with a pair of parallel sides, then $\mathbf{P}$ is an isosceles trapezoid. The line $\ell$ coincides with the Simson line, i.e., the line joining the midpoints of the bases of $\mathbf{P}$, and passes through the anticenter of $\mathbf{P}$. In this case the Simson line contains the segment $SH$.

We now find the points $\mathbf{P}$ whose pedal quadrilaterals have at least one pair of parallel sides.

If $\mathbf{P}$ is a parallelogram, then the points $\mathbf{P}$ whose pedal quadrilaterals have at least one pair of parallel sides are all and only the points of the diagonals of $\mathbf{P}$.

Suppose now that $\mathbf{P}$ is not a parallelogram. We prove that the locus of the point $\mathbf{P}$ whose pedal quadrilateral has the sides $H_1H_4$ and $H_2H_3$ parallel is the circle $A_1A_3S$ (see Figure 9).

First observe that $S$ is a point with respect to whom $H_1H_4$ and $H_2H_3$ are parallel because the points $H_i$ are collinear. Set up now an orthogonal coordinate system such that $A_1 \equiv (-1, 0)$ and $A_3 \equiv (1, 0)$; let $A_2 \equiv (a, b)$, $A_4 \equiv (c, d)$ and
If $H_1H_4$ and $H_2H_3$ are parallel, then $P$ lies on the circle $\gamma$ of equation:

$$(hd + kb)x^2 + (hd + kb)y^2 - (hk - 4bd)y = hd + kb,$$

where $h = a^2 + b^2 - 1$ and $k = c^2 + d^2 - 1$.

Note that the points $A_1$ and $A_3$ are on $\gamma$, and $\gamma$ is the circle $A_1A_3S$.

Analogously we can prove that the points $P$ whose pedal quadrilateral has the sides $H_1H_2$ and $H_3H_4$ parallel is the circle $A_2A_4S$. Therefore we have established the following theorem.

**Theorem 6.** The points $P$ whose pedal quadrilaterals have at least one pair of parallel sides are precisely those on the circles $A_1A_3S$ and $A_2A_4S$.

In general, the circles $A_1A_3S$ and $A_2A_4S$ intersect at two points, the Simson point $S$ and one other point $P^*$ (see Figure 10). The pedal quadrilateral of $P^*$ is
a parallelogram. We call $P^*$ the parallelogram point of $P$. Observe that if $P$ is a parallelogram the parallelogram point is the intersection of the diagonals of $P$.

If $P$ is cyclic, the pedal quadrilateral of the circumcenter $O$ of $P$ is the Varignon parallelogram of $P$. Therefore, the parallelogram point of $P$ is $O$. It follows that if $P$ is cyclic, the Simson point is the intersection point of the circles $A_1A_3O$ and $A_2A_4O$, other than $O$.

4. Some properties of the pedal polygon

Let $P$ be a polygon with $n$ sides. Consider the pedal polygon $H$ of a point $P$ with respect to $P$. We denote by $Q_i$ the quadrilateral $PH_iA_{i+1}H_{i+1}$, for $i = 1, 2, \ldots, n$. Since the angles in $H_i$ and in $H_{i+1}$ are right, $Q_i$ cannot be concave.

**Lemma 7.** If $ABCD$ is a convex or a crossed quadrilateral such that $ABC$ and $CDA$ are right angles, then it is cyclic. Moreover, its circumcenter is the midpoint of $AC$ and its anticenter is the midpoint of $BD$.

![Figure 11](image_url)

**Proof.** Let $ABCD$ be a convex or a crossed quadrilateral with $ABC$ and $CDA$ right angles (see Figure 11). Then, it is cyclic with the diagonal $AC$ as diameter. (When it is a crossed quadrilateral it is inscribed in the semicircle with diameter $AC$). Then its circumcenter $O$ is the midpoint of $AC$.

Consider the maltitudes with respect to the diagonals $AC$ and $BD$. The maltitude through $O$ is perpendicular to the chord $BD$ of the circumcircle, then it passes through the midpoint $H$ of $BD$. But also the maltitude relative to $AC$ passes through $H$. Then, the anticenter of the quadrilateral is $H$. Note that the maltitudes of a crossed quadrilateral $ABCD$ are concurrent because they are also the maltitudes of the cyclic convex quadrilateral $ACBD$. □
By Lemma 7, the quadrilaterals $Q_i$ are cyclic. Denote by $O_i$ and $A'_i$ the circumcenter and the anticenter of $Q_i$ respectively. We call $O_1O_2\ldots O_n$ the polygon of the circumcenters of $P$ with respect to $P$ and denote it by $P_c(P)$. We call $A'_1A'_2\ldots A'_n$ the polygon of the anticenters of $P$ with respect to $P$ and we denote it with $P_a(P)$.

**Theorem 8.** The polygon $P_c(P)$ is the image of $P$ under the homothety $h(P, \frac{1}{2})$.

**Proof.** By Lemma 7, the circumcenter $O_i$ of $Q_i$ is the midpoint of $A_iP$ (see Figure 12 for a pentagon $P$).

![Figure 12](image12.png)

Figure 12

Note that by varying $P$ the polygons $P_c(P)$ are all congruent to each other (by translation).

**Theorem 9.** The polygon $P_a(P)$ is the medial polygon of $H$, with vertices the midpoints of the segments $H_iH_{i+1}$ for $i = 1, 2, \ldots, n$.

![Figure 13](image13.png)

Figure 13
Proof. By Lemma 7, the anticenter $A'_i$ of $Q_i$ is the midpoint of $H_iH_{i+3}$ (see Figure 13 for a pentagon).

Corollary 10. (a) If $P$ is a triangle, $P_a(P)$ is the medial triangle of $H$.
(b) If $P$ is a quadrilateral, $P_a(P)$ is the Varignon parallelogram of $H$.

Theorem 11. If $H$ is cyclic, the Euler lines of the quadrilaterals $Q_i$ are concurrent at the circumcenter of $H$.

Proof. The Euler line of the quadrilateral $Q_i$ passes through the circumcenter $O_i$ of $Q_i$ and through the anticenter $A'_i$ of $Q_i$, that is the midpoint of $H_iH_{i+3}$, then it is the perpendicular bisector of a side of $H$ (see Figure 14 for a quadrilateral).

Corollary 12. If $P$ is a triangle, the Euler lines of the quadrilaterals $Q_i$ are concurrent at the circumcenter of $H$ (see Figure 15).

Remark. If $P$ is a quadrilateral and $H$ is not cyclic, the Euler lines of the quadrilaterals $Q_i$ bound a quadrilateral affine to $H$ ([4, p.471]).

References


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Special Inscribed Trapezoids in a Triangle

Nikolaos Dergiades

Abstract. We give a generalization Floor van Lamoen’s recent result on jigsawing a quadrangle in a triangle.

1. Construction of an inscribed trapezoid

This note is a generalization of a recent result of Floor van Lamoen [1].

For an arbitrary point $A'$ on the side $BC$ of a given triangle $ABC$, we want to find on $BC$ two points $P, P'$ isotomic with respect to $B$ and $C$ such that the parallels from $P, P'$ to $AA'$ meet their closest sides $AB, AC$ at the points $Q, Q'$ and we have in the trapezoid $QP P'Q'$,

$$QQ' = PQ + P'Q'.$$  \hfill (1)

Let $A_1$ be the midpoint of $BC$. The parallel from $A_1$ to $AA'$ meets $QQ'$ at its midpoint $A_2$, the line $AB$ at $Q_1$, and the parallel from $B$ to $AC$ at a point $S_1$.

The symmetric of $Q'$ in $A_1$ is the point $S$, the intersection of $BS_1$ and $QP$. Since $A_1A_2 = \frac{PQ+P'Q'}{2} = \frac{QQ'}{2}$, the triangle $A_1QQ'$ is right angled, and the same holds for the triangle $A_1QS$. The parallel from $Q_1$ to $QA_1$ meets the line $BC$ at a point...
$D$ and from $\frac{BS}{BS_1} = \frac{BQ}{BQ_1} = \frac{BA_1}{BD}$, we conclude that $DS_1$ is parallel to $A_1S$. Hence angle $Q_1DS_1$ is a right angle and the point $D$ lies on the circle with diameter $S_1Q_1$. This leads to the following construction.

Construct

1. the parallel from $A_1$ to $AA'$ to intersect the line $AB$ at $Q_1$ and the parallel from $B$ to $AC$ at $S_1$,
2. the circle with diameter $Q_1S_1$ to intersect $BC$ at the point $D$ such that $A_1$ is between $B, D$,
3. the parallels from $A_1$ to $DQ_1$ and $DS_1$ to intersect $AB$ at $Q$ and $AC$ at $Q'$ respectively,
4. the parallels to $AA'$ from $Q$ and $Q'$ to intersect $BC$ at $P$ and $P''$ respectively.

The points $P, P'$ are isotomic with respect to $BC$, and the trapezoid $QPP'Q'$ satisfies (1) (see Figure 2).

2. An interesting property

Rotate triangle $QBP$ about $Q$ to $QA''P''$ with $P''$ on $QQ'$. Since $Q'P'' = Q'P'$ and $P''C = BP$, a rotation about $Q'$ will bring triangle $Q'C'P'$ to $Q'A''P''$. The lines $QP''$ and $QA''$ are the reflections of $QB$ and $QP$ in the bisector of angle $Q$ of triangle $AQQ'$. Since $QP$ is parallel to $AA'$, the line $QA''$ contains the isogonal conjugate of the infinite point of $AA'$ in triangle $AQQ'$. For the same reason, the line $Q'A''$ also contains the isogonal conjugate of the same infinite point. It follows that $A''$ is this isogonal conjugate, and it lies on the circumcircle of $AQQ'$ (see Figure 3).
In particular, if $AA'$ is an altitude of triangle $ABC$, then the line $AA''$ passes through the circumcenter of the triangle.

Reference


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Gossard’s Perspector and Projective Consequences

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Abstract. Considering as starting point a geometric configuration studied, among others, by Gossard, we pursue the projective study of a triangle in the Euclidean plane, its Euler line and its nine-point circle, and we relate Pappus’ Theorem to the nine-point circle and Euler line.

1. Introduction

The relative position of Euler’s line with respect to the sides of a triangle has raised the geometers’ interest since the very first paper on this topic, Leonhard Euler’s classical work [10].

In 1997, problem A1 from the W. L. Putnam competition explored the case when Euler’s line is parallel to one of the sides of a triangle. Amer. Math. Monthly published Problem 10980 proposed by Ye Zhong Hao and Wu Wei Chao, whose statement is the following. Consider four distinct straight lines in the same plane, with the property that no two of them are parallel, no three are concurrent, and no three form an equilateral triangle. Prove that, if one of the lines is parallel to the Euler line of the triangle formed by the other three, then each of the four given lines is parallel to the Euler line of the triangle formed by the other three. In the Editorial Comment following the solution of problem 10980 (see vol. 111 (2004), pp. 824), the editors have pointed out the meaningful contributions to the history of this problem, especially Gossard’s presentation at an A. M. S. conference in 1915. A generalization from 1999, given by Paul Yiu, is mentioned in [13].

In the Bulletin of the A. M. S. from 1916, we find O. D. Kellogg’s report on Gossard’s 1915 talk at the AMS Southwestern Section Conference (see [15]). As far as we know, Gossard’s paper has not been published, although we know from the report what he proved and what methods he used. The summary, as published by the Bulletin, is the following: "Euler proved that orthocenter, circumcenter, and centroid of a triangle are collinear, and the line through them has received the name Euler line. He also proved that the Euler line of a given triangle together with two of its sides forms a triangle whose Euler line is parallel with the third side of the given triangle. By the use of vector coordinates or ordinary projective coordinates, Professor Gossard proves the following theorem: the three Euler lines of
triangle, form a triangle triply perspective with the given triangle and having the same Euler line. The orthocenters, circumcenters and centroids of these two triangles are symmetrically placed as to the center of perspective."

Our goal in the present note goes beyond providing elementary proofs for these facts, and aims to explore the deeper geometric meaning of a phenomenon seen in the above mentioned results. Application 1 is Ye and Wu’s problem. Applications 3 and 4, proved below, are just particular cases of Application 1. Proposition 1 is Gossard result from 1916, with a different proof. Furthermore, the original tools in Gossard’s work were ordinary projective coordinates. That’s why it would be natural to explore from a projective viewpoint the geometric structure inspired by Euler’s original contribution, which made the substance in Gossard’s work. In the last part of our paper, we discuss the projective viewpoint on the relative position of the Euler line and the three lines forming a given triangle. We will show how Euler’s line can be regarded as the axis of a projectivity between two sides of a triangle. This result was also proved by D. Barbilian (see [4]), and it appears in a note unpublished during Barbilian’s life. The result is presented below in our Proposition 4. We have been able to reconstruct the context of Barbilian’s work and we have obtained incidence results that complete the discussion on Ye and Wu’s problem.

Finally, with Propositions 3 and 4, which as far as we know appear for the first time here, we extend the projective analysis on this geometric structure (i.e., a triangle, its Euler line and its nine-point circle) and will relate Pappus’ Theorem to the nine-point circle and Euler line. We also study the parallelism of Euler’s line with one of the sides of the triangle from the projective viewpoint. In conclusion, one of the most important consequences of our investigation is that we are able to better understand the geometric connections between Euler’s line and the nine-point circle using projective methods. Our geometric motivation was the belief that beyond the synthetic and analytic methods, one can fathom the entire depth of a geometry problem by understanding the projective background of a certain geometric structure.

2. Synthetic and analytic viewpoint

First, we prove a Lemma which will become our main tool of investigation. This Lemma was inspired by Ye and Wu’s problem. Consider the Euclidean plane and a Cartesian frame. Let \( A, B, C \) be three arbitrary points in the Euclidean plane.

**Lemma 1.** Denote by \( m_E \) the slope of Euler’s line in \( \Delta ABC \) and by \( m_1, m_2, m_3 \) the slopes of the lines \( BC, AC, \) and \( AB \), respectively. Then

\[
m_E = -\frac{m_1m_2 + m_3m_1 + m_2m_3 + 3}{m_1 + m_2 + m_3 + 3m_1m_2m_3}
\]

**Proof.** Measuring the slope of the angle between \( BC \) and the Euler’s line of \( \Delta ABC \), we have (see Figure 1):
\[ \frac{m_1 - m_E}{1 + m_1 m_E} = \tan \angle(HOS) = \frac{HS}{MN} = \frac{AM - AH - ON}{BN - BM} \]
\[ = \frac{2R \sin B \sin C - 2R \cos A - R \cos A}{R \sin A - 2R \sin C \cos B} \]
\[ = \frac{2 \sin B \sin C + 3 \cos(B + C)}{\sin(B + C) - 2 \sin C \cos B} \]
\[ = \frac{3 \cos B \cos C - \sin B \sin C}{\sin B \cos C - \sin C \cos B} \]
\[ = 3 - \tan B \tan C \]

Replacing in the last relation the following expressions
\[ \tan B = \frac{m_3 - m_1}{1 + m_1 m_3}, \quad \tan C = \frac{m_1 - m_2}{1 + m_1 m_2}, \]
we get the equality
\[ \frac{m_1 - m_E}{1 + m_1 m_E} = 3 - \frac{m_3 - m_1}{1 + m_1 m_3} \cdot \frac{m_1 - m_2}{1 + m_1 m_2}. \]

Cross-multiplying and collecting the like-terms, we obtain:
\[ m_1 m_2 + m_1 m_3 + m_1 m_E + m_2 m_3 + m_2 m_E + m_3 m_E + 3 m_E m_1 m_2 m_3 + 3 = 0. \]

Solving for \( m_E \) in this relation immediately yields the relation from the statement of our lemma.

We should remark here that any other relative positions of the points \( A, B, C \) yield the same result. Now, we present several applications of this lemma.
Application 1. (Problem 10980, *American Mathematical Monthly*, proposed by Ye Zhong Hao and Wu Wei Chao, 109 (2002) 921, solution, 110 (2004) 823–824.) Consider four distinct straight lines in the same plane, with the property that no two of them are parallel, no three are concurrent, and no three form an equilateral triangle. Prove that, if one of the lines is parallel to the Euler line of the triangle formed by the other three, then each of the four given lines is parallel to the Euler line of the triangle formed by the other three.

**Solution:** Denote by $m_1, m_2, m_3,$ and $m_4$ the slopes of the four lines $d_1, d_2, d_3,$ and $d_4$, respectively. Suppose that Euler’s line of the triangle formed by the lines $d_1, d_2, d_3$ is parallel to $d_4$ and has slope $m_E$. Then $m_E = m_4$ and we get

$$m_1m_2 + m_1m_3 + m_2m_4 + m_3m_4 + 3m_4m_1m_2m_3 + 3 = 0.$$

This relation is symmetric in any one of the slopes and the conclusion follows immediately.

Application 2. Consider $\triangle ABC$ and $\triangle A'B'C'$ such that the measure of the oriented angles between the straight lines $AB$ and $A'B'$, $AC$ and $A'C'$, and $BC$ and $B'C'$, respectively, are equal to $\alpha$. Then the measure of the angle between Euler’s line of $\triangle ABC$ and Euler’s line of $\triangle A'B'C'$ is also $\alpha$.

**Solution:** We consider the following construction (see Figure 2). On the circum-circle of $\triangle ABC$, we consider the points $A''$, $B''$, and $C''$ such that $A''B'' \parallel A'B'$, $A''C'' \parallel A'C'$, and $B''C'' \parallel B'C'$. More precisely, we choose $A''$ such that the angle $\hat{(AOA''}$ is $\alpha$. Let us consider now the rotation $R^\alpha_O$ of center $O$ and oriented angle $\alpha$. Then $m(\angle AB, A''B'') = m(\angle AC, A''C'') = m(\angle BC, B''C'')$ yields $A'' = R^\alpha_O(A)$, $B'' = R^\alpha_O(B)$, $C'' = R^\alpha_O(C)$. We denote by $e, e'$ and $e''$ Euler’s lines of $\triangle ABC$, $\triangle A'B'C'$, and respectively $\triangle A''B''C''$. Then $\triangle A''B''C''$ is obtained by rotating $\triangle ABC$ about $O$ by $\alpha$. Thus, all the elements of $\triangle ABC$ rotate about $O$. This means $e'' = R^\alpha_O(e)$, or $m(\angle(e, e'')) = \alpha$. Since the slopes satisfy the following equalities $m_{A''B''} = m_{A'B'}$, $m_{A''C''} = \ldots$
the slopes of the straight lines $AC$, $AB$, and from Lemma 1:

Let $\Delta ABC$ and $\Delta A'B'C'$ be two triangles with the property $AB \perp A'B'$, $AC \perp A'C'$, $BC \perp B'C'$. Then the Euler lines of the two triangles are perpendicular.

We can prove this Remark directly from Lemma 1. However, we can also provide a direct argument for its proof. Denote by $m_1', m_2', m_3'$ the slopes of the side and by $m_E'$ the slope of Euler’s line of $\Delta A'B'C'$. Then

$$m_E' = \frac{m_1'm_2' + m_3'm_1' + m_2'm_3' + 3}{m_1' + m_2' + m_3' + 3m_1'm_2'm_3'} = \frac{-\frac{1}{m_1}(-\frac{1}{m_2}) + (-\frac{1}{m_1}) + (-\frac{1}{m_2}) + (-\frac{1}{m_1})}{(-\frac{1}{m_1}) + (-\frac{1}{m_2}) + (-\frac{1}{m_3}) + 3(-\frac{1}{m_1})(-\frac{1}{m_2})(-\frac{1}{m_3})} = \frac{m_1 + m_2 + m_3 + 3m_1m_2m_3}{m_1m_2 + m_3m_1 + m_2m_3 + 3} = -\frac{1}{m_E}.$$

This proves that the two lines are perpendicular.

**Application 3.** In the acute triangle $ABC$, Euler’s line is parallel to $BC$ if and only if $\tan B \tan C = 3$.

**Note:** In [17], it is mentioned that this problem was proposed by Dan Br˘anzei. We have discussed this application in [6]. The solution uses a direct trigonometric argument. We present here the analytic argument based on Lemma 1.

**Solution:** Choose a coordinate system so that the $x$-axis is parallel to $BC$. If we denote by $m_1$ the slope of the straight line $BC$, then $m_1 = 0$. Denoting $m_2, m_3, m_e$ the slopes of the straight lines $AC$, $AB$, and Euler’s line $e$, respectively, we get from Lemma 1:

$$m_e = \frac{-m_2m_3 + 3}{m_2 + m_3}.$$

Thus, Euler’s line $e$ of $\Delta ABC$ is parallel to $BC$ if and only if $m_e = 0$, which is equivalent to $m_2m_3 = -3$. Now we take into account that $m_2 = -\tan C$ and $m_3 = \tan B$ (or, depending on the position of $\Delta ABC$, we could have $m_2 = \tan C$ and $m_3 = -\tan B$). Consequently, $\tan B \tan C = 3$.

For an interesting connection between the formula obtained here for $m_e$ and Tzitzeica surfaces, a topic studied in depth in affine differential geometry, see [2]. For a graphical study of Tzitzeica surfaces by using Mathematica, see [3]. For the importance of Tzitzeica’s surfaces in the development of differential geometry at the beginning of the 20th century, see [1].

**Application 4.** (W. L. Putnam Competition, 1997) A rectangle, $HOMF$, has sides $HQ = 11$ and $OM = 5$. A triangle $ABC$ has $H$ as the intersection of the altitudes, $O$ the center of the circumscribed circle, $M$ the midpoint of $BC$, and $F$ the foot of the altitude from $A$. What is the length of $BC$?
Solution: Since Euler’s line is parallel to \( BC \), by the previous application, we have \( \tan B \tan C = 3 \). This is just a consequence of the previous application. We can continue our argument as in [14], pg.233, or [6]. Expressing \( \tan B \) and \( \tan C \) from triangles \( ABF \) and \( AFC \), respectively, we get

\[
\frac{h_a}{BF} \cdot \frac{h_a}{FC} = 3.
\]

Since \( HG \parallel BC \), we have \( h_a = AF = 3FH = 3 \cdot 5 = 15 \). Therefore, \( BF \cdot FC = \frac{15 \cdot 15}{2} = 75 \). Namely, we express \( BC^2 = (BF+FC)^2 = (FC-BF)^2 + 4BF \cdot FC \).

To compute the first term in the last expression we write \( FC-BF = FM+MC- (BM-FM) = 2FM = 2OH = 22 \). Therefore, \( BC^2 = 22^2 + 4 \cdot 75 = 784 \), thus \( BC = 28 \). \( \square \)

**Lemma 2.** Euler’s line of \( \Delta ABC \) intersects the lines \( AB \) and \( AC \) in \( M \), respectively \( N \). Then Euler’s line of \( \Delta AMN \) is parallel to \( BC \).

\[
\text{Figure 3}
\]

Proof. Choose a coordinate system so that the \( x \)-axis is parallel to \( BC \), as in Application 3 (see Figure 3). If we denote by \( m_1 \) the slope of the straight line \( BC \), then \( m_1 = 0 \). Denoting \( m_2, m_3, m_e \) the slopes of the straight lines \( AC, AB \), and respectively Euler’s line \( e \). By Lemma 1:

\[
m_e = -\frac{m_2m_3 + 3}{m_2 + m_3},
\]

and the slope of Euler’s line of \( \Delta AMN \) is

\[
m_{e'} = -\frac{m_em_2 + m_em_3 + m_2m_3 + 3}{m_e + m_2 + m_3 + 3m_em_2m_3}.
\]
In fact, the numerator of the last expression is
\[
m_\epsilon m_2 + m_\epsilon m_3 + m_2 m_3 + 3
\]
\[
= m_\epsilon (m_2 + m_3) + m_2 m_3 + 3
\]
\[
= \left( -\frac{m_2 m_3 + 3}{m_2 + m_3} \right) (m_2 + m_3) + m_2 m_3 + 3 = 0.
\]
In fact, we proved that \( m_\epsilon' = 0 \), which means that \( e' || BC \).

Application 5. Consider two triangles such that \( \triangle ABC \equiv \triangle A'B'C' \) and they have the same Euler’s line. Then \( \triangle A'B'C' \) is obtained from \( \triangle ABC \) either by a translation, or by a central symmetry.

Example 1. Problem 244 in [19] states the following. Let \( H \) be the orthocenter of \( \triangle ABC \), and \( O_a, O_b, O_c \) the circumcenters of triangles \( BHC, CHA, AHB \). Then \( \triangle ABC \equiv \triangle O_aO_bO_c \) have the same nine-point circle and the same Euler’s line. This provides us an example of two triangles that have the same Euler’s line (see Figure 4).

Example 2. Now we describe two triangles of interest that have the same Euler’s line. Consider \( \triangle ABC \) and its circumcircle \( C \). Consider also the incircle tangent to \( BC, AC \) and \( AB \) respectively in \( D, E, \) and \( F \). On the straight lines \( AI, BI, CI \) we consider the excenters (i.e., the centers of the excircles) \( I_a, I_b, \) and \( I_c \). Remark that

---

Figure 4.
the circumcircle of $\triangle ABC$ is the nine-point circle of $\triangle I_a I_b I_c$, because $A, B, C$ are the feet of the altitudes (e.g. $AI_a \perp I_b I_c$).

Thus, $I$ is the orthocenter in $\triangle I_a I_b I_c$, and $O$ is the center of the nine-point circle in $\triangle I_a I_b I_c$. Therefore, $OI$ is Euler’s line in $\triangle I_a I_b I_c$. Remark that $\triangle DEF$ and $\triangle I_a I_b I_c$ have parallel sides. Therefore their Euler’s lines must be parallel (we may say that this is a consequence of Application 2). But the circumcenter of $\triangle DEF$ is the point $I$. This means that the Euler’s line of $\triangle DEF$ passes through $I$ and, being parallel to $OI$, must be $OI$.

3. Gossard’s perspector

In this section we present an elementary proof of Gossard’s result cited in [15].

**Proposition 3** (Gossard, [15]). Denote by $e$ the Euler line of an arbitrary $\triangle ABC$ in the Euclidean plane. Suppose that $e$ intersects $BC$, $AB$, $AC$ in $M$, $N$, and respectively $P$. Denote by $e_1, e_2, e_3$ Euler’s lines of $\triangle ANP$, $\triangle BMN$, and $\triangle CPM$, respectively. Denote $A', B', C'$ the intersection of the following pair of lines: $e_2 \cap e_3$, $e_1 \cap e_3$, and $e_1 \cap e_2$, respectively. Then $\triangle A'B'C' \equiv \triangle ABC$, and $\triangle A'B'C'$ has the same Euler line $e$, and there exists a point $I_G$ (called Gossard’s perspector) on the line $e$ such that $\triangle A'B'C'$ is the symmetric of $\triangle ABC$ by the symmetry centered in $I_G$.

The proof presented below is based on Lemma 1. Thus, we claim that it may be more elementary than Gossard’s original proof, as it is presented by Kellogg in [15]. An important rôle in the proof is played by the conditions $e_1 \parallel BC$, $e_2 \parallel AC$, $e_3 \parallel AB$. 
Proof. We choose coordinate axis such that the vertices of $\triangle ABC$ have the coordinates $A(0, 1)$, $B(b, 0)$, $C(c, 0)$ (see Figure 4). Let $G$ be the gravity center of $\triangle ABC$; then $G\left(\frac{b+c}{3}, \frac{1}{3}\right)$. The slope of Euler’s line in $\triangle ABC$ is given by

\[
m_e = \frac{m_2m_3 + 3}{m_2 + m_3} = \frac{-\left(-\frac{1}{c}\right)\left(-\frac{1}{b}\right) + 3}{-\frac{1}{c} - \frac{1}{b}} = \frac{3bc + 1}{b + c}.
\]

Thus, the equation of Euler’s line is $y = \frac{3bc + 1}{b + c} x - bc$. The coordinates of the points $M$, $N$, and $P$ are:

- $M \left(\frac{bc(b + c)}{3bc + 1}, 0\right)$,
- $N \left(\frac{b(b + c)(bc + 1)}{3b^2c + 2b + c}, \frac{2b^2c - bc^2 + b}{3b^2c + 2b + c}\right)$,
- $P \left(\frac{c(b + c)(bc + 1)}{3bc^2 + 2c + b}, \frac{2bc^2 - b^2c + c}{3bc^2 + 2c + b}\right)$.

The line $e_1$ passes through the center of gravity of $\triangle ANP$ and is parallel to $BC$, therefore it has the equation

\[
(e_1) : y = \frac{y_N + y_P + y_A}{3}.
\]

At the intersection of lines $e$ and $e_1$ we have the point $Q$ whose coordinates are

\[
Q \left(\frac{b + c}{3bc + 1}, \frac{1}{3}(E + 3bc), \frac{1}{3}E\right),
\]
where we have denoted by
\[ E = \frac{2b^2c - bc^2 + b}{3b^2c + 2b + c} + \frac{2bc^2 - b^2c + c}{3bc^2 + 2c + b} + 1. \]

The center of gravity of \( \triangle BMN \), denoted \( R \), has the coordinates
\[
(x_R, y_R) = \left( \frac{1}{3} \left( \frac{bc(b + c)}{3bc + 1} + b + \frac{b(b + c)(bc + 1)}{3b^2c + 2b + c} \right), \frac{1}{3} \left( \frac{2b^2c - bc^2 + b}{3b^2c + 2b + c} \right) \right).
\]

Euler’s line in \( \triangle BMN \) passes through \( R \) and is parallel to \( AC \), thus it has the equation
\[
(e_2) : \quad y - y_R = -\frac{1}{c}(x - x_R).
\]

Denote by \( S \) the intersection of the lines \( e \) and \( e_2 \). We get
\[
y_S = \frac{(3bc + 1)(x_R + cy_R) - bc(b + c)}{3bc^2 + 2c + b}.
\]

To emphasize the transformation by symmetry (as described in [15]), we claim that \( y_S + y_P = y_Q + y_M \). This is equivalent to
\[
\frac{(3bc + 1)(x_R + cy_R) - bc(b + c)}{3bc^2 + 2c + b} + \frac{2bc^2 - b^2c + c}{3bc^2 + 2c + b} = \frac{1}{3} \left( \frac{2b^2c - bc^2 + b}{3b^2c + 2b + c} + \frac{2bc^2 - b^2c + c}{3bc^2 + 2c + b} + 1 \right).
\]

By replacing \( x_R \) and \( y_R \) and simplifying the relation, we obtain the desired equality. Therefore, the segments \([PS]\) and \([QM]\) have the same midpoint. (It is not necessary to check also that \( x_P + x_S = x_Q + x_M \), since \( P, S, Q \) and \( M \) are collinear.)

Denote by \( I_G \) the common midpoint of those two segments. As above, one can prove that \( I_G \) is the midpoint of the segment \([NT]\), where \( \{T\} = e_3 \cap e \). The analogy of the computation can be further seen since the coordinates of \( I_G \) are symmetric in \( b \) and \( c \). Thus, with the above notation for \( E \), \( I_G \) has the coordinates
\[
(x_{ig}, y_{ig}) = \left( \frac{1}{2} \left( \frac{bc(b + c)}{3bc + 1} + \frac{b + c}{3bc + 1} \cdot \frac{1}{3}(E + 3bc) \right), \frac{1}{6}E \right).
\]

We can write the coordinates in the form
\[
I_G \left( \frac{1}{6} \cdot \frac{b + c}{3bc + 1}(E + 6bc), \frac{1}{6}E \right).
\]

This is the point called the Gossard perspector. Denote \( S_{ig} \) the symmetry of center \( I_G \) in the Euclidean plane. Since \( e_1 \parallel BC \), \( Q \in e_1 M \in BC \), and \( I_G \) is the midpoint of \([QM]\), we have \( e_1 = S_{ig}(BC) \). Similarly \( e_2 = S_{ig}(AC) \), \( e_3 = S_{ig}(AB) \).

Then, we have obtained the following:
\[
\{A'\} = e_2 \cap e_3 = S_{ig}(AC) \cap S_{ig}(AB) = S_{ig}(AC \cap AB) = S_{ig}(\{A\}).
\]

Similarly, \( \{B'\} = S_{ig}(\{B\}) \), and \( \{C'\} = S_{ig}(C) \).

Consequently, \( \Delta A'B'C' \equiv \Delta ABC \), and \( \Delta A'B'C'' = S_{ig}(\Delta ABC) \).
Denoting \( G \) and \( G' \) the gravity centers of \( \triangle ABC \) and \( \triangle A'B'C' \), we have \( \{G'\} = S_{IG}(\{G\}) \). For the orthocenters we get a similar correspondence: \( \{H'\} = S_{IH}(\{H\}) \). Thus, \( e' = S_{IG}(e) \), where \( e' \) is Euler’s line of \( \triangle A'B'C' \). But \( I_G \in e \). Thus, Euler’s line \( e \) passes through the center of symmetry. We deduce that \( S_{IG}(e) = e \), or \( e' = e \). Finally, we proved that \( \triangle ABC \) and \( \triangle A'B'C' \) have the same Euler’s line. This completes the analytic proof of Gossard’s perspector theorem, as mentioned in our introduction (see [15]). □

Example 3. We have seen in Example 1 (see [19], 244) that if \( H \) is the orthocenter of \( \triangle ABC \), and \( O_a, O_b, O_c \) are the circumcenters of triangles \( BHC, CHA, AHB \), then \( \triangle ABC \) and \( \triangle O_aO_bO_c \) have the same Euler’s line (see Figure 4). In fact, \( O_a, O_b, \) and \( O_c \) are the symmetric points of \( O \) with respect to the sides \( BC, AC \) and, respectively, \( AB \). Denote by \( A_1, B_1, \) and \( C_1 \) the midpoints of the sides \( BC, AC \) and, respectively, \( AB \).

Then \( H \) is the circumcenter of \( \triangle O_aO_bO_c \). Actually, \( \triangle O_aO_bO_c \) is the homothetic of \( \triangle A_1B_1C_1 \) by homothety of center \( O \) and ratio 2. Thus, \( \triangle O_aO_bO_c \) has the sides parallel and congruent to the sides of \( \triangle ABC \), and, furthermore, \( OO_a \perp BC \), and also \( OO_a \perp O_bO_c \), (and the similar relations). This proves that \( O \) is the orthocenter of \( \triangle O_aO_bO_c \). Therefore \( \triangle ABC \) and \( \triangle O_aO_bO_c \) interchanged among them the orthocenters and the circumcenters. This is the argument to see that the Euler’s lines in the two triangles are the same and the two triangles have the same center of the nine-point circle, since \( O_9 \) is the midpoint of \( OH \). Further, \( \triangle ABC \) and \( \triangle O_aO_bO_c \) are symmetric with respect to \( O_9 \). Therefore, Gossard’s perspector in \( \triangle O_aO_bO_c \) is the symmetric of Gossard perspector in \( \triangle ABC \) with respect to \( O_9 \), the center of the nine-point circle.

4. Projective viewpoint

Consider now a projectivity \( f : d_1 \to d_2 \). (See also [7, pp.39 ff], [8, pp.9-11]) The geometric locus of the points from which the the projectivity is seen as an involution of pencils of lines is called axis of the projectivity.
intersection of the cross-joints of any pairs of corresponding points (see [8, pp.36-37]). This result is known as the axis theorem. To illustrate it, if \( M_1 \rightarrow N_1 \) and \( M_2 \rightarrow N_2 \), then the point \( \{P\} = M_1N_2 \cap M_2N_1 \) lies on the axis of the projectivity, since we have the mapping \( r_1 = PM_1 \rightarrow PN_1 = r_2 \) and \( r_2 = PM_2 \rightarrow PN_1 = r_1 \). Thus, \( r_1 \rightarrow r_2 \) and \( r_2 \rightarrow r_1 \), which means that the projectivity \( f : d_1 \rightarrow d_2 \) is seen as an involution. As a consequence, we remind here the well-known geometric structure called Pappus’ line.

![Figure 8.](https://via.placeholder.com/150)

Let \( A, B, C \in d_1 \) and \( A', B', C' \in d_2 \). Then the points \( \{M\} = AB' \cap BA' \), \( \{N\} = AC' \cap AC' \cap CA' \), \( \{P\} = BC' \cap CB' \), are collinear (see Figure 8). This result can be viewed as an immediate consequence of the axis theorem. Indeed, consider the projectivity \( f : d_1 \rightarrow d_2 \) uniquely determined by \( A \rightarrow A', B \rightarrow B', C \rightarrow C' \). By the axis theorem, we get immediately that the points \( \{M\} = AB' \cap BA' \), \( \{N\} = AC' \cap AC' \cap CA' \), \( \{P\} = BC' \cap CB' \) are collinear. With this preparation, we are able to show that the Euler’s line of a triangle \( ABC \) can be regarded as the axis of projectivity for three suitable projectivities between the sides of \( \Delta ABC \) (see Figure 9).

![Figure 9.](https://via.placeholder.com/150)
Denote by $A', B', C'$ the midpoints of the sides $BC, AC$, and respectively $AB$. Denote by $A_1, B_1, C_1$ the feet of altitudes from $A, B, C$. We use the standard notations for $O$, the circumcircle, $G$ the center of gravity, and $H$ the orthocenter of $\triangle ABC$. There are three projectivities, each one between two sides of $\triangle ABC$. One of them is $f_C : BC \to AC$, the projectivity determined by $B \to A, A_1 \to B_1, A' \to B'$. Since $H$ and $G$ appear as cross-joints points, they lie on the axis of projectivity of $f_C$. Specifically, $\{H\} = AA_1 \cap BB_1, \{G\} = BB' \cap AA'$. Since two points determine uniquely a line, and since $G$ and $H$ determine Euler’s line, this means that the Euler’s line is identified with the axis of projectivity $f_C$. Furthermore, on the Euler’s line we get a new point: $\{\Omega_{AB}\} = A_1 B' \cap A'B_1$. We can also emphasize the pair of homologous points that determine $O$, the circumcenter, in this projectivity. Extend the line determined by the vertex $A$ and by $O$ and denote $\{X\} = AO \cap BC$. Similarly, $\{Y\} = BO \cap AC$. Since in our projectivity $B \to A$, then $X \to Y$. Thus, on the axis of projectivity we obtain $\{O\} = AX \cap BY$.

Considering similar constructions for the projectivities $f_A$ and $f_B$, we obtain the following fact.

**Proposition 4** (Barbilian [4]). In $\triangle ABC$, let $A', B', C'$ be the midpoints of the sides $BC, AC$, and respectively $AB$. Denote by $A_1, B_1, C_1$ the feet of altitudes from $A, B, C$. Then the points $\{\Omega_{AB}\} = A_1 B' \cap A'B_1, \{\Omega_{CB}\} = C_1 B' \cap C'B_1, \{\Omega_{AC}\} = A_1 C' \cap A'C_1$ are collinear and they lie on Euler’s line of $\triangle ABC$.

In the first part, we have presented Applications 3 and 4, where we give synthetic and trigonometric characterizations of the fact that Euler’s line is parallel to a side of the triangle. We study here the following question: What is projective condition that the projectivity $f_C : BC \to AC$ must satisfy such that Euler’s
line is parallel to $BC$? Denote by $(e)$ Euler’s line in $\triangle ABC$. (See Figure 10.) Let $\{T\} = AC \cap (e), \{U\} = BC \cap (e)$. We need to determine the pairs of straight lines that characterize in a projectivity the points $T$ and $U$. Recall that the projectivity $f_C$ has as homologous points $B \rightarrow A$. To get $T$, consider the pair $C \rightarrow (e) \cap AC$. Similarly, we get $U$ by the pair $(e) \cap BC \rightarrow C$. Therefore we have obtained the projective characterization of the fact that the Euler line is parallel to a side of the triangle. Thus, we are able to state the projective counterpart of Application 3, which is the trigonometric characterization of this parallelism.

**Proposition 5.** In $\triangle ABC$, let $(e)$ be the Euler’s line. The sufficient condition that $(e) \parallel BC$, is that the projectivity $f_C$ has $\infty \rightarrow C$ as pair of homologous points. Similarly, to have $(e) \parallel AC$, it is sufficient that $f_C$ has $C \rightarrow \infty$ as pairs of homologous points.

Four our next step, we need to recall here Pappus’ Theorem on the circle. Let $A, B, C$ and $A', B', C'$ six points on the circle $C$. Then the intersection points $AB' \cap A'B, AC' \cap A'C$ and $BC' \cap B'C$ are collinear. To recall the idea of the most direct proof, consider the projectivity $f : C \rightarrow C$ uniquely determined by $A \rightarrow A', B \rightarrow B', C \rightarrow C'$. Then, the intersection points mentioned in the statement lie precisely on the axis of the projectivity. With this observation, we obtain that Euler’s line is the axis of projectivity of a certain projectivity within the nine-point circle. The result is the following.

**Proposition 6.** Consider $A', B', C'$ the midpoints of the sides $BC, AC$ and respectively $AB$. Let $A_1, B_1$ and $C_1$ the feet of the altitudes. Consider the projectivity $\phi$ uniquely determined by $A_1 \rightarrow B_1, A' \rightarrow B', B_2 \rightarrow A_2$. Then the points $A_1A_2 \cap B_1B_2 = \{H\}$ (the orthocenter of $\triangle ABC$), $A_1B' \cap A'B_1 = \{\Omega_{AB}\}$ and $A_1A_2' \cap B'B_2 = \{O_9\}$ (the center of the nine-point circle) are collinear on the axis of projectivity of $\phi$. 

![Figure 11.](image-url)
The proof is just a direct application of Pappus’ Theorem on the circle, for the geometric structure described in the statement. Since $H$ and $\Omega_{AB}$ are on Euler’s line, the axis of projectivity and Euler’s line must be the same straight line. As a consequence, the third point, $O_9$, the center of the nine-point circle, must be on the axis of projectivity, thus on Euler’s line.

Proposition 4 appears in [4, pp. 40]. Actually, Dan Barbilian collected in an undated note, published in the cited collection of posthumous works, several projective properties of the nine-point circle and its connection with Euler’s line. He focused mainly on the projective properties, which represent, as we can see, an important part of the more complex phenomenon whose overall picture we tried to present here.

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A Sangaku-Type Problem with Regular Polygons, Triangles, and Congruent Incircles

Naoharu Ito and Harald K. Wimmer

Abstract. We consider a dissection problem of a regular \( n \)-sided polygon that generalizes Suzuki’s problem of four congruent incircles in an equilateral triangle.

1. Introduction

The geometrical problem that is the starting point of our note is due to Denzaburo Suzuki. It was engraved on a wooden tablet and dedicated 1886 to the Miwatari Shrine in Fukushima prefecture [4, p. 6], [1, p. 24]. Referring to Figure 1 below we state the problem in the following equivalent form.

![Figure 1. Four incircles in an equilateral triangle](image)

Suzuki’s problem of four congruent incircles in an equilateral triangle. Let \( ABC \) be an equilateral triangle. The side \( AC \) is extended to the point \( B' \), the side \( BA \) is extended to \( C' \), and \( CB \) to \( A' \), such that the triangles \( AB'C' \), \( BC'A' \), \( CA'B' \) and \( ABC' \) have congruent incircles. Find the length of \( A'B' \) of the exterior equilateral triangle in terms of the length of \( AB \).

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Suzuki’s problem is an example of a unique brand of mathematics that flourished in Japan in the 18th and 19th century. Amateur mathematicians crafted geometric theorems on wooden tablets (called sangaku), which were displayed in the precincts of a shrine or temple. Remarkably, some of those theorems predate work of Western mathematicians (see [5]). In addition to [1] we also mention the monographs [2] and [3] as sources of sangaku problems. An excellent survey of Japanese temple geometry is Rothman’s article [6] in the Scientific American.

In this note we generalize Suzuki’s four-congruent-incircles problem. Instead of an equilateral triangle we now consider a regular $n$-sided polygon. To illustrate the general case we choose $n = 5$. Figure 2 shows the configuration of six congruent incircles in a regular pentagon.

![Figure 2. Six congruent incircles in a regular pentagon](image)

Let $P_o$ be the regular $n$-sided polygon that forms the exterior boundary of Figure 2 and let $P_i$ be the regular $n$-sided polygon in the interior. Our main result is the following.

**Theorem 1.** If $a_o$ and $a_i$ are the lengths of the sides of $P_o$ and $P_i$ respectively, then

$$a_o = a_i \left(1 + \sqrt{1 + \left(\sin \frac{\pi}{n}\right)^{-2}}\right).$$

The proof of Theorem 1 will be given in Section 2. In Section 4 we assume that the exterior polygon $P_o$ is given. We derive a result that leads to a simple construction of the interior polygon $P_i$. The special cases with $3 \leq n \leq 6$ will be considered in Section 5.
2. Incircles

The proof of Theorem 1 is based on a relation between inradius and area of a triangle.

Consider a triangle $ABC$ in Figure 3 with sides $a = BC$, $b = CA$, $c = AB$, and $\angle BAC = \alpha$. We denote the area, the incenter, and the inradius of the triangle by $\Delta$, $I$, and $r$, respectively. Then

$$\Delta = r^2 \cot \frac{\alpha}{2} + ra. \quad (2)$$

This can be seen as follows.

Suppose the incircle touches $CA$, $AB$ and $BC$ at $P$, $Q$ and $R$, respectively. Since $AP = r \cot \frac{\alpha}{2}$, the area of the kite $AQIP$ is given by $r \cdot AP = r^2 \cot \frac{\alpha}{2}$. The area of the triangle $BIC$ is equal to $\frac{1}{2}ra$. Thus the area of the polygon $QBCPIQ$ is $ra$. Combining the areas of the kite and the polygon, we obtain (2).

The following notation refers to Figure 2. Let $T$ be one of the $n$ triangles that bound the interior polygon $P_i$. Then $T$ has sides $a_o$, $v$, $v + a_i$. Let $\psi$ be the angle between $v$ and $a_o$. We set $\theta = \frac{\pi}{n}$. Then the angle opposite to $a_o$ is equal to $2\theta = \frac{2\pi}{n}$.

3. Proof of Theorem 1

If $r_i$ is the radius of the incircle of $P_i$, then $r_i = \frac{1}{2}a_i \cot \theta$. Let $F_o$, $F_i$ and $F$ be the areas of $P_o$, $P_i$ and $T$ respectively. Then

$$F_o = \frac{n a_o^2}{4} \cot \theta \quad \text{and} \quad F_i = \frac{n a_i^2}{4} \cot \theta,$$

and $nF + F_i = F_o$. Therefore,

$$F = \frac{1}{n}(F_o - F_i) = \frac{1}{4} (a_o^2 - a_i^2) \cot \theta.$$
Let $\rho$ be the inradius of $T$. Then (2) implies $F = \rho^2 \cot \theta + a_o \rho$. Hence,

$$\frac{a_o^2 - a_i^2}{4} \cot \theta = \rho^2 \cot \theta + a_o \rho.$$ 

Now assume that the incircles of $P_i$ and $T$ are equal, i.e.,

$$r_i = \rho = \frac{1}{2} a_i \cot \theta.$$ 

We obtain $a_o^2 - a_i^2 = a_i^2 \cot^2 \theta + 2 a_o a_i$. This yields the quadratic equation

$$a_o^2 - 2 a_i a_o = a_i^2 (1 + \cot^2 \theta) = a_i^2 (\sin \theta)^{-2}.$$ 

Then $a_o > 0$ implies (1).

4. The triangle $T$

Suppose the exterior polygon $P_o$ is given. How can one construct the triangle $T$ and subsequently the interior polygon $P_i$? The angle opposite to $a_o$ is equal to $\frac{2 \pi}{n}$. Hence it suffices to know how to obtain the angle $\psi$ between $a_o$ and $v$. For that purpose we derive the following result due to P. Yiu.

**Theorem 2.** The angle $\psi$ between $a_o$ and $v$ is given by

$$\cos \psi = \sin^2 \frac{\pi}{n}. \quad (3)$$

**Proof.** It follows from (2) that the area of the triangle $T$ is given by

$$F = \rho^2 \cot \frac{\psi}{2} + \rho(v + a_i) \quad (4)$$

and

$$F = \rho^2 \cot \frac{\pi - (2\theta + \psi)}{2} + \rho v.$$ 

Substituting $a_i = 2 \rho \tan \theta$ in (4) leads to

$$\cot \frac{\psi}{2} + 2 \tan \theta = \tan(\theta + \frac{\psi}{2}) = \frac{\tan \theta + \tan \frac{\psi}{2}}{1 - \tan \theta \tan \frac{\psi}{2}},$$

which is equivalent to

$$\cot \frac{\psi}{2} - 2 \tan^2 \theta \tan \frac{\psi}{2} = \tan \frac{\psi}{2},$$

and

$$\tan^2 \frac{\psi}{2} = \frac{1}{1 + 2 \tan^2 \theta}. \quad (5)$$

From this,

$$\cos \psi = \frac{1 - \tan^2 \frac{\psi}{2}}{1 + \tan^2 \frac{\psi}{2}} = \frac{1 - \frac{1}{1 + 2 \tan^2 \theta}}{1 + \frac{1}{1 + 2 \tan^2 \theta}} = \frac{\tan^2 \theta}{1 + \tan^2 \theta} = \sin^2 \theta.$$

□
5. Special cases

We apply Theorem 1 and Theorem 2 to numbers \( n \leq 6 \). Note that the righthand side of (1) becomes infinitely large as \( n \) goes to infinity. In the case \( n = 6 \) (and also in a less conspicuous form with \( n = 5 \)) we encounter the golden ratio \( \phi = \frac{1}{2}(1 + \sqrt{5}) \). To check the case \( n = 5 \) we recall

\[
\cos \frac{2\pi}{5} = \frac{\sqrt{5} - 1}{4} \quad \text{and} \quad \sin \frac{2\pi}{5} = \frac{\sqrt{10 + 2\sqrt{5}}}{4}.
\]

In the case \( n = 3 \) we obtain the known solution of Suzuki’s 4-congruent-incircles problem of Section 1. The following table summarizes the data for \( n = 3, 4, 5, 6 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \cos \psi )</th>
<th>( \frac{a_3}{a_4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 ( \frac{3}{7} )</td>
<td>( \frac{3+\sqrt{21}}{4} )</td>
<td></td>
</tr>
<tr>
<td>4 ( \frac{1}{2} )</td>
<td>( 1 + \sqrt{3} )</td>
<td></td>
</tr>
<tr>
<td>5 ( \frac{5-\sqrt{5}}{8} = \frac{\sqrt{5}}{4\phi} )</td>
<td>( 1 + \sqrt{3 + \frac{2}{\sqrt{5}}} )</td>
<td></td>
</tr>
<tr>
<td>6 ( \frac{1}{4} )</td>
<td>( 1 + \sqrt{5} = 2\phi )</td>
<td></td>
</tr>
</tbody>
</table>

Figure 4. Five congruent incircles in a square

Figure 5. Seven congruent incircles in a regular hexagon

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A Generalization of the Conway Circle

Francisco Javier García Capitán

Abstract. For any point in the plane of the triangle we show a conic that becomes the Conway circle in the case of the incenter. We give some properties of the conic and of the configuration of the six points that define it.

Let \( ABC \) be a triangle and \( I \) its incenter. Call \( B_\alpha \) the point on line \( CA \) in the opposite direction to \( AC \) such that \( AB_\alpha = BC = a \) and \( C_\alpha \) the point on line \( BA \) in the opposite direction to \( AB \) such that \( AC_\alpha = a \). Define \( C_\beta, A_\beta \) and \( B_\alpha, A_\alpha, C_\alpha \) cyclically. The six points \( A_\beta, A_\alpha, B_\beta, B_\alpha, C_\beta, C_\alpha \) lie in a circle called the Conway circle with \( I \) as center and squared radius \( r^2 + s^2 \) as indicated in Figure 1. This configuration also appeared in Problem 6 in the 1992 Iberoamerican Mathematical Olympiad. The problem asks to establish that the area of the hexagon \( C_\alpha B_\alpha A_\beta B_\beta C_\beta A_\alpha \) is at least \( 13\Delta(ABC) \) (see [4]).

Figure 1. The Conway circle

Figure 2 shows a construction of these points which can be readily generalized. The lines \( BI \) and \( CI \) intersect the parallel of \( BC \) through \( A \) at \( B_0 \) and \( C_0 \). The points \( A_\beta \) and \( A_\alpha \) are obtained by completing the parallelograms \( BAC_0A_\beta \) and \( CAB_0A_\alpha \). If we take an arbitrary point \( P = (u : v : w) \) instead of \( I \), then \( B_0 = (u : -w : w) \) and \( C_0 = (u : v : -v) \). From these, we determine the points

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The author is grateful to José Hernández Santiago for telling about the Conway circle.
$A_b$, $A_c$, and analogously the other four points (see Figure 3). In homogeneous barycentric coordinates, these are

$$A_b = (0 : u + v : -v), \quad A_c = (0 : -w : w + u);$$

$$B_c = (-w : 0 : v + w), \quad B_a = (u + v : 0 : -u);$$

$$C_a = (w + u : -u : 0), \quad C_b = (-v : v + w : 0).$$

**Proposition 1.** The area of $ABC$ is the geometric mean of the areas of the triangles $AB_aC_a$, $BC_bA_b$, $CA_cB_c$.

**Proof.** From the coordinates of these points, we have

$$\frac{\Delta(AB_aC_a)}{\Delta(ABC)} = \frac{u^2}{vw}, \quad \frac{\Delta(BC_bA_b)}{\Delta(ABC)} = \frac{u^2}{vw}, \quad \frac{\Delta(CA_cB_c)}{\Delta(ABC)} = \frac{u^2}{vw}.$$ 

Therefore, $\Delta(ABC)^3 = \Delta(AB_aC_a) \cdot \Delta(BC_bA_b) \cdot \Delta(CA_cB_c).$ \qed

**Theorem 2.** For any point $P$, the six points $A_b, A_c, B_c, B_a, C_a, C_b$ always lie on a conic.

![Figure 3](image_url)

**Proof.** They all lie on the conic $\Gamma(P)$ with equation

$$u(v + w)x^2 + v(w + u)y^2 + w(u + v)z^2 + (vw + (u + v)(u + w))yz$$

$$+ (wu + (v + w)(v + u))zx + (uv + (w + u)(w + v))xy = 0.$$ 

\qed
From the matrix form $X M X^t = 0$ for $\Gamma(P)$ where
\[
M = \begin{pmatrix}
2u(v+w) & (u+w)(v+w) + uv & (u+v)(v+w) + uw \\
(u+w)(v+w) + uw & 2v(u+w) & (u+v)(u+w) + vw \\
(u+v)(v+w) + uw & (u+v)(u+w) + vw & 2w(u+v)
\end{pmatrix}
\]
we have
\[|M| = 2(v+w-u)(u+v-w)(u-v+w)(uvw+u^2v+uv^2+v^2w+w^2+vw^2),\]
and $\Gamma(P)$ has discriminant
\[\delta(P) = -\frac{1}{4}(v+w-u)(u+v-w)(u-v+w)(u+v+w).\]

Therefore we can get the following results:

**Proposition 3.** (a) The conic $\Gamma(P)$ is degenerate when $P$ lies on the sides of the medial triangle or in the cubic $K_{327}$.

(b) If the conic $\Gamma(P)$ is nondegenerate, it is homothetic to the circumconic with perspector $(u^2 : v^2 : w^2)$ and has center $P$.

*Proof.* (b) follows from rewriting the equation of the conic $\Gamma(P)$ in the form
\[u^2yz + v^2zx + w^2xy + (x+y+z)(u(v+w)x + v(w+u)y + w(u+v)z) = 0.\]

**Proposition 4.** The conic $\Gamma(P)$ and the circumconic with perspector $P^2$ have the same infinite points.

If $P = (u : v : w)$ and $u = v + w$, then $\Gamma(P)$ factors as
\[(vx + wx + vy + 2wy + wz)(vx + wx + vy + 2vz + wz) = 0,
\]
two parallel lines to the cevian $AP$. If $P$ lies on $K_{327}$, then $\Gamma(P)$ factors as two lines through $P$ in the directions of the asymptotes of the circumconic with perspector $P^2$.

If $P$ lies outside the sidelines of the medial triangle and the cubic $K_{327}$, the conic $\Gamma(P)$ is an ellipse when $P$ is interior to the medial triangle and a hyperbola otherwise.

**Corollary 5.** The conic $\Gamma(P)$ is
(a) a circle if and only if $P$ is the incenter or an excenter of the triangle,
(b) a rectangular hyperbola if and only if $P$ lies on the polar circle.

**Remarks.** (1) If $P = I_a$, the center of the excircle on the side $BC$, then $\Gamma(I_a)$ is a circle with squared radius $r_a^2 + (s-a)^2$.

(2) The polar circle $S_Ax^2 + S_By^2 + S_Cz^2 = 0$ contains real points only if triangle $ABC$ has an obtuse angle.

**Proposition 6.** (a) If $A', B', C'$ are the midpoints of $B_aC_a$, $C_bA_b$, $A_cB_c$ respectively, the lines $AA'$, $BB'$, $CC'$ concur at $P$.

(b) If $A''$, $B''$, $C''$ are the midpoints of $B_cC_a$, $C_aA_c$, $A_bB_a$ respectively, the lines $AA''$, $BB''$, $CC''$ concur at $P$. 

A generalization of the Conway circle
**Figure 4**

**Proof.** In homogeneous barycentric coordinates,
\[ A' = (2vw + wu + uv : -uw : -uv), \]
\[ A'' = -(v^2 + w^2) : v(v + w) : w(v + w). \]
These points clearly lie on the line \( AP: wy - vz = 0 \). Similarly, \( B', B'' \) lie on \( BP \) and \( C', C'' \) lie on \( CP \). □

**Proposition 7.** If \( P = (u : v : w) \), both of the triangles formed by the lines \( B_aC_a, C_bA_b, A_cB_c \) and the lines \( B_cC_b, C_aA_a, A_bB_a \) are perspective with \( ABC \) at the isotomic conjugate of the anticomplement of \( P \), i.e.,
\[ Q = \left( \frac{1}{v + w - u} : \frac{1}{w + u - v} : \frac{1}{u + v - w} \right). \]

**Proof.** (a) The lines \( B_aC_a, C_bA_b, A_cB_c \) have equations
\[ ux + (w + u)y + (u + v)z = 0, \]
\[ (v + w)x + vy + (u + v)z = 0, \]
\[ (v + w)x + (w + u)y + wz = 0. \]
They bound a triangle with vertices
\[ X = (-u(u + v + w) : (v + w)(u + v - w) : (v + w)(w + u - v)), \]
\[ Y = ((w + u)(v + w - u) : -v(u + v + w) : (w + u)(v + w - u)), \]
\[ Z = ((u + v)(w + u - v) : (u + v)(w + u - v) : -w(u + v + w)). \]
See Figure 5. This is perspective with $ABC$ at

$$Q = \left( \frac{1}{v + w - u} : \frac{1}{w + u - v} : \frac{1}{u + v - w} \right).$$

(b) The lines $B_cC_b, C_AA_c, A_bB_a$ have equations

- $(v + w)x + vy + wz = 0,$
- $ux + (w + u)y + wz = 0,$
- $ux + vy + (u + v)z = 0.$

They bound a triangle with vertices

- $X' = (-(u + v + w) : u + v - w : w + u - v),$
- $Y' = (v + w - u : -(u + v + w) : v + w - u),$
- $Z' = (w + u - v : w + u - v : -(u + v + w)).$

The triangle $X'Y'Z'$ is clearly perspective at the same point $Q.$ \hfill $\Box$

In the case of the Conway configuration, this is the Gergonne point.

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On Polygons Admitting a Simson Line
as Discrete Analogs of Parabolas

Emmanuel Tsukerman

Abstract. We call a polygon which admits a Simson line a Simson polygon. In this paper, we show that there is a strong connection between Simson polygons and the seemingly unrelated parabola. We begin by proving a few general facts about Simson polygons. We use an inductive argument to show that no convex \( n \)-gon, \( n \geq 5 \), admits a Simson line. We then determine a property which characterizes Simson \( n \)-gons and show that one can be constructed for every \( n \geq 3 \). We proceed to show that a parabola can be viewed as a limit of special Simson polygons, which we call equidistant Simson polygons, and that these polygons provide the best piecewise linear continuous approximation to the parabola. Finally, we show that equidistant Simson polygons can be viewed as discrete analogs of parabolas and that they satisfy a number of results analogous to the pedal property, optical property, properties of Archimedes triangles and Lambert’s Theorem of parabolas. The corresponding results for parabolas are easily obtained by applying a limit process to the equidistant Simson polygons.

1. Introduction

The Simson-Wallace Theorem (see, e.g., [5]) is a classical result in plane geometry. It states that

Theorem 1. (Simson-Wallace Theorem\(^1\)). Given a triangle \( ABC \) and a point \( P \) in the plane, the pedal points of \( P \) (That is, the feet of the perpendiculars dropped from \( P \) to the sides of the triangle) are collinear if and only if \( P \) is on the circum-circle of triangle \( ABC \).

Such a line is called a Simson line of \( P \) with respect to triangle \( ABC \).

A natural question is whether an \( n \)-gon with \( n \geq 4 \) can admit a Simson line. In [3, pp.137–144] and [4], it is shown that every quadrilateral possesses a unique

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\(^{1}\)One remark concerning the theorem is that the Simson-Wallace Theorem is most commonly known as “Simson’s Theorem”, even though “Wallace is known to have published the theorem in 1799 while no evidence exists to support Simson’s having studied or discovered the lines that now bear his name” [5]. This is perhaps one of the many examples of Stigler’s law of eponymy.
Simson Line, called “the Simson Line of a complete\(^2\) quadrilateral”. We call a polygon which admits a Simson line a Simson polygon. In this paper, we show that there is a strong connection between Simson polygons and the seemingly unrelated parabola.

We begin by proving a few general facts about Simson polygons. We use an inductive argument to show that no convex \(n\)-gon, \(n \geq 5\), admits a Simson Line. We then determine a property which characterizes Simson \(n\)-gons and show that one can be constructed for every \(n \geq 3\). We proceed to show that a parabola can be viewed as a limit of special Simson polygons, called equidistant Simson polygons, and that these polygons provide the best piecewise linear continuous approximation to the parabola. Finally, we show that equidistant Simson polygons can be viewed as discrete analogs of parabolas and that they satisfy a number of results analogous to the pedal property, optical property, properties of Archimedes triangles and Lambert’s Theorem of parabolas. The corresponding results for parabolas are easily obtained by applying a limit process to the equidistant Simson polygons.

2. General Properties of Simson Polygons

We begin with an easy Lemma. Throughout, we will use the notation that \((XYZ)\) is the circle through points \(X, Y, Z\).

**Lemma 2.** Let \(S\) be a point in the interior of two rays \(AB\) and \(AC\). Suppose that \(ABSC\) is cyclic, and let \(X\) be a point on ray \(AB\) such that \(|AX| < |AB|\). Let \(Y = (AXS) \cap AC\). Then \(|AY| > |AC|\).

![Figure 1. Lemma 2 and Lemma 3](image)

**Proof.** Since \(|AX| < |AB|\), \(\angle AXS > \angle ABS\). Since \(ABSC\) and \(AXSY\) are cyclic, \(\angle ACS = \pi - \angle ABS\) and \(\angle AYS = \pi - \angle AXS\). Therefore \(\angle AYS < \angle ABS\) so that \(|AY| > |AC|\). \(\Box\)

\(^2\)A complete quadrilateral is the configuration formed by 4 lines in general position and their 6 intersections. When it comes to pedals, we are only concerned with the sides making up the polygon. Since we extend these, the pedal of a quadrilateral is equivalent to that of its complete counterpart. For this reason, we will refer to a polygon simply by the number of sides it has.
As mentioned in the introduction, in the case of a quadrilateral there is always a unique Simson point defined as a point from which the projections into the sides are collinear. Let \( A, B, C, D, E, F \) denote the vertices of the complete quadrilateral, as in fig. 1. It is shown in [3] that the Simson point is the unique intersection of \((AFC) \cap (ABE) \cap (BCD) \cap (DEF)\), also known as the Miquel point of a complete quadrilateral. Using Lemma 2, we can conclude the following:

**Lemma 3.** Let \( ABCDEF \) be a complete quadrilateral where points in each of the triples \( A, B, C; B, D, E, \) etc. as in fig. 1 are collinear and angle \( \angle CDE \) is obtuse. Denote the Miquel point of \( ABCDEF \) by \( S \). There exist no two points \( X \) and \( Y \) on rays \( AF, AB \) respectively with \( |AX| < |AF|, |AY| < |AB| \) such that \((AXY)\) passes through \( S \).

**Proof.** The Miquel point \( S \) lies on \((AFC)\) and \((ABE)\). By Lemma 2, no such \( X \) and \( Y \) exist. \( \square \)

We call a polygon for which no three vertices lie on a line nondegenerate. In Lemma 4 and Theorems 5 and 6 we will assume that the polygon is nondegenerate.

**Lemma 4.** If \( \Pi = V_1 \cdots V_n, n \geq 5 \) is a convex Simson polygon, then \( \Pi \) has no pair of parallel sides.

**Proof.** By the nondegeneracy assumption, it is clear that no two consecutive sides can be parallel. So suppose that \( V_iV_{i+1} \parallel V_jV_{j+1}, i \notin \{1, 2, n\} \). Then \( S \) lies on the Simson line \( L \) orthogonal to \( V_iV_2 \) and \( V_iV_{i+1} \). The projection of \( S \) into each other side \( V_jV_{j+1} \) must also lie on \( L \), so that either \( V_jV_{j+1} \) is parallel to \( V_iV_2 \) or it passes through \( S \). By the nondegeneracy assumption, no two consecutive sides can pass through \( S \). Therefore the sides of \( \Pi \) must alternate between being parallel to \( V_iV_2 \) and passing through \( S \). It is easy to see that no such polygon can be convex. \( \square \)

It is worth noting that both the convexity hypothesis and the restriction to \( n \geq 5 \) in the last result are necessary, for one can construct a non-convex \( n \)-gon, \( n \geq 5 \) having pairs of parallel sides and the trapezoid (if not a parallelogram) is a convex Simson polygon with \( n = 4 \) having a pair of parallel sides. Using the above result, we can prove:

**Theorem 5.** A convex pentagon does not admit a Simson point.

**Proof.** Let \( \Pi = ABCDE \) be a nondegenerate convex pentagon. Suppose that \( S \) is a point for which the pedal in \( \Pi \) is a line. Then in particular the pedal is a line for every 4 sides of the pentagon. Therefore if \( BC \cap DE = F \), then \( S \) must be a Simson point for \( ABFE \), so that \( S \) is the Miquel point of \( ABFE \). This implies that

\[ S = (GAB) \cap (GFE) \cap (HAE) \cap (HBF), \]

where \( BC \cap AE = G \) and \( AB \cap DE = H \). By the same reasoning applied to quadrilateral \( CGED \), \( S \) must be the Miquel point of \( CGED \). Therefore \( S \) lies on \((FCD)\). Because \( \Pi \) is convex, \(|FC| < |FB|\) and \(|FD| < |FE|\). We can now apply corollary 3 with \( C \) and \( D \) playing the role of points \( X \) and \( Y \) to conclude that \( S \) cannot lie on \((FCD)\) - a contradiction. \( \square \)
Consider a convex polygon $\Pi$ as the boundary of the intersection of half planes $H_1, H_2, \ldots, H_n$. Then the polygon formed from the boundary of $\bigcap_{i \neq k}^n H_i$ for $k \in \{1, 2, \ldots, n\}$ is also convex.

We are now ready to prove the following result by induction:

**Theorem 6.** A convex $n$-gon with $n \geq 5$ does not admit a Simson point.

**Proof.** The base case has been established. Assume the hypothesis for $n \geq 5$, and consider the case for an $(n + 1)$-gon $\Pi$ with vertices $V_1, \ldots, V_{n+1}$. Suppose that $\Pi$ admits a Simson point. Let $V_{n-1}V_n \cap V_{n+1}V_1 = V'$. This intersection exists by Lemma 4. Since $\Pi$ admits a Simson point, $\Pi' = V_1 \ldots V_{n-1}V'$ must also admit one. By the preceding remark, $\Pi'$ is convex, and since it has $n$ sides, the hypothesis is contradicted. Therefore $\Pi$ cannot admit a Simson line, completing the induction. \qed

Now that we have established that no convex $n$-gon (with $n \geq 5$) admits a Simson line, we will proceed to find a necessary and sufficient condition for an $n$-gon $\Pi = V_1V_2 \ldots V_n$ to have a Simson point. Let $W_i = V_{i-1}V_i \cap V_{i+1}V_{i+2}$ for each $i$, with $V_{n+k} = V_k$. In case that $V_{i-1}V_i$ and $V_{i+1}V_{i+2}$ are parallel, view $W_i$ as a point at infinity and $(V_iW_iV_{i+1})$ as the line $V_iV_{i+1}$. For example, in a right-angled trapezoid with $AB \perp BC$ and $AB \perp AD$, $S$ will necessarily lie on the line $AB$ (in fact $S = AB \cap CD$).

**Theorem 7.** An $n$-gon $\Pi = V_1 \cdots V_n$ admits a Simson point $S$ if and only if all circles $(V_iW_iV_{i+1})$ have a common intersection.

**Proof.** Assume first that $S$ is a Simson point for $\Pi$. The projections of $S$ into $V_{i-1}V_i$, $V_iV_{i+1}$ and $V_{i+1}V_{i+2}$ are collinear. By the Simson-Wallace Theorem (Theorem 1), $S$ is on the circumcircle of $V_iW_iV_{i+1}$.

Conversely, let $S = \bigcap_i (V_iW_iV_{i+1})$. For each $i$, this implies that the projections of $S$ into $V_{i-1}V_i$ and $V_{i+1}V_{i+2}$ are collinear. As $i$ ranges from 1 to $n$ we see that all projections of $S$ into the sides are collinear. \qed

To construct an $n$-gon with a given Simson point $S$ and Simson line $L$, let $X_1, X_2, \ldots, X_n$ be $n$ points on $L$. The $n$ lines the $i$th of which is perpendicular to $SX_i$ and passing through $X_i$, $i = 1, \ldots, n$ are the sides of an $n$-gon with Simson point $S$ and Simson line $L$. The $X_i$ are the projections of $S$ into the sides of the $n$-gon and the vertices $V_i$ are the intersections of consecutive pairs of sides.

### 3. Simson Polygons and Parabolas

In this section we will show that there is a strong connection between Simson polygons and parabolas. In particular, we may view a special type of Simson polygons, which we call equidistant Simson polygons, as discrete analogs of the parabola.
Definition. Let $\Pi = V_1 \cdots V_n$ be a Simson polygon with Simson point $S$ and projections $X_1, \ldots, X_n$ of $S$ into its sides. In the special case that $|X_iX_{i+1}| = \Delta$ for each $i = 1, \ldots, n - 1$, we call such a polygon $\Pi$ an equidistant Simson polygon.

The following result shows that all but one of the vertices of an equidistant Simson polygon lie on a parabola. Moreover, the parabola is independent of the position of $X_1$ (but depends on $\Delta$).

Theorem 8. Let $S$ be a point and $L$ a line not passing through $S$. Suppose that $X_1, \ldots, X_n$ are points on $L$ such that $|X_iX_{i+1}| = \Delta$ for all $i = 1, \ldots, n - 1$ and let $\Pi = V_1 \cdots V_n$ be the equidistant Simson polygon with Simson point $S$ and projections $X_1, \ldots, X_n$ of $S$ into its sides. Then $V_1, \ldots, V_n$ lie on a parabola $C$. Moreover, $C$ is independent of the position of $X_1$ on $L$.

Proof. Without loss of generality, let $S = (0, s)$, $L$ be the $x$-axis, $X_i = (X + (i - 1)\Delta, 0)$ and $X_{i+1} = (X + i\Delta, 0)$. A calculation shows that the perpendiculars at $X_i$ and $X_{i+1}$ to the segments $SX_i$ and $SX_{i+1}$, respectively, intersect at the point $(2X + (2i - 1)\Delta, \frac{(X + (i - 1)\Delta)(X + i\Delta)}{s})$. Therefore the coordinates of the intersection satisfy $y = \frac{x^2 - \Delta^2}{2s}$ independently of $X$. It follows that $V_1, \ldots, V_{n-1}$ lie on the parabola $y = \frac{x^2 - \Delta^2}{2s}$. \hfill $\Box$

The fact that $C$ is independent of the position of $X_1$ on $L$ can be illustrated on figure 3 by supposing that $X_1, X_2, \ldots, X_8$ are being translated on $L$ as a rigid body. Then the independence of $C$ from $X_1$ implies that $C$ remains fixed and $V_1, \ldots, V_7$ slide together about $C$.

Corollary 9. Let $C$ be a parabola with focus $F$. The locus of projections of $F$ into the lines tangent to $C$ is the tangent to $C$ at its vertex.
Points $V_1, \ldots, V_8$ are the vertices of an equidistant Simson octagon with a Simson point $S$, Simson line $L$ and projections $X_1, \ldots, X_8$. By Theorem 8, $V_1, \ldots, V_7$ lie on a parabola.

Figure 4. Corollary 9: $X$ is a variable point of $C$, $F$ is the focus, $P$ is the projection of $F$ into the tangent at $X$ and $L$ is the tangent to $C$ at its vertex.

Proof. As seen in the proof of Theorem 8, the coordinates of the $V_i$, $i = 1, \ldots, n$ are continuous functions of $\Delta$. Therefore as $n \to \infty$ and $\Delta \to 0$ in Theorem 8, the limit of the polygon is a parabola with focus $S$ and tangent line at the vertex equal to $L$.

This property can be equivalently stated as: “the pedal curve of the focus of a parabola with respect to the parabola is the line tangent to it at its vertex”. This property is by no means new, but its derivation does give a nice connection between the pedal of a polygon and the pedal of the parabola. Specifically, we can view the focus $F$ as the Simson point of a parabola (considered as a polygon with infinitely many points) and the tangent at the vertex as the Simson line of the parabola.
Let \( V_1 \cdots V_{n+2} \) be an equidistant Simson polygon. We will now prove that the sides connecting the vertices \( V_1, V_2, \ldots, V_{n+1} \) form an optimal piecewise linear continuous approximation of the parabola. To be precise, we show that it is a solution to the following problem:

**Problem.** Consider a continuous piecewise linear approximation \( l(x) \) of a parabola \( f(x), x \in [a, b] \) obtained by connecting several points on the parabola. That is, let

\[
l(x) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} (x - x_{i+1}) + f(x_{i+1}) \text{ for } x \in [x_i, x_{i+1}]
\]

where \( a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b \). Find \( x_1, x_2, \ldots, x_{n-1} \in (a, b) \) such that the error

\[
\int_a^b |f(x) - l(x)|dx
\]

is minimal.

The points \( (x_i, f(x_i)), i = 0, \ldots, n \) are called knot points and a continuous piecewise linear approximation which solves the problem is called optimal. Since all parabolas are similar, it suffices to consider \( f(x) = \frac{x^2 - \Delta^2}{4s} \).

**Theorem 10.** The optimal piecewise-continuous linear approximation to \( f(x) \) with the setup above is given by the sides \( V_1 V_2, V_2 V_3, \ldots, V_n V_{n+1} \) of an equidistant Simson \( (n+2) \)-gon with \( X_1 = \frac{b}{2}, \Delta = \frac{b-a}{n} \) and \( V_i = (a + (i-1)\Delta, f(a + (i-1)\Delta)) \). The knot points \( (x_0, f(x_0)), \ldots, (x_n, f(x_n)) \) are the vertices \( V_1, V_2, \ldots, V_{n+1} \).

**Proof.** The equation of the \( i \)th line segment simplifies to

\[
l(x) = \frac{x(x_{i+1} + x_i) - x_i x_{i+1} - \Delta^2}{4s}, \text{ for } x \in [x_i, x_{i+1}],
\]

Therefore \( f(x) - l(x) = \frac{(x-x_{i+1})(x-x_i)}{4s} \) for \( x \in [x_i, x_{i+1}] \). Integrating \( |f(x) - l(x)| \) from \( x_i \) to \( x_{i+1} \) we get

\[
\int_{x_i}^{x_{i+1}} |f(x) - l_i(x)|dx = \frac{(x_{i+1} - x_i)^3}{24|s|}.
\]

It is enough to minimize

\[
S(x_1, \ldots, x_{n-1}) = 24|s| \int_a^b |f(x) - l(x)|dx = \sum_{i=0}^{n-1} (x_{i+1} - x_i)^3.
\]

Taking the partial derivative with respect to \( x_i \) for \( 1 \leq i \leq n - 1 \) and setting to zero, we get

\[
\frac{\partial}{\partial x_i} S(x_1, \ldots, x_{n-1}) = 3(x_i - x_{i-1})^2 - 3(x_{i+1} - x_i)^2 = 0
\]

\[\iff (x_i - x_{i-1})^2 = (x_{i+1} - x_i)^2.\]
Since the points are ordered and distinct, \( x_i = \frac{x_{i+1} + x_{i-1}}{2} \), so that the \( x_i \)'s form an arithmetic progression. The \( x \)-coordinates of the vertices \( V_i \) satisfy this relation, and by uniqueness, the theorem is proved.

\[ \square \]

By similar reasoning, one can see that the same sides of the \((n+2)\)-gon are also optimal if the problem is modified to solving the least-squares problem

\[ \min_{x_1, \ldots, x_{n-1}} \int_a^b (f(x) - l(x))^2 \, dx. \]

From the proof of Theorem 10, we have the following interesting result about parabolas.

**Corollary 11.** Let \( f(x) \) be the equation of parabola, \( \Delta \) be a real number and let \( l(x) \) be the line segment with end points \((y, f(y)), (y + \Delta, f(y + \Delta))\). Then the area

\[ \int_y^{y+\Delta} |f(x) - l(x)| \, dx \]

bounded by \( f(x) \) and \( l(x) \) is independent of \( y \).

This property also explains why the \( x \)-coordinates of the knot points of the optimal piecewise linear continuous approximation of the parabola are at equal intervals.

We now list some of the properties of equidistant Simson polygons:

**Theorem 12.** An equidistant Simson polygon \( V_1V_2 \ldots V_n \) with projections \( X_1, X_2, \ldots, X_n \) has the following properties:

1. If \( j - i > 0 \) is odd, the segments \( V_iV_j, V_{i+1}V_{j-1}, \ldots, V_{j+i-1}V_{j+i-1} \) are parallel for every \( i, j \in \{1, 2, \ldots, n-1\} \).
2. If \( j - i > 0 \) is even, the segments \( V_iV_j, V_{i+1}V_{j-1}, \ldots, V_{j+i-1}V_{j+i-1} \) and the tangent to the parabola at \( V_{j+i-2} \) are parallel for every \( i, j \in \{1, 2, \ldots, n-1\} \).
3. The midpoints of the parallel segments in (1) (respectively (2)) lie on a line orthogonal to the Simson line \( L \).

**Proof.** (1). The slope between \( V_i \) and \( V_j \) is easily calculated to be \( \frac{2X + (i+j-1)\Delta}{2s} \).

(2). Recall that the parabola is given by \( y = \frac{x^2 - \Delta^2}{4s} \) so that its slope at \( V_{j+i} \) is \( \frac{2X + (j+i-1)\Delta}{2} \).

(3). The \( x \)-coordinate of the midpoint of \( V_iV_j \) is \( 2X + (i+j-1)\Delta \). \[ \square \]

The following property of Simson polygons can be viewed as a discrete analog of the isogonal property of the parabola.
Property 1. Let $S$ and $L$ be the Simson point and Simon line of a Simson polygon (not necessarily equidistant) with vertices $V_1, \ldots, V_n$ and define $X_1, \ldots, X_n$ as before. Let $V'_i$ be the reflection of $V_i$ in $L$. Then the lines $V_iX_i$ and $V_iX_{i+1}$ are isogonal with respect to the lines $V_iV'_i$ and $V_iS$ (i.e. $\angle V'_iV_iX_i = \angle X_{i+1}V_iS$) for $i = 1, \ldots, n$.

Proof. The proof is by a straightforward angle count. \hfill $\square$

In the case when the Simson polygon in Proposition 1 is equidistant, we can take limits to obtain the isogonal property of the parabola:

Corollary 13. Let $C$ be a parabola with focus $F$ and tangent line $L$ at its vertex. Let $X$ be any point on $C$ and $K$ the tangent at $X$. Furthermore, let $X'$ be the reflection of $X$ in $L$. Then $K$ forms equal angles with $X'X$ and $FX$.

Using the same setup as in Theorem 8 for an equidistant Simson polygon,
**Theorem 14.** Let $M_i$ be the midpoint of $V_i V_{i+1}$, $i = 1, \ldots, n - 2$. Then the midpoints $M_i$ lie on a parabola $C'$ with focus $S$ and tangent line at its vertex $L$.

**Proof.** Since $V_i = (2X + (2i - 1)\Delta, \frac{(x+(i-1)\Delta)(x+i\Delta)}{s})$,

$$M_i = (2(X + i\Delta), \frac{(X + i\Delta)^2}{s}).$$

Therefore the $M_i$ lie on the parabola $p(x) = \frac{x^2}{4s}$ with focus $S$. The slope of $V_i V_{i+1}$ is $\frac{X + i\Delta}{s}$, which is the same as that of $p(x)$ at $M_i$.

\[ \square \]

In a coordinate system where $S$ lies above $L$, the parabolas $C$ and $C'$ form sharp upper and lower bounds to the piecewise linear curve $f(x)$ formed by the sides connecting $V_1, \ldots, V_{n-1}$ (discussed in Theorem 10). Informally, one can think of $C$ and $C'$ as “sandwiching” $f(x)$, and in the limit $n \to \infty$ and $\Delta \to 0$, the two curves coincide and equal the limit of the polygon.

The following result is a discrete analog of the famous optical reflection property of the parabola.

**Corollary 15.** Let $M_i$ be the midpoints of $V_i V_{i+1}$ as in Theorem 14 and $p_i$ be the line passing through $M_i$ orthogonal to $L$ for $i = 1, 2, \ldots, n-2$. Then the reflection $p_i'$ of $p_i$ in $V_i V_{i+1}$ passes through $S$ for each $i = 1, 2, \ldots, n-2$.

Figure 7. Corollary 15: $\Pi = V_1 \cdots V_8$ is an equidistant Simson polygon. The reflections at the midpoints of the sides of $\Pi$ of rays orthogonal to $L$ pass through $S$.

Let $X$ and $Y$ be two points on a parabola $C$. The triangle formed by the two tangents at $X$ and $Y$ and the chord connecting $X$ and $Y$ is called an *Archimedes Triangle* [2]. The chord of the parabola is called the triangle’s base. One of the
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results stated in Archimedes’ Lemma is that if \( Z \) is the vertex opposite to the base of an Archimedes triangle and \( M \) is the midpoint of the base, then the median \( MZ \) is parallel to the axis of the parabola. The following result yields a discrete analog to Archimedes’ Lemma. Let \( V_1 \cdots V_n \) be an equidistant Simson polygon.

**Theorem 16.** Let \( W_{i,j} = V_i V_{i+1} \cap V_j V_{j+1} \) for each \( i, j \in \{1, 2, \ldots, n-2\} \) and \( i \neq j \). Let \( M_{i,j+1} \) and \( M_{i+1,j} \) be the respective midpoints of chords \( V_i V_{j+1} \) and \( V_{i+1} V_j \). Then \( W_{i,j} M_{i,j+1} \) and \( W_{i,j} M_{i+1,j} \) are orthogonal to \( L \).

**Proof.** As shown in the proof of Theorem 12, the \( x \)-coordinate of \( M_{i,j+1} \) is \( 2X + (i + j)\Delta \) and that of \( M_{i+1,j} \) is the same. The point \( W_{i,j} \) is the intersection of the line \( V_i V_{i+1} \) given by \( y = \frac{X + i\Delta}{s}x - \frac{(X + i\Delta)^2}{s} \) and the line \( V_j V_{j+1} \) given by \( y = \frac{X + j\Delta}{s}x - \frac{(X + j\Delta)^2}{s} \), so that \( W_{i,j} = (2X + (i + j)\Delta, \frac{(X + i\Delta)(X + j\Delta)}{s}) \). \( \square \)

**Corollary 17.** The points \( W_{i,j+1}, W_{i+1,j}, W_{i+2,j-1} \), etc. and the points \( M_{i,j+1}, M_{i+1,j}, M_{i+2,j-1} \), are collinear. The line on which they lie is orthogonal to \( L \).

Taking limits, we get the following Corollary which includes the part of Archimedes’ Lemma stated previously:

**Corollary 18.** The vertices opposite to the bases of all Archimedes triangles with parallel bases lie on a single line parallel to the axis of the parabola and passing through the midpoints of the bases.

![Figure 8. Corollary 18: Triangles \( X_1Y_1Z_1 \) and \( X_2Y_2Z_2 \) are two Archimedes triangles with parallel bases \( X_1Y_1, X_2Y_2 \). Points \( Z_1, Z_2 \) and the midpoints of the bases \( M_1, M_2 \) all lie on a line parallel to the axis of the parabola.](image)

The final theorem to which we give generalization is Lambert’s Theorem, which states that the circumcircle of a triangle formed by three tangents to a parabola passes through the focus of the parabola [2]. We can prove it using the Simson-Wallace Theorem.

**Theorem 19.** Let \( V_1 \cdots V_n \) be a Simson polygon (not necessarily equidistant) with Simson point \( S \). Let \( i, j, k \in \{1, 2, \ldots, n\} \), be distinct. Then the circumcircle of the triangle \( T \) formed from lines \( V_i V_{i+1}, V_j V_{j+1} \) and \( V_k V_{k+1} \) passes through \( S \).
Proof. Since the projections of $S$ into $V_iV_{i+1}$, $V_jV_{j+1}$ and $V_kV_{k+1}$ are collinear, $S$ is a Simson point of the triangle $T$. Therefore by the Simson-Wallace Theorem (Theorem 1), $S$ lies on the circumcircle of $T$. \hfill \Box

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure9}
\caption{Theorem 19 and Corollary 20.}
\end{figure}

Corollary 20. (Lambert’s Theorem). The focus of a parabola lies on the circumcircle of a triangle formed by any three tangents to the parabola.

Proof. Taking the limit of a sequence of equidistant Simson polygons gives Lambert’s Theorem for a parabola, since the lines $V_iV_{i+1}$, $V_jV_{j+1}$, $V_kV_{k+1}$ become tangents in the limit. \hfill \Box

References

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Three Natural Homoteties of The Nine-Point Circle

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Abstract. Given a triangle with the reflections of its vertices in the opposite sides, we prove that the pedal circles of these reflections are the images of nine-point circle under specific homoteties, and that their centers form the anticevian triangle of the nine-point center. We also construct two concentric circles associated with the pedals of these reflections on the sidelines, and study the triangle bounded by the radical axes of these pedal circles with the nine-point circle.

1. Three pedal circles

Given a triangle $ABC$ with angles $\alpha, \beta, \gamma$, circumcenter $O$, orthocenter $H$, and nine-point center $N$, we let $M_a, M_b, M_c$ be the midpoints of the sides $BC, CA, AB$, $H_a, H_b, H_c$ the pedals of $A$ on $BC, B$ on $CA, C$ on $AB$ respectively. Consider also the reflections $A'$ of $A$ in $BC, B'$ of $B$ in $CA, C'$ of $C$ in $AB$.

Our first result (Theorem 3 below) is about the pedal circles of $A', B', C'$ with respect to triangle $ABC$.

Construct the circle through $O, B, C$, and let $O_a$ be the second intersection of this circle with the line $AO$.

**Proposition 1.** $O_a$ and $A'$ are the isogonal conjugates in triangle $ABC$.

![Figure 1](image-url)

*Proof.* Clearly the lines $AO_a$ and $AH$ are isogonal with respect to angle $A$, since $O$ and $H$ are isogonal conjugates. Also,

$$\angle A'BC_a = 2\angle A'Ca = 2\angle HAB = 2\angle OAC = \angle OaOC = \angle OaBC.$$
Therefore, the lines $A'B$ and $O^a B$ are symmetric in the external bisector of angle $B$, and so are isogonal with respect to angle $B$. Similarly, $A'C$ and $O^a B$ are isogonal with respect to angle $C$. This shows that $A'$ and $O^a$ are isogonal conjugates.

The points $A'$ and $O_a$ have a common pedal circle, with center at the midpoint $N^a$ of $A'O_a$.

**Proposition 2.** $O_a A'$ is parallel to $OH$.

**Proof.** Let $X_a, B'_a, C'_a$ be the pedals of $O_a$ on $BC, CA, AB$ respectively. From

$$\frac{AM_b}{AB'_a} = \frac{AO}{AO_a} = \frac{AM_c}{AC'_a},$$

we have $B'_a C'_a / / M_b M_c / / BC$. Therefore the cyclic quadrilateral $B'_a C'_a H_a X_a$, having a pair of parallel sides, must be a symmetric trapezoid. Now,

$$\angle C'_a X_a O_a = \angle C'_a BO_a = \angle CBA' = \beta = \angle C'_a O_a X_a.$$ 

The second equality is valid because $O_a B$ and $A'B$ are isogonal with respect to $B$, and the last one because $B, X_a, O_a, C'_a$ are concyclic. It follows that $C'_a O_a = C'_a X_a = B'_a H_a$. Similarly, $B'_a O_a = C'_a H_a$. Therefore, $C'_a O_a B'_a H_a$ is a parallelogram, and $B'_a H_a$ is parallel to $O_a C'_a$, and also to $CH$, being all perpendicular to $AB$.

Since $M_b$ and $M_c$ are the midpoints of $AC$ and $AB$, we have

$$\frac{AO}{AO_a} = \frac{AM_b}{AB'_a} = \frac{AC}{2 \cdot AB'_a} = \frac{AH}{2 \cdot AH_a} = \frac{AH}{AA'}.$$

Therefore, $O_a A'$ is parallel to $OH$. \qed
Theorem 3. The pedal circle of $A'$ (and $O_a$) is the image of the nine-point circle of $ABC$ under the homothety $h(A, t_a)$, where $t_a = \frac{2 \sin \beta \sin \gamma}{\cos \alpha}$.

Proof. The circle $B_aB'_aC'_a$ is homothetic to the nine-point circle $H\nu$ at $A$ since
\[
\frac{AB_a}{AH_b} = \frac{AA'}{AH} = \frac{AO_a}{AO} = \frac{AB'_a}{AM_b} = \frac{AC'_a}{AM_c}.
\]
The ratio of homothety is
\[
t_a = \frac{AA'}{AH} = \frac{2 \cdot AH_a}{2 \cdot OM_a} = \frac{2R \sin \beta \sin \gamma}{2R \cos \alpha} = \frac{\sin \beta \sin \gamma}{\cos \alpha}.
\]
Since the center $N^a$ of the pedal circle of $A'$ and $O_a$ is the midpoint of $O_aA'$, the line $AN^a$ intersects $OH$ at its midpoint $N$, the nine-point center of $ABC$. \hfill \square

Analogously let $O_b$, $O_c$ be the second intersections of the circles $OCA$, $OAB$ with the lines $BO$, $CO$ respectively. The common pedal circle of $B'$ and $O_b$ has center $N^b$ the midpoint of $O_bB'$ and that of $C'$ and $O_c$ has center $N^c$ the midpoint of $O_cC'$. These pedal circles are images of the nine-point circle under the homotheties $h(B, t_b)$ and $h(C, t_c)$ with $t_b = \frac{\sin \gamma \sin \alpha}{\cos \beta}$ and $t_c = \frac{\sin \alpha \sin \beta}{\cos \gamma}$ respectively.

Theorem 4. $N^aN^bN^c$ is the anticevian triangle of the nine-point $N$.

Proof. Since $N^a$ is the midpoint of $O_aA'$ and the nine-point center $N$ is the midpoint of $OH$, by Proposition 2, $A$, $N$, $N_a$ are collinear. Similarly, $N^b$ and $N^c$ are on the cevians $BN$ and $CN$ respectively. We show that the line $N^bN^c$, $N^cN^a$, $N^aN^b$.
contain $A$, $B$, $C$ respectively. From this the result follows. It is enough to show that $N^b N^c$ contains $A$. For this, note that $B'$, $A$, $O_c$ are collinear because

$$\angle B'AB + \angle BAO_c = 2\angle B'AC + \angle BOO_c$$

$$= 2\alpha + (180^\circ - \angle BOC)$$

$$= 2\alpha + (180^\circ - 2\alpha)$$

$$= 180^\circ.$$

Similarly, $O_b$, $A$, and $C'$ are collinear. Therefore, the midpoints of $O_b B'$ and $O_c C'$, namely, $N^b$ and $N^c$, are collinear with $A$. □

2. Two concentric circles associated with six pedals

Let $A''B''C''$ be the triangle bounded by the lines $B_aC_a$, $C_bA_b$, and $A_cB_c$.

**Theorem 5.** The incenter of triangle $A''B''C''$ is the orthocenter of the orthic triangle $H_aH_bH_c$, and the incircle touches the sides at the midpoints $P_a$, $P_b$, $P_c$ of the segments $B_aC_a$, $C_bA_b$, $A_cB_c$ respectively.

![Figure 4.](image-url)
We first claim that the segments \( B_a C_a, C_b A_b, A_c B_c \) have equal lengths. Note that the homothety \( h(A, t_a) \) maps \( H_b, H_c \) to \( B_a, C_a \) respectively. Hence,

\[
B_a C_a = t_a \cdot H_b H_c = \frac{\sin \beta \sin \gamma}{\cos \alpha} \cdot 2R \sin \alpha \cos \alpha = 4R \sin \alpha \sin \beta \sin \gamma.
\]

Since this expression is symmetric in \( \alpha, \beta, \gamma \), it also gives the lengths of \( C_b A_b \) and \( A_c B_c \).

Note that the corresponding sidelines of triangles \( A''B''C'' \) and the orthic triangle \( H_a H_b H_c \) are parallel. The two triangles are homothetic. By parallelism,

\[
\angle A''A_b A_c = \angle H_c H_a A = \angle C H_a H_b = \angle A B_c C''.
\]

Therefore, \( A''A_c A_b \) is an isosceles triangle with \( A''A_b = A''A_c \). Since \( A_b C_b = A_c B_c \), we deduce that \( A''P_b = A''P_c \). Similarly, \( B''P_c = B''P_a \) and \( C''P_a = C''P_b \). Hence, \( P_a, P_b, P_c \) are the points of tangency of the incircle of triangle \( A''B''C'' \) with its sides.

Next we claim that \( H_a, N_a \) and \( P_a \) all lie on a line perpendicular to \( B_b C_a \). Let \( U_a \) be the midpoint of \( AH \). Since \( NU_a \) is parallel to \( OA \), it is perpendicular to \( B_b H_c \). As \( NH_b = NH_c \), the line \( NU_a \) is the perpendicular bisector of \( H_b H_c \). The homothety \( h(A, t_a) \) maps \( U_a H_b NH_c \) into \( H_a B_a N_a A_c \), and \( H_a N_a \) is the perpendicular bisector of \( B_a C_a \). Therefore, it passes through the midpoint \( P_a \) of \( B_a C_a \). Since \( H_a N_a \) is perpendicular to \( H_b H_c \), it passes through the orthocenter of the orthic triangle \( H_a H_b H_c \). The same is true for the other two lines \( H_b N_b \) and \( H_c N_c \), which are the perpendiculars to the sides \( C''A'' \) and \( A''B'' \) at the points \( P_b \) and \( P_c \) respectively. Therefore, the incenter of \( A''B''C'' \) is the orthocenter of the orthic triangle.

**Remarks.** (1) The common length of \( B_a C_a, C_b A_b, A_c B_c \) is also the perimeter of the orthic triangle, being \( 4R \sin \alpha \sin \beta \sin \gamma = R(\sin 2\alpha + \sin 2\beta + \sin 2\gamma) \).

(2) The orthocenter of the orthic triangle is the triangle center \( X(52) \) in [3].

**Corollary 6.** The lines \( N^a H_a, N^b H_b, N^c H_c \) are concurrent at \( H_o \).

**Theorem 7.** The six pedals \( A_b, A_c, B_c, B_a, C_a, C_b \) lie on a circle with center \( H_o \).

**Proof.** From Theorem 5, we have \( H_o P_a = H_o P_b = H_o P_c \). Also recall from the proof of the same theorem, the segments \( B_a C_a, C_b A_b, A_c B_c \) have equal lengths. Therefore, \( H_o B_a C_a, H_o C_b A_b, H_o A_c B_c \) are congruent isosceles triangles, and \( H_o \) is the center of a circle containing these six pedals (see Figure 4).

**Theorem 8.** The triangles \( ABC, A''B''C'' \), and \( P_a P_b P_c \) are perspective at the symmedian point of triangle \( ABC \).

**Proof.** (1) Since \( AA_c A_b \) and \( AB_c B_b \) are isosceles triangles, \( B_c C_b \) and \( A_c A_b \) are parallel, and the triangles \( AC_b B_c \) and \( ABC \) are homothetic (see Figure 5). Now,

\[
\angle A''B_c C_b = \angle A.''A_c A_b = \angle H_b H_a C = \alpha = \angle B_c A C_b.
\]

Similarly, \( \angle A''C_b B_c = \angle B_c A C_b \). Therefore, \( A''B_c \) and \( A''C_b \) are tangents from \( A'' \) to the circumcircle of triangle \( AC_b B_c \). The line \( A''A \) is a symmedian of triangle \( AC_b B_c \). Since \( ABC \) and \( AC_b B_c \) are homothetic at \( A \), the same line \( A''A \) is a
symmedian of triangle $ABC$, and it contains the symmedian point $K$ of triangle $ABC$. The same reasoning shows that $B''B$ and $C''C$ also contain $K$. Therefore, triangles $A''B''C''$ and $ABC$ are perspective at $K$.

(2) In triangle $ABC$, $B_aC_a$ is antiparallel to $BC$ since

$$\angle C_aB_aA = \angle C_aC_bC = \angle BH_aH_b$$

The reflection of triangles $AC_aB_a$ in the bisector of angle $A$ is homothetic to $ABC$. Therefore, the median $AP_a$ of triangle $AC_aB_a$ is the same as the symmedian $AK$; similarly for $BP_b$ and $CP_c$. The three lines are concurrent at the symmedian point $K$. \hfill \Box$

3. A triangle bounded by three radical axes

Let $L_a, L_b, L_c$ be the radical axes of the nine-point circle with the pedal circles of $A', B', C'$ respectively. These lines bound a triangle $Q_aQ_bQ_c$. The vertex $Q_a$ is the radical center of the nine-point circle and the pedal circles of $B'$ and $C'$; similarly for the vertices $Q_b$ and $Q_c$. 
Lemma 9. Let $J_a$ be the midpoint of $OA$. The line $J_aM_a$ is perpendicular to $Q_bQ_c$ and contains the midpoint of $ON$.

Proof. Since $N$ is the midpoint of $OH$, the segment $J_aN$ is parallel to $AH$ and therefore to $OM_a$. Furthermore, $J_aN = \frac{1}{2}AH = OM_a$. It follows that $J_aM_a$ intersects $ON$ at its midpoint.
Proposition 10. Given triangle $ABC$ with incircle triangle $DEF$, extend $AB$ and $AC$ to $P$ and $Q$ such that $BP = BC = CQ$. Let $T$ be the midpoint $PQ$, and $M$ the midpoint of the arc $BAC$ of the circumcircle.

(a) The line $TM$ is perpendicular to $EF$.

(b) $BT$ and $CT$ are parallel to $DF$ and $DE$ respectively.

Proof. (a) By the angle bisector theorem, $AE = \frac{bc}{a+b}$, $AF = \frac{bc}{a+c}$. Therefore, $\frac{AE}{AP} = \frac{a+c}{a+b} = \frac{AQ}{AP}$, showing that $PQ$ is parallel to $EF$ (see Figure 8). On the other hand, the circumcircles of $ABC$ and $APQ$ intersect at $A$ and $M$, which is the center of the rotation taking the oriented segments $BP$ and $CQ$ into each other (see [4, p.5]). Since $MB = MC$, $M$ is the center of this rotation. Hence, $MT$ is the perpendicular bisector of $PQ$. We conclude that $MT$ and $EF$ are perpendicular to each other.

(b) We show that $BT$ is parallel to $DF$.

Let $Y$ be the intersection of the lines $BT$ and $AC$. Applying Menelaus’ theorem to triangle $APQ$ with transversal $BTY$, we have

$$\frac{AY}{YQ} \cdot \frac{QT}{TP} \cdot \frac{PB}{BA} = -1 \Rightarrow \frac{AY}{YQ} = -\frac{AB}{BP} = -\frac{c}{a} \Rightarrow \frac{AY}{AQ} = \frac{c}{c-a}.$$ 

Therefore, $AY = \frac{c(a+b)}{c-a}$. Now, $DF$ intersects $AC$ at $E'$ such that $BE'$ is the external bisector of angle $E$. $\frac{AE'}{EC} = -\frac{c}{a} \Rightarrow \frac{AE'}{AC} = \frac{c}{c-a}$. It follows that $AE' = \frac{c}{c-a} \cdot b$. From these, $\frac{AE'}{AY} = \frac{b}{a+b} = \frac{AF}{AY}$. Therefore, $BT$ is parallel to $DF$.

The same reasoning shows that $CT$ is parallel to $DE$. □

Remark. Proposition 10 remains valid if $P$ and $Q$ are chosen on the rays $BA$ and $CA$ instead, and $BE$, $BF$ are external bisectors.
Theorem 11. The orthocenter of triangle $Q_aQ_bQ_c$ is the midpoint of $ON$.

Figure 9

Proof. It is enough to prove that $Q_aM_a$ is parallel to $AN$.

Let $D_a = AH \cap H_bH_c$, $D_b = BH \cap H_cH_a$, $D_c = CH \cap H_aH_b$. We claim that $D_bD_c$ is perpendicular to $AN$. The points $D_b$ and $D_c$ have equal powers with respect to the nine-point circle of $ABC$ and the circumcircle of $HBC$. Therefore, the line $D_bD_c$ is the radical axis of the these two circles. The circumcenter of $HBC$ is the reflection of $O$ in $BC$, and form a parallelogram with $O$, $A$, $H$, with $N$ as the common midpoint of the diagonals. Therefore $AN$ is the line joining the centers of the nine-point circle and the center of the circle $HBC$, and is perpendicular to the radical axis $D_bD_c$.

The line $AN$ also contains the center $N^o$ of the circle $\Gamma_a$. Therefore the radical axes $Q_bQ_c$ and $D_bD_c$ are parallel, and $AN$ is perpendicular to $Q_bQ_c$.

Now we show that $Q_aM_a$ is parallel to $AN$. 
It is easy to see that $D_aD_bD_c$ is the incentral triangle of $H_aH_bH_c$. (If triangle $ABC$ is obtuse, then the two bisectors not corresponding to obtuse angle have to be replaced by external bisectors; see Remark following Proposition 10). Applying Proposition 10 to the orthic triangle $H_aH_bH_c$, the lines $Q_aQ_b$ and $Q_aQ_c$ are parallel to $D_aD_b$ and $D_aD_c$ respectively, and the midpoint of the arc $H_bH_aH_c$ is $M_a$, the midpoint of $BC$. Therefore, $Q_aM_a$ is perpendicular to $D_bD_c$, which is parallel to $Q_bQ_c$.

The lines $Q_aM_a$, $Q_bM_b$, $Q_cM_c$ are the altitudes of the triangle $Q_aQ_bQ_c$. But these lines are parallel to $AN$, $BN$, $CN$ respectively. They are concurrent at the midpoint of $ON$. □

**Remark.** The midpoint of $ON$ is the triangle center $X(140)$ in [3].

**References**


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Intersecting Equilateral Triangles

Colleen Nielsen and Christa Powers

Abstract. In 1980, J. Fickett proposed the following problem: Assume two congruent rectangles $R_1$ and $R_2$ intersect in at least one point. Let $a$ be the length of the part of the boundary of $R_1$ that lies inside $R_2$ and let $b$ be the length of the part of the boundary of $R_2$ that lies inside $R_1$. The conjecture was that the ratio $\frac{a}{b}$ is no smaller than $\frac{1}{3}$ and no larger than $3$. This paper presents the solution to the problem when $R_1$ and $R_2$ are replaced by equilateral triangles of the same size. We have proved that the ratio $\frac{a}{b}$ is no smaller than $\frac{1}{2}$ and no larger than $2$.

In [1], Fickett proposed a problem involving congruent rectangles $R_1$ and $R_2$ (including their interiors) in the Euclidean plane and their boundaries $\partial R_1$ and $\partial R_2$. The conjecture was that

$$\frac{1}{3} \leq \frac{\text{length } \partial R_1 \cap R_2}{\text{length } \partial R_2 \cap R_1} \leq 3.$$

We show a similar result for equilateral triangles and include a generalization for regular polygons.

**Theorem 1.** Let $P$ and $Q$ (including their interiors) be congruent regular $n$-gons, $3 \leq n$, in the Euclidean plane with respective boundaries $\partial P$ and $\partial Q$.

(a) If $P$ and $Q$ intersect in exactly $2n - 1$ or $2n$ boundary points, then $\text{length } \partial P \cap Q = \text{length } \partial Q \cap P$.

(b) If $n = 3$, then $\frac{1}{2} \leq \frac{\text{length } \partial P \cap Q}{\text{length } \partial Q \cap P} \leq 2$.

We begin with a useful Lemma.

**Lemma 2.** If $a_i$, $b_i$ are positive real numbers, $2 \leq n$ and $1 \leq i \leq n - 1$, and

$$\frac{a_i}{b_i} = \frac{a_{i+1}}{b_{i+1}}$$

then

$$\frac{a_i}{b_i} = \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n}.$$

**Proof.** $\frac{a_i}{b_i} = \frac{a_{i+1}}{b_{i+1}}$ implies that $\frac{a_i}{b_i} = \frac{a_j}{b_j}$ for $1 \leq i, j \leq n$ and $a_ib_j = b_ia_j$. Thus,
a_i b_1 + a_i b_2 + \cdots + a_i b_n = b_i a_1 + b_i a_2 + \cdots + b_i a_n, \\
a_i (b_1 + b_2 + \cdots + b_n) = b_i (a_1 + a_2 + \cdots + a_n), \\
\frac{a_i}{b_i} = \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n}.

We now prove part (a) of Theorem 1 for $2n$ points. Only an adjustment in the subscripts is needed for the $2n - 1$ case. Assume $P$ and $Q$ intersect in $2n$ boundary points with sides labeled as in Figure 1. We must show that

\[ b_0 + b_2 + \cdots + b_{2n-2} = b_1 + b_3 + \cdots + b_{2n-1}. \]

\[ a_0 + a_2 + \cdots + a_{2n-2} = b_0 + b_2 + \cdots + b_{2n-2}. \]

\[ c_0 + c_2 + \cdots + c_{2n-2} = c_1 + c_3 + \cdots + c_{2n-1}. \]

\[ \Delta T_i \text{ and } \Delta T_{i+1} \text{ both have a } \frac{180(n-2)}{n} \text{ degree angle and a pair of congruent vertical angles and so they are similar and more generally, } \Delta T_i \sim \Delta T_k. \]

This implies that

\[ \frac{a_i}{a_j} = \frac{b_i}{b_j} = \frac{c_i}{c_j} \quad \text{and therefore} \quad \frac{a_i}{b_i} = \frac{a_1}{b_1} = \frac{a_3}{b_3} = \cdots = \frac{a_{2n-1}}{b_{2n-1}} \]

and

\[ \frac{a_i}{b_i} = \frac{a_0}{b_0} = \frac{a_2}{b_2} = \cdots = \frac{a_{2n-2}}{b_{2n-2}}. \]

Using Lemma 1 yields

\[ \frac{a_1 + a_3 + \cdots + a_{2n-1}}{b_1 + b_3 + \cdots + b_{2n-1}} = \frac{a_i}{b_i} = \frac{a_0 + a_2 + \cdots + a_{2n-2}}{b_0 + b_2 + \cdots + b_{2n-2}}. \]

Similarly,

\[ \frac{a_1 + a_3 + \cdots + a_{2n-1}}{a_0 + a_2 + \cdots + a_{2n-2}} = \frac{b_1 + b_3 + \cdots + b_{2n-1}}{b_0 + b_2 + \cdots + b_{2n-2}} = \frac{c_1 + c_3 + \cdots + c_{2n-1}}{c_0 + c_2 + \cdots + c_{2n-2}}. \]

Again, applying Lemma 1,
Lemma 3. Assume the sides of the regular polygon are of length 1 and so \( a_i + b_i (\text{mod } n) + c_{i+2} (\text{mod } n) = 1 \) for \( 1 \leq i \leq n - 1 \). We find the sum of the sides of each polygon.

\[
\begin{align*}
&\quad a_0 + b_1 + c_2 + a_2 + b_3 + c_4 + \cdots + a_{2n-2} + b_{2n-1} + c_0 = n, \\
&\quad a_0 + a_2 + \cdots + a_{2n-2} + c_0 + c_2 + \cdots + c_{2n-2} = n - (b_1 + b_3 + \cdots + b_{2n-1}); \\
&\quad a_1 + b_2 + c_3 + a_3 + b_4 + c_5 + \cdots + a_{2n-1} + b_0 + c_1 = n, \\
&\quad a_0 + a_2 + \cdots + a_{2n-1} + c_1 + c_3 + \cdots + c_{2n-1} = n - (b_0 + b_2 + \cdots + b_{2n-2}).
\end{align*}
\]

These along with (1) yield,

\[
\begin{align*}
&\quad \frac{b_1 + b_2 + \cdots + b_{2n-1}}{b_0 + b_2 + \cdots + b_{2n-2}} = \frac{a_1 + a_3 + \cdots + a_{2n-1} + c_1 + c_3 + \cdots + c_{2n-1}}{a_0 + a_2 + \cdots + a_{2n-2} + c_0 + c_2 + \cdots + c_{2n-2}} \\
&\quad = \frac{n - (b_0 + b_2 + \cdots + b_{2n-2})}{n - (b_1 + b_3 + \cdots + b_{2n-1})}.
\end{align*}
\]

Let \( X = b_1 + b_3 + \cdots + b_{2n-1} \) and \( Y = b_0 + b_2 + \cdots + b_{2n-2} \). Substituting \( X = \frac{n - Y}{n - X} \), we obtain \( nX - X^2 = nY - Y^2, (X - Y)(X + Y - n) = 0 \).

Now, for each \( \Delta T_i, b_i < a_i + c_i \), and so

\[
2(X + Y) = 2 \sum_{k=0}^{2n-1} b_k + \sum_{k=0}^{2n-1} a_i + b_i + c_i = 2n,
\]

which implies that \( X + Y < n \). Thus, \( X + Y - n \neq 0 \), and \( X - Y = 0 \) or \( X = Y \).

Later, we shall need Theorem 1(a) in the special case where \( n = 3 \) and the two equilateral triangles intersect in exactly five points. We now prove part (b) of Theorem 1 and begin with two preliminary results.

**Lemma 3.** If \( 0 \leq x \leq \frac{\pi}{3} \) then \( f(x) = \sin \left( x + \frac{\pi}{3} \right) + \sin x \) has a minimum value of \( \frac{\sqrt{3}}{2} \) at \( x = 0 \) and a maximum value of \( \sqrt{3} \) at \( x = \frac{\pi}{3} \).

**Proof.** For \( 0 \leq x \leq \frac{\pi}{3} \),

\[
f'(x) = \cos \left( x + \frac{\pi}{3} \right) + \cos x = \sqrt{3} \cos \left( x + \frac{\pi}{6} \right) \geq 0,
\]

which implies that \( f \) is an increasing function on \( [0, \frac{\pi}{3}] \). Thus, \( f(0) = \frac{\sqrt{3}}{2} \) is the minimum value of \( f \), and \( f \left( \frac{\pi}{3} \right) = \sqrt{3} \) is the maximum value. \( \square \)

**Proposition 4.** Assume triangle \( ABC \) has sides \( a, b, c \) and opposite vertices \( A, B, C \) respectively. If \( \angle C = 60^\circ \), then \( \frac{1}{2} \leq \frac{c}{a+b} < 1 \).
Proof. Using the law of cosines, \( c^2 = a^2 + b^2 - 2ab \cos 60^\circ = a^2 + b^2 - ab \). Now, 
\[ (2c)^2 - (a+b)^2 = 4(a^2 - ab + b^2) - (a^2 + 2ab + b^2) = 3a^2 - 6ab + 3b^2 = 3(a-b)^2 \geq 0. \]
Therefore, \( 2c \geq a + b \). Since also \( c < a + b \), we have \( \frac{1}{2} \leq \frac{c}{a + b} < 1 \). \( \square \)

To prove part (b) of Theorem 1, we must consider the number of intersection points that are not vertices of a triangle. An intersection point that is a vertex of a triangle will be called a vertex intersection. These points will not affect the underlying geometry of what is to follow. If the triangles intersect in fewer than five points, we have four cases to consider (Figure 2).

Case 1: Two intersection points forming a triangle; possibly one vertex intersection.

Case 2: Two intersection points forming a quadrilateral; possibly one or two vertex intersections.

Case 3: Four intersection points forming a quadrilateral.

Case 4: Four intersection points forming a pentagon; possibly two vertex intersections.

Case 1: In Figure 3, the result follows directly from Proposition 4.

Case 2: We apply the proposition to \( \triangle ABC \) and \( \triangle ABD \) in Figure 4.

Case 3: This case is represented by Figure 5.
We must show that $\frac{1}{2} \leq \frac{a + b}{d + e} \leq 2$. The angles $\gamma$ and $\phi$ are external angles of triangles $AFD$ and $BED$ respectively. Since $\angle FDA = \angle BDE$, $\gamma = \phi$. Applying the law of sines and Lemma 1,

$$\frac{a}{\sin \alpha} = \frac{d}{\sin \delta} = \frac{c}{\sin \gamma} = \frac{b}{\sin \phi} = \frac{e}{\sin \theta},$$

$$\frac{a + b}{\sin \alpha + \sin \beta} = \frac{c}{\sin \gamma} = \frac{d + e}{\sin \delta + \sin \theta},$$

$$\frac{a + b}{d + e} = \frac{\sin (\beta + 60^\circ) + \sin \beta}{\sin \delta + \sin (\delta + 60^\circ)}.$$

Since $\alpha$ is an external angle of triangle $CDB$, $\alpha = \beta + 60^\circ$. Similarly, $\theta = \delta + 60^\circ$. Substitution yields

$$\frac{a + b}{d + e} = \frac{\sin (\beta + 60^\circ) + \sin \beta}{\sin \delta + \sin (\delta + 60^\circ)}.$$

Since $\alpha + \delta < 180^\circ$, $\beta + \delta \leq 60^\circ$, and by Lemma 2, $\frac{\sin (\beta + 60^\circ) + \sin \beta}{\sin \delta + \sin (\delta + 60^\circ)}$ will have a minimum value when $\beta = 0$ and $\delta = 60^\circ$ and a maximum value when $\beta = 60^\circ$ and $\delta = 0$. Thus,

$$\frac{\sqrt{3}}{2} \leq \frac{\sin (\beta + 60^\circ) + \sin \beta}{\sin \delta + \sin (\delta + 60^\circ)} \leq \frac{\sqrt{3}}{2} \Rightarrow \frac{1}{2} \leq \frac{a + b}{d + e} \leq 2.$$

Case 4: We must now show that $\frac{1}{2} \leq \frac{a + b + d}{c + e} \leq 2$ and there are two subcases to consider. The first is illustrated by Figure 6.

In Figure 6(a), construct a segment through point $A$ that is parallel to $ED$ and that intersects $FD$ and $FE$ in points $D_1$ and $E_1$, respectively. Similarly, construct a segment that is parallel to $BC$ and that intersects $AB$ and $AC$ in points $B_1$ and $C_1$, respectively so that triangles $AB_1C_1$ and $D_1E_1F$ are congruent (Figure 6(b)).

By construction, $c_1 < c$, $d_1 > d$, and $e_1 < e$. We apply Theorem 1(a) for
equilateral triangles having five intersection points to Figure 6(b), and obtain

\[
\frac{a + b + d}{c + e} < \frac{a + b + d_1}{c_1 + e_1} = 1 \leq 2.
\]

The second case is shown in Figure 7(a). Extend \(AB\) and \(AC\) so that each intersects the line through \(F\) parallel to \(BC\) at points \(B_2\) and \(C_2\), respectively. Construct a segment that is parallel to \(ED\) and that intersects \(FE\) and \(FD\) in points \(E_2\) and \(D_2\) respectively, so that triangles \(AB_2C_2\) and \(D_2E_2F\) are congruent (Figure 7(b)). By construction, \(c < c_2\) and \(e < e_2\). We apply Case 2 to Figure 7(b) and obtain

\[
\frac{1}{2} \leq \frac{a + b}{c_2 + e_2} < \frac{a + b + d}{c + e}.
\]
Combining the inequalities yields $\frac{1}{2} \leq \frac{a + b + d}{c + e} \leq 2$. This completes the proof of Theorem 1.

Fickett’s rectangle problem also appeared in a conjecture that if the congruent polygons were triangles then the maximum ratio would be $\csc \frac{\theta}{2}$ where $\theta$ is the smallest angle of the triangle. For equilateral triangles, $\theta = 60^\circ$ and $\csc \frac{\theta}{2} = 2$. This corresponds to our result.

References


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Some Simple Results on Cevian Quotients

Francisco Javier García Capitán

Abstract. We find the loci of the cevian quotients \( P/Q \) and \( Q/P \) when one of the points is fixed and the other moves along a given line. We also show that, for a given point \( P \), the locus of \( Q \) for which the line joining \( P/Q \) and \( Q/P \) is parallel to \( PQ \) is a conic through \( P \) and \( G/P \), and give two simple constructions of the conic.

The term cevian quotient was due to John Conway [1]. Given two points \( P = (u : v : w) \) and \( Q = (x : y : z) \) in homogeneous barycentric coordinates with reference to a triangle \( ABC \), the cevian quotient \( P/Q \) is the perspector of the cevian triangle of \( P \) and the anticevian triangle of \( Q \). It is the point

\[
P/Q = \left( x \left( -\frac{x}{u} + \frac{y}{v} + \frac{z}{w} \right) : y \left( \frac{x}{u} - \frac{y}{v} + \frac{z}{w} \right) : z \left( \frac{x}{u} + \frac{y}{v} - \frac{z}{w} \right) \right).
\]

A most basic property of cevian quotient is the following theorem.

Theorem 1 ([2, §2.12], [5, §8.3]). \( P/Q = Q' \) if and only if \( P/Q' = Q \).

This is equivalent to \( P/(P/Q) = Q \). It can be proved by direct verification with coordinates. We offer an indirect proof, with the advantage of an explicit construction, for given \( Q \) and \( Q' \), of a point \( P \) with \( P/Q = Q' \) and \( P/Q' = Q \).

For \( Q = (x : y : z) \) and \( Q' = (x' : y' : z') \) with anticevian triangles \( XYZ \) and \( X'Y'Z' \), it is easy to check that the lines \( QX' \) and \( Q'X \) intersect on the sideline \( BC \), at the point \( \left( 0 : xy' + x'y : zx' + z'x \right) = \left( 0 : \frac{1}{x'y' + x'y} : 1 \right) \) (see Figure 1). Similarly, the lines \( QY' \) and \( Q'Y \) intersect on \( CA \) at \( \left( \frac{1}{y'z' + y'z} : 0 : \frac{1}{x'y + x'y} \right) \), and the lines \( QZ' \) and \( Q'Z \) intersect on \( AB \) at \( \left( \frac{1}{y'z' + y'z} : 1 : \frac{1}{x'y + x'y} \right) \). These form the cevian triangle of the point \( P = \left( \frac{1}{y'z' + y'z} : \frac{1}{x'y' + x'y} : 1 \right) \). It is clear that \( P/Q = Q' \) and \( P/Q' = Q \).

Remark. The point \( P = \left( \frac{1}{y'z' + y'z} : \frac{1}{x'y' + x'y} : 1 \right) \) is called the cevian product \( Q \ast Q' \) of \( Q \) and \( Q' \). Clearly, \( Q \ast Q' = Q' \ast Q \).

Proposition 2. Let \( P \) be a fixed point. If \( Q \) moves along a line \( \mathcal{L} \), then the quotient \( P/Q \) traverses the bicevian conic of \( P \) and the trilinear pole of \( \mathcal{L} \).
Proof. Let $P = (u : v : w)$ and $Q$ move along the line $\mathcal{L}$ with line coordinates $[p : q : r]$. If $Q' = P/Q = (x : y : z)$, then $Q = P/Q'$ is on the line $\mathcal{L}$, and

$$px\left(\frac{-x}{u} + \frac{y}{v} + \frac{z}{w}\right) + qy\left(\frac{x}{u} - \frac{y}{v} + \frac{z}{w}\right) + rz\left(\frac{x}{u} + \frac{y}{v} - \frac{z}{w}\right) = 0.$$

Clearing denominators and simplifying, we obtain

$$pwvwx^2 + qwuy^2 + ruvwz^2 - u(qv + rw)yz - v(rw + pu)zx - w(pu +qv)xy = 0.$$

If $x = 0$, this becomes $u(uy - vz)(wy - uz) = 0$. The conic intersects the line $BC$ at $(0 : v : w)$ and $(0 : r : q)$. Similarly, it intersects $CA$ at $(u : 0 : v)$ and $(r : 0 : p)$, and $AB$ at $(u : v : 0)$ and $(q : p : 0)$. This is the bicevian conic through the traces of $P$ and $\left(\frac{1}{p} : \frac{1}{q} : \frac{1}{r}\right)$, the trilinear pole of $\mathcal{L}$. 

Corollary 3 ([4]). Let $P$ be a fixed point. The locus of $Q$ for which the cevian quotient $P/Q$ lies on the tripolar of $P$ is the inscribed conic with perspector $P$.

Proposition 4. Let $P$ be a fixed point. If $Q$ moves along a line $\mathcal{L}$, then the cevian quotient $Q/P$ traverses the circumconic of the anticevian triangle of $P$ with perspector $P_x/P$, where $P_x$ is the trilinear pole of $\mathcal{L}$.

Proof. Let $P = (u : v : w)$ and $Q$ move along the line $\mathcal{L}$ with line coordinates $[p : q : r]$. If $Q'' = Q/P = (x : y : z)$, then $Q = \left(\frac{1}{wy + vz} : \frac{1}{uz + wx} : \frac{1}{vx + uy}\right)$ is on the line $\mathcal{L}$, and

$$\frac{p}{wy + vz} + \frac{q}{uz + wx} + \frac{r}{vx + uy} = 0.$$

Clearing denominators and simplifying, we obtain

$$pvwx^2 + qwuy^2 + ruvwz^2 + (pu + qv + rw)(uyz + vzx + wxy) = 0.$$
It is easy to verify that this conic passes through $A' = (-u : v : w)$, $B' = (u : -v : w)$, $C' = (u : v : -w)$. It is a circumconic of the anticevian triangle of $P$. The tangents to the conic at $A'$, $B'$, $C'$ are the lines $L_a : (qv + rw)x + quy +ruz = 0$, $L_b : pvx + (pu + rw)y + rvz = 0$, $L_c : pwx + qwy + (pu +qv)z = 0$ which intersects $\mathcal{L}$ on $BC$, $CA$, $AB$ respectively. This is the conic tangent to the lines $A'X'$, $B'Y'$, $C'Z'$ at $A'$, $B'$, $C'$ respectively. These lines bound a triangle with vertices
\[
\left(\frac{qv + rw}{u} : q : r\right), \quad \left(p : \frac{rv + pu}{v} : r\right), \quad \left(p : q : \frac{pu +qv}{w}\right).
\]
These form a triangle perspective with the anticevian triangle of $P$ at
\[\left(u(-pu +qv +rw) : v(pu -qv +rw) : w(pu +qv -rw)\right),\]
the cevian quotient of $\left(\frac{1}{p} : \frac{1}{q} : \frac{1}{r}\right)$ (the trilinear pole of $\mathcal{L}$) by $P$. □

Let $\mathcal{L}'/P$ be the conic in Proposition 4. This conic is a circle if and only if it is the circumcircle of the anticevian triangle of $P$. The line $\mathcal{L}'$ is the one containing the intercepts of the tangents to this circle at $A'$, $B'$, $C'$ on the respective sidelines of triangle $ABC$ (see Figure 2). This has line coordinates
\[p : q : r = (u(v + w - u)(c^2v^2 - (b^2 + c^2 - a^2)vw + b^2w^2))\]
\[v(w + u - v)(a^2w^2 - (c^2 + a^2 - b^2)wu + c^2u^2)\]
\[w(u + v - w)(b^2u^2 - (a^2 + b^2 - c^2)uv + a^2v^2).\]
Here are some simple examples in which $\mathcal{L}/P$ is the circumcircle of the anticevian triangle of $P$:

<table>
<thead>
<tr>
<th>$P$</th>
<th>$\mathcal{L}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>centroid</td>
<td>$a^2x + b^2y + c^2z = 0$</td>
</tr>
<tr>
<td>incenter</td>
<td>line at infinity</td>
</tr>
<tr>
<td>symmedian point</td>
<td>$\sum_{\text{cyclic}} a^2(b^2 + c^2 - a^2)x = 0$</td>
</tr>
</tbody>
</table>

**Proposition 5.** Let $P$ be a fixed point. The locus of $Q$ for which the line joining $(P/Q)$ to $(Q/P)$ is parallel to $PQ$ is the union of the cevian lines $AP$, $BP$, $CP$ and a conic $\Gamma(P)$

(1) homothetic to the circumconic with perspector $P$,

(2) passing through $P$ and the cevian quotient $G/P$, and has

(3) the midpoint of $P$ and $G/P$ as center.

**Proof.** If $P = (u : v : w)$ and $Q = (x : y : z)$, the line joining $P/Q$ and $Q/P$ contains the infinite point of $PQ$ if and only if

$$\begin{vmatrix}
  x(-\frac{v}{u} + \frac{v}{v} + \frac{z}{w}) & y(\frac{v}{u} - \frac{v}{v} + \frac{z}{w}) & z(\frac{v}{u} - \frac{v}{v} - \frac{z}{w}) \\
  u - \frac{v}{u} + \frac{v}{v} + \frac{w}{v} & v - \frac{z}{u} - \frac{z}{v} + \frac{w}{u} & w(\frac{z}{u} + \frac{z}{v} - \frac{z}{w}) \\
  (v + w)x - u(y + z) & (w + u)y - v(z + x) & (u + v)z - w(x + y)
\end{vmatrix} = 0.$$

Clearing denominators and simplifying, we obtain

$$2(wy - vz)(uz - wx)(vx - uy)(vwx^2 + wuy^2 + uvz^2 - u^2yz - v^2zx - w^2xy) = 0.$$

Therefore $Q$ lies on one of the lines $AP$, $BP$, $CP$ or the conic $\Gamma(P)$ defined by

$$vwx^2 + wuy^2 + uvz^2 - u^2yz - v^2zx - w^2xy = 0.$$

Rewriting this as

$$\Gamma(P) : (u + v + w)(uz + vzx + wxy) - (x + y + z)(vwx + wuy + uvz) = 0,$$

it is clear that $\Gamma(P)$ is homothetic to the circumconic with perspector $P$, and it is routine to verify that it contains $P$ and the cevian quotient $G/P = (u(-u + v + w) : v(u - v + w) : (u + v - w)w)$. The center of the conic is the midpoint of $P$ and $G/P$, namely,

$$(u(u^2 - uv - uw - 2vw) : v(v^2 - uv - 2uw - vw) : w(w^2 - 2uv - uw - vw)).$$

$\Box$

**Remarks.** (1) If $P$ is the symmedian point, then $\Gamma(P)$ is the Brocard circle with diameter $OK$.

(2) If the line $PQ$ contains $A$, then both cevian quotients $P/Q$ and $Q/P$ are on the same line.

It is easy to note that the conic $\Gamma(P)$ contains the points

$$A_1 = (-u + v + w : v : w), \quad B_1 = (u : u - v + w : w) = (u : v : u + v - w).$$

We present two simple constructions of these points, one by Peter Moses [3], and another by Paul Yiu [6].
**Construction** (Moses). Intersect the cevians $AP$, $BP$, $CP$ with the parallels through the centroid $G$ to $BC$, $CA$, $AB$, at $X$, $Y$, $Z$ respectively. $A_1$, $B_1$, $C_1$ are the harmonic conjugates of $P$ in $AX$, $BY$, $CZ$ respectively.

**Construction** (Yiu). Let $P'$ be the superior of $P$, i.e., the point dividing $PG$ in the ratio $PP' : P'G = 3 : -2$. Construct the parallels of the line $PG$ through the vertices $A$, $B$, $C$, to intersect the sidelines $BC$, $CA$, $AB$ at $X'$, $Y'$, $Z'$ respectively. Then $A_1 = AP \cap X'P'$, $B_1 = BP \cap Y'P'$, and $C_1 = CP \cap Z'P'$. See Figure 3.

**References**

A Vector-based Proof of Morley’s Trisector Theorem

Cesare Donolato

Abstract. A proof is given of Morley’s trisector theorem using elementary vector analysis and trigonometry. The known expression for the side of Morley’s equilateral triangle is also obtained.

Since its formulation in 1899, many proofs of Morley’s trisector theorem have appeared, typically based on plane geometry or involving trigonometry; a historical overview of this theorem with numerous references up to the year 1977 can be found in [4]. Some of the more recent geometric proofs are of the “backward” type [2, 5]; a group-theoretic proof was also given [3]. About fifteen different methods that were used to prove Morley’s theorem are described in detail in [1], with comments on their specific characteristics. The website [1] also provides the related references, which span from the year 1909 to 2010.

In this note we prove the theorem in two stages. First a lemma is proved by use of the dot product of vectors and trigonometry, then the theorem itself easily follows from elementary geometry.

Morley’s Theorem. In any triangle, the three points of intersection of the adjacent angle trisectors form an equilateral triangle.

Let the angles of triangle $ABC$ be of amplitude $3\alpha$, $3\beta$, $3\gamma$, then $\alpha + \beta + \gamma = 60^\circ$. The adjacent trisectors meet to form Morley’s triangle $PQR$; the line extensions of $BR$ and $CQ$ intersect at $D$. In triangle $BDC$ the bisector of angle

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$D$ is concurrent with the other two bisectors $BP$ and $CP$ at $P$, the incenter of the triangle. First, a lemma is proved, from which the theorem easily follows.

**Lemma.** The line $DP$ is perpendicular to the line $RQ$.

**Proof.** Here use is made of the vector method in conjunction with trigonometry. Let $\mathbf{e}$ be the unit vector along $DP$ and $s_1$ the vector representing the side $QR$ of triangle $PQR$. Then the lemma can be restated as saying that the scalar product $s_1 \cdot \mathbf{e}$ vanishes. Figure 1 shows that $s_1 = v_3 - v_2$ so that we must prove that

$$(v_3 - v_2) \cdot \mathbf{e} = 0.$$  

In triangle $BDC$ we have $2\beta + 2\gamma = 120^\circ - 2\alpha$, therefore $\angle D = 60^\circ + \alpha$. This angle is bisected by the line $DP$, hence $\angle QDP = \angle RDP = \angle DRP = 30^\circ + \alpha$. By the exterior angle theorem $\angle AQP = \alpha + \gamma$ and $\angle ARD = \alpha + \beta$. The angle between the vectors $v_3$ and $\mathbf{e}$, being the difference between angles $RDP$ and $ARD$, is $30^\circ - \beta$. Similarly, the angle between $v_2$ and $\mathbf{e}$ is $30^\circ - \gamma$. From these,

$$(v_3 - v_2) \cdot \mathbf{e} = v_3 \cdot \mathbf{e} - v_2 \cdot \mathbf{e} = v_3 \cos(30^\circ - \beta) - v_2 \cos(30^\circ - \gamma) = v_3 \sin(60^\circ + \beta) - v_2 \sin(60^\circ + \gamma). \tag{1}$$

The magnitudes of $v_3$ and $v_2$ can be found by applying the law of sines to triangles $ARB$ and $AQC$ respectively:

$$v_3 = \frac{c \sin \beta}{\sin(\alpha + \beta)} = \frac{c \sin \beta}{\sin(60^\circ - \gamma)}, \quad v_2 = \frac{b \sin \gamma}{\sin(\alpha + \gamma)} = \frac{b \sin \gamma}{\sin(60^\circ + \beta)}.$$ 

Substituting these expressions into (1), we obtain

$$(v_3 - v_2) \cdot \mathbf{e} = \frac{c \sin \beta \sin(60^\circ + \beta)}{\sin(60^\circ - \gamma)} - \frac{b \sin \gamma \sin(60^\circ + \gamma)}{\sin(60^\circ - \beta)} \sin(60^\circ - \gamma) = \frac{1}{4} \cdot \frac{c \sin 3\beta - b \sin 3\gamma}{\sin(60^\circ - \beta) \sin(60^\circ - \gamma)}$$

with the aid of the identity

$$\sin x \sin(60^\circ + x) \sin(60^\circ - x) = \frac{1}{4} \sin 3x, \tag{2}$$

which can be easily proved through the product-to-sum trigonometric formulas.

The law of sines for triangle $ABC$ yields $c \sin 3\beta - b \sin 3\gamma = 0$. Therefore, $(v_3 - v_2) \cdot \mathbf{e} = 0$. 

**Proof of Morley’s Theorem.** Knowing that $DP \perp RQ$, we see that $DP$ divides $DQR$ into two congruent right triangles (with a common leg and a pair of equal acute angles) so that $DQ = DR$. Consequently, triangles $DPQ$ and $DRP$ are also congruent (by SAS), and $s_2 = s_3$. The whole procedure can be used to prove that $s_1 = s_2$. It follows that $s_1 = s_2 = s_3$, and triangle $PQR$ is equilateral. This completes the proof of Morley’s theorem.
Remark. Since triangle $DQR$ is composed of two congruent right triangles, and $\angle QDR = 30^\circ + \alpha$, its complement $\angle DQR = 60^\circ - \alpha$.

The side of Morley’s triangle. The side length $s$ of the equilateral triangle $PQR$ can be calculated by applying the law of sines to triangle $AQR$, whose angles are now known. Since $\angle RAQ = \alpha$, and $\angle AQR = \angle AQD + \angle DQR = (\alpha + \gamma) + (60^\circ - \alpha) = 60^\circ + \gamma$, we find that

$$s = \frac{v_3 \sin \alpha}{\sin(60^\circ + \gamma)} = \frac{c \sin \alpha \sin \beta}{\sin(60^\circ + \gamma) \sin(60^\circ - \gamma)}.$$

By multiplying both terms of the last fraction by $\sin \gamma$, and using in the denominator the identity (2), we get the known expression for the side of Morley’s triangle

$$s = \frac{4c \sin \alpha \sin \beta \sin \gamma}{\sin 3\gamma} = 8R \sin \alpha \sin \beta \sin \gamma,$$

where $R = \frac{c}{2 \sin 3\gamma}$ is the radius of the circumcircle of triangle $ABC$.

References


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Corrigendum

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A vector-based proof of Morley’s trisector theorem,

Page 235, line 2: “∠QDR” should read “∠QDP”.
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