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Dynamics of the Nested Triangles Formed by the Tops of the Perpendicular Bisectors

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Abstract. Given a triangle, we construct a new triangle by taking as vertices the tops of the interior perpendicular bisectors. We describe the dynamics of this transformation exhaustively up to similarity. An acute initial triangle generates a sequence with constant largest angle: except for the equilateral case, the transformation is then ergodic and amounts to a surjective tent map of the interval. An obtuse initial triangle either becomes acute or degenerates by reaching a right-angled state.

1. Introduction

The midpoints of the sides of a triangle ABC are the vertices of the *medial* triangle, which is obtained from ABC by a homothety of ratio $-1/2$ about the centroid. By iterating this transformation, one obtains a sequence of directly similar nested triangles that converges to the common centroid. The feet of the medians and of the perpendicular bisectors generate thus a boring sequence! The feet of the angle bisectors are more interesting: the iterated transformation produces a sequence of nested triangles that always converges to an equilateral shape, as shown in [8] for isosceles initial triangles and in [2] for the general case (in 2006). In the 1990s, four papers [3, 4, 9, 1] analyzed the *pedal* or *orthic* sequence defined by the feet of the altitudes: Peter Lax [4] proved the ergodicity of the construction. We considered in [5] reflection triangles and their iterates: the vertices of the new triangle are obtained by reflecting each vertex of ABC in the opposite side. We were able to decrypt the complex fractal structure of this mapping completely: the sequences generated by acute or right-angled triangles behave nicely, as they always converge to an equilateral shape.

The tops of the medians, angle bisectors, and altitudes are the vertices of the original triangle. But what about the tops of the interior perpendicular bisectors of ABC as new vertices? We found no trace of this problem in the literature. The solution presented here offers an elementary and concrete approach to chaos and requires almost no calculations. Note that another kind of triangle and polygon transformations have a long and rich history, those given by circulant linear combinations of the old vertices, like Napoleon's configuration (see the references in

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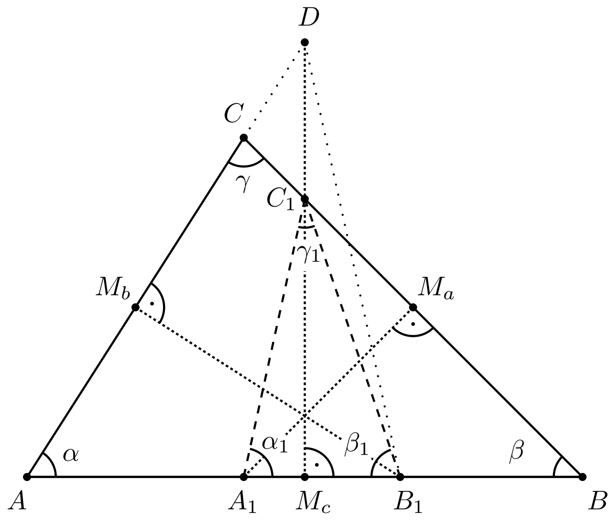

Figure 1. Triangle $A_1B_1C_1$ of the tops of the interior perpendicular bisectors

[6]): their dynamics can be best described by using convolution products and a shape function relying on the discrete Fourier transform [6, 7].

2. Triangle of the tops of the perpendicular bisectors

Let $\Delta = ABC$ be a triangle with corresponding angles α, β, γ and opposite sides a, b, c. The interior perpendicular bisectors issued from the side midpoints M_a, M_b, and M_c end at A_1, B_1, and C_1, respectively (Figure 1). The triangle $\Delta_1 = A_1B_1C_1 = T(\Delta)$ is the *child* of Δ, and Δ a *parent* of Δ_1. We denote the mth iterate of the transformation T by T^m, $m \in \mathbf{Z}$. We are interested in the *shape* of the descendants and ancestors of Δ, *i.e.*, in their angles. In this paper, we provide an exhaustive description of the dynamics of T with respect to shape.

We consider only two types of degenerate triangles, the "isosceles" ones: we assign angles 0°, 0°, 180° to every nontrivial segment with midpoint and angles 90°, 90°, 0° to every nontrivial segment with one double endpoint. A nondegenerate triangle is *proper*. We identify the shape of a triangle Δ with the point of the set

$$\mathcal{S} = \big\{(\alpha, \beta) \mid 0^\circ < \beta \leq \alpha \leq 90^\circ - \beta/2\big\} \cup \{(0^\circ, 0^\circ), (90^\circ, 0^\circ)\}$$

given by the two smallest angles of Δ (Figure 2). $\mathcal{S}$ is the disjoint union of the subsets $\mathcal{O}$, $\mathcal{R}$, and $\mathcal{A}$ of the obtuse, right-angled, and acute shapes, respectively. We denote the shape of an isosceles triangle with equal angles α by I_α, $0^\circ \leq \alpha \leq 90^\circ$. The shapes of the isosceles triangles form the *roof* of $\mathcal{S}$, whose top is the equilateral shape I_{60°. The transformation T induces a transformation τ of $\mathcal{S}$. We set $\tau(I_{90^\circ}) = I_{90^\circ}$ by definition. The children of right-angled triangles are degenerate with two vertices at the midpoint of the hypotenuse: τ maps the whole segment $\mathcal{R}$ to I_{90°. There are no other proper triangles with a degenerate child.

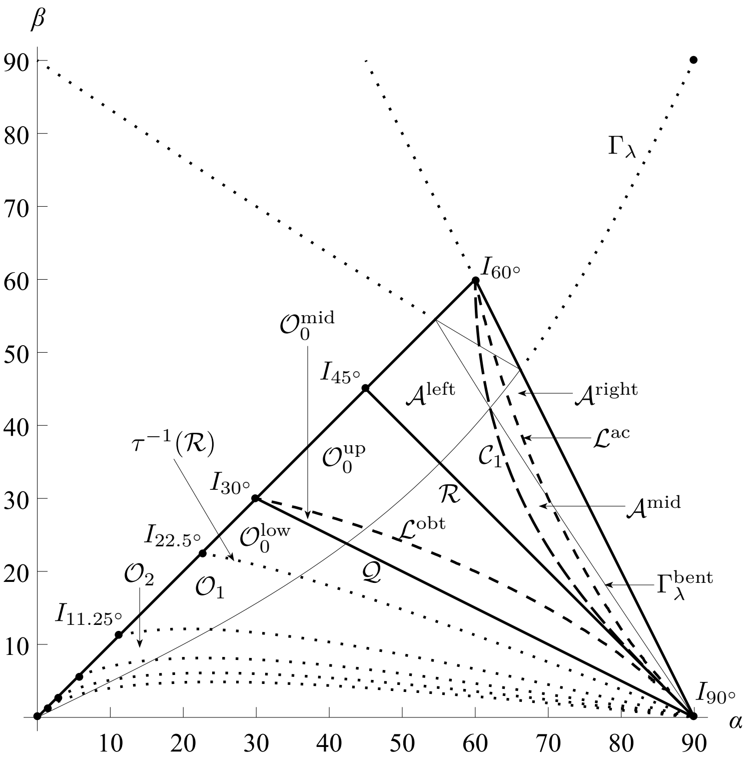

Figure 2. Set $\mathcal{S}$ of the triangle shapes and a curve Γ_λ with corresponding $\Gamma_\lambda^{\mathrm{bent}}$

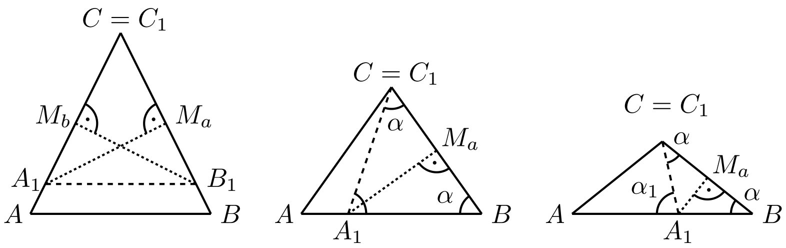

Figure 3. Isosceles parent triangles with angles $\alpha = \beta$ for $\alpha > 60^\circ$, $45^\circ < \alpha < 60^\circ$, and $\alpha < 45^\circ$

The roof is invariant under τ (Figure 3). I_{0° and the points of the right roof side are fixed points, and

$$\tau(I_\alpha) = \begin{cases} I_{2\alpha} & \text{if } 0^\circ \le \alpha \le 45^\circ \\ I_{180^\circ - 2\alpha} & \text{if } 45^\circ \le \alpha \le 60^\circ \\ I_\alpha & \text{if } 60^\circ \le \alpha \le 90^\circ \end{cases} .$$

When α grows, $\tau(I_\alpha)$ travels on the roof as follows: from I_{0° to I_{90° for $0^\circ \le \alpha \le 45^\circ$, then back to I_{60° for $45^\circ \le \alpha \le 60^\circ$ before descending the right roof side for $60^\circ \le \alpha \le 90^\circ$. Each shape I_α has thus one, two, or three isosceles parents according as it lies on the left roof side, at I_{60° or I_{90°, or on the rest of the right roof side, respectively. For $60^\circ < \alpha < 90^\circ$, the three isosceles parents of I_α are itself, the point where the parallel to $\mathcal{R}$ through I_α cuts the left roof side, and the reflection of this parent in the line of $\mathcal{R}$. An equilateral triangle has four isosceles parents: itself and three of shape I_{30°. I_α degenerates eventually to I_{90° (and this after m steps) if and only if $\alpha = 90^\circ/2^m$ for some integer $m \ge 0$. Otherwise, $I_\alpha \ne I_{0^\circ}$ reaches its final nondegenerate state $I_{2^n\alpha}$ or $I_{180^\circ - 2^n\alpha}$ after n steps, where $n \ge 0$ is given by $30^\circ/2^{n-1} \le \alpha < 45^\circ/2^{n-1}$ in the first and by $45^\circ/2^{n-1} < \alpha < 60^\circ/2^{n-1}$ in the second case.

We consider the tangents of the angles of Δ:

$$u = \tan\alpha, \quad v = \tan\beta, \quad w = \tan\gamma = \frac{u+v}{uv-1}.$$

We write $\tan(\alpha,\beta)$ for $(\tan\alpha, \tan\beta)$. In the (u,v)-plane, the transformation induced by τ is essentially the radial stretch $(u,v) \mapsto \dfrac{|w|}{u}(u,v)$. More precisely, we have the following result.

Theorem 1. *When (α,β) is the shape of the proper obtuse or acute triangle Δ, the angles α_1 and β_1 of $T(\Delta)$ are acute with $\alpha_1 \ge \beta_1$ and $\tan(\alpha_1,\beta_1) = \dfrac{|w|}{u}(u,v)$. According as the parent is obtuse or acute, its child has a smaller or the same largest angle.*

Proof. We first look at the acute case $90^\circ > \gamma \ge \alpha \ge \beta > 0^\circ$ (Figure 1). Let D be the intersection of side b with the perpendicular bisector of c. By Thales' theorem, $A_1M_cM_aC_1$ and $M_cB_1DM_b$ are convex cyclic quadrilaterals with circumcircles of diameters A_1C_1 and B_1D, respectively. One has thus $\alpha_1 = \angle M_cM_aB = \gamma = \angle M_cB_1D$ and

$$\tan\beta_1 = \frac{C_1M_c}{M_cB_1} = \frac{C_1M_c}{c/2} \cdot \frac{c/2}{DM_c} \cdot \frac{DM_c}{M_cB_1} = \frac{\tan\beta}{\tan\alpha}\tan\gamma.$$

Since $\alpha_1 = \gamma$, one has $\gamma_1 \le \angle A_1C_1B = \alpha \le \gamma$.

In the obtuse case $\gamma > 90^\circ > \alpha \ge \beta > 0^\circ$, A_1 lies on the right of M_c and B_1 on the left. One considers the quadrilaterals $M_cA_1M_aC_1$ and $B_1M_cDM_b$ to get $\alpha_1 = \angle M_cM_aC_1 = 180^\circ - \gamma = \angle M_cB_1D$ and $\tan\beta_1 = \tan\beta \cdot \tan(180^\circ - \gamma)/\tan\alpha$. One has $\gamma_1 = 180^\circ - (\alpha_1 + \beta_1) = \gamma - \beta_1 < \gamma$. □

An acute shape and all its descendants lie thus on a parallel to the line of the right-angled shapes: for every $\varphi \in (0^\circ, 15^\circ]$, the segment of acute shapes $\mathcal{P}_\varphi^{\text{ac}}$ parallel to $\mathcal{R}$ from $I_{45^\circ+\varphi}$ to $I_{90^\circ-2\varphi}$ is invariant under τ (Figure 4).

The shape of an obtuse or right-angled child $\tau(\alpha,\beta)$ is (α_1,β_1) since $\gamma_1 > \alpha_1$ (with equality for $(\alpha,\beta) = I_{90^\circ}$). The shape of an acute child is (α_1,β_1), (γ_1,β_1), or (β_1,γ_1) since $\alpha_1 \ge \beta_1$. We describe below the conditions of each occurrence.

Note that (γ_1, β_1) is the horizontal reflection of (α_1, β_1) in the line of the right roof side and that (β_1, γ_1) is the reflection of (γ_1, β_1) in the line of the left roof side.

Theorem 2. *The acute shape (α, β) and its reflection in the line of $\mathcal{R}$, the obtuse shape $(90^\circ - \beta, 90^\circ - \alpha)$, share the same child.*

Proof. The tangents of the shapes are (u, v) and $(u', v') = (1/v, 1/u)$. Since $w' = -w$ and $v'/u' = v/u$, the children have the same angles. □

As a consequence, the segment $\mathcal{Q}$ of obtuse shapes joining I_{30° and I_{90° is reflected in $\mathcal{R}$ by τ to the right roof side consisting of fixed points (Figures 2 and 4). (When an isosceles child triangle has a scalene parent, note that a reflection in the child's axis gives a second parent triangle.) And for $\varphi \in (0^\circ, 15^\circ]$, the action of τ on the segment $\mathcal{P}_\varphi^{\mathrm{obt}}$ of obtuse shapes parallel to $\mathcal{R}$ from $I_{45^\circ-\varphi}$ to $(90^\circ - 4\varphi, 2\varphi) \in \mathcal{Q}$ is the reflection to $\mathcal{P}_\varphi^{\mathrm{ac}}$ followed by τ: since $\mathcal{P}_\varphi^{\mathrm{ac}}$ is invariant under τ, τ maps $\mathcal{P}_\varphi^{\mathrm{obt}}$ to $\mathcal{P}_\varphi^{\mathrm{ac}}$.

3. Dynamics of the transformation

Consider a slope λ with $0 < \lambda \leq 1$. On the segment $v = \lambda u$, $0 \leq u < \sqrt{1/\lambda}$, $|w|$ is given by $(1 + \lambda)u/(1 - \lambda u^2)$ and is a strictly growing convex function of u with image $[0, +\infty)$. The segment $v = \lambda u$, $0 \leq u < \sqrt{1/\lambda}$, is thus stretched bijectively and continuously to the whole half-line $v = \lambda u$, $u \geq 0$, by the map $(u, v) \mapsto \dfrac{|w|}{u}(u, v)$, whose only fixed point is the origin. The strictly growing curves Γ_λ corresponding to these segments are given by

$$\Gamma_\lambda\colon \quad \beta = \arctan(\lambda \tan \alpha), \quad 0^\circ \leq \alpha \leq 90^\circ,\ 0 < \lambda \leq 1,$$

which is $\beta = \alpha$ for $\lambda = 1$: they link the origin I_{0° and the point $(90^\circ, 90^\circ)$, are symmetric in the line of $\mathcal{R}$, and provide a simple covering of $\{(\alpha, \beta) \mid 0^\circ < \beta \leq \alpha < 90^\circ\}$ (Figures 2 and 4). The lower parts $\Gamma_\lambda^{\mathrm{obt}}$ of the Γ_λ's for $0^\circ \leq \alpha \leq \arctan \sqrt{1/\lambda}$ cover the set of the obtuse and proper right-angled shapes. The middle parts $\Gamma_\lambda^{\mathrm{ac}}$ for $\arctan \sqrt{1/\lambda} < \alpha \leq \arctan \sqrt{1 + 2/\lambda}$ cover the set of the acute shapes. By Theorem 1, the transformation τ stretches each $\Gamma_\lambda^{\mathrm{obt}}$ to the doubly bent curve $\Gamma_\lambda^{\mathrm{bent}}$ linking I_{0° and I_{90° (Figure 2): $\Gamma_\lambda^{\mathrm{bent}}$ is obtained from Γ_λ by a horizontal reflection of its upper part in the line of the right roof side followed by a partial reflection in the left roof side. By Theorem 2, τ maps each path $\Gamma_\lambda^{\mathrm{ac}}$ (joining a right-angled shape and an isosceles fixed point of τ) to the way back along $\Gamma_\lambda^{\mathrm{bent}}$ from I_{90° to the right roof side.

τ provides thus a simple covering of $\mathcal{O} \cup \mathcal{R} \setminus \{I_{90^\circ}\}$ (by obtuse shapes exclusively), a fivefold covering of $\mathcal{A}$ without roof, a triple covering of the section $\{I_\alpha \mid 45^\circ < \alpha < 60^\circ\}$ of the left roof side, a double covering of I_{60° and a quadruple covering of the right roof side without endpoints. The covering of $\mathcal{A}$ without roof consists of three layers of obtuse and two layers of acute shapes. More precisely, each region $\mathcal{O}_n \subset \mathcal{O}$ bordered by the curve $\tau^{-n}(\mathcal{R})$ and its parent curve $\tau^{-n-1}(\mathcal{R})$ is mapped by τ bijectively and upwardly (along the curves Γ_λ) to the next less obtuse region $\mathcal{O}_{n-1}$ for all integers $n \geq 1$ (Figure 2). The curve

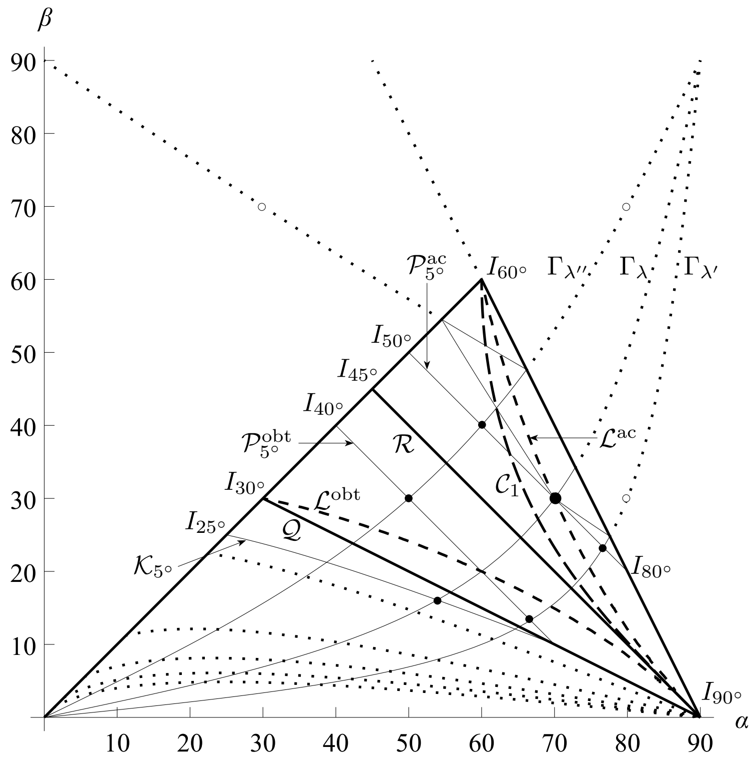

Figure 4. The five parents of the acute shape $(70^\circ, 30^\circ) \in \mathcal{L}^{\mathrm{ac}}$ and the parent curves of the segment $\mathcal{P}^{\mathrm{ac}}_{5^\circ}$

$\tau^{-n}(\mathcal{R})$, $n \geq 0$, joins $I_{45^\circ/2^n}$ and I_{90°. Every neighborhood of the α-axis contains all but finitely many ancestor regions of $\mathcal{O}_0$, the region of the *slightly obtuse shapes*. $\mathcal{O}_0 \setminus \{I_{30^\circ}\}$ is the disjoint union of three layers of the fivefold covering of $\mathcal{A}$ without roof and of a double covering of the acute roof section without I_{60° and I_{90°. Each of the following three subregions of $\mathcal{O}_0$ is mapped bijectively to $\mathcal{A}$ without roof (along the three smooth sections of the curves $\Gamma^{\mathrm{bent}}_{\lambda}$) : the lower subregion $\mathcal{O}^{\mathrm{low}}_0$, the middle lens $\mathcal{O}^{\mathrm{mid}}_0$, and the upper subregion $\mathcal{O}^{\mathrm{up}}_0$ delimited by the curves $\tau^{-1}(\mathcal{R})$, $\mathcal{Q}$, $\mathcal{L}^{\mathrm{obt}}$, and $\mathcal{R}$. The curve $\mathcal{L}^{\mathrm{obt}}$ consists of the obtuse parent shapes mapped to a bend point of some curve $\Gamma^{\mathrm{bent}}_{\lambda}$ on the left roof side. The reflections of $\mathcal{O}^{\mathrm{mid}}_0$ and $\mathcal{O}^{\mathrm{up}}_0$ in the line of $\mathcal{R}$, separated by the curve $\mathcal{L}^{\mathrm{ac}}$, constitute the remaining two layers of acute shapes in the covering of $\mathcal{A}$.

The shape of an acute child $\tau(\alpha, \beta)$ is (α_1, β_1) if the parent (α, β) lies in $\mathcal{O}^{\mathrm{low}}_0$, (γ_1, β_1) if the parent is in $\mathcal{O}^{\mathrm{mid}}_0$ or $\mathcal{A}^{\mathrm{right}}$, and (β_1, γ_1) if the parent is located between $\mathcal{L}^{\mathrm{obt}}$ and $\mathcal{L}^{\mathrm{ac}}$ (Figure 2).

The bijective stretch

$$p^{-1}\colon \{(u, v) \mid u \geq v > 0, uv < 1\} \to \{(u, v) \mid u \geq v > 0\},\ (u, v) \mapsto \frac{|w|}{u}(u, v)$$

is the inverse function of

$$p(u,v) = \frac{\sqrt{(u+v)^2+4u^3v}-(u+v)}{2u^2v}(u,v).$$

One parent (α,β) of the proper shape (α_1,β_1) is thus the obtuse shape given by $\tan(\alpha,\beta) = p(\tan(\alpha_1,\beta_1))$ (Figure 4). When (α_1,β_1) is an acute shape, the parent (α,β) lies in $\mathcal{O}_0^{\text{low}}$ and the other parents are the obtuse shapes (α',β'), (α'',β'') given by $\tan(\alpha',\beta') = p(\tan(\gamma_1,\beta_1))$, $\tan(\alpha'',\beta'') = p(\tan(\gamma_1,\alpha_1))$ and their acute reflections $(90^\circ-\beta', 90^\circ-\alpha')$ and $(90^\circ-\beta'', 90^\circ-\alpha'')$ on $\mathcal{P}^{\text{ac}}_{(\alpha_1+\beta_1-90^\circ)/2}$. Except (α,β), the parents are the vertices of a rectangle. The parent (α,β) and (α_1,β_1) lie on the same curve Γ_λ ($\lambda = \tan\beta_1/\tan\alpha_1$), whereas (α',β') and (α'',β'') reach their child by traveling on bent curves $\Gamma^{\text{bent}}_{\lambda'}$ and $\Gamma^{\text{bent}}_{\lambda''}$ ($\lambda' = \tan\beta_1/\tan\gamma_1$, $\lambda'' = \tan\alpha_1/\tan\gamma_1$), after the first bend for $(\alpha',\beta') \in \mathcal{O}_0^{\text{mid}}$ and the second for $(\alpha'',\beta'') \in \mathcal{O}_0^{\text{up}}$.

Each acute nonequilateral triangle has exactly five differently placed parents: by axial symmetry, an isosceles child has one or two pairs of inversely congruent scalene parents according as its equal angles are larger or smaller than 60°. We already mentioned the four parents of an equilateral triangle: itself and three of shape I_{30°. The formula for $p(u,v)$ shows that the parents of a given proper triangle are all constructible by straightedge and compass.

The parents (α,β) of the proper right-angled shapes (α_1,β_1) with $\tan(\alpha_1,\beta_1)=(u_1,v_1)$ are the obtuse solutions of $u_1v_1 = 1$ and constitute thus the curve

$$\tau^{-1}(\mathcal{R}) \setminus \{I_{90^\circ}\}: \quad \tan^2(\alpha+\beta) = \tan\alpha/\tan\beta, \quad 0^\circ < \beta \le \alpha,\ \alpha+\beta < 90^\circ$$

(Figure 2). Since $u_1/v_1 = u/v$, the proper shapes of $\tau^{-2}(\mathcal{R})$ are the shapes (α,β) below $\tau^{-1}(\mathcal{R})$ with $\tan^2(\alpha_1+\beta_1) = \tan\alpha/\tan\beta$, and so on. For each integer $n \ge 1$, a parametric representation of the curve $\tau^{-n}(\mathcal{R})$ without I_{90° is given by $\tan(\alpha,\beta) = p^n(\hat{u}, 1/\hat{u})$ with $\tan\beta = \hat{u}^{-2}\tan\alpha$ and $(\hat{u},1/\hat{u}) = \tan(\hat{\alpha}, 90^\circ-\hat{\alpha})$, $45^\circ \le \hat{\alpha} < 90^\circ$, where p^n denotes the nth iterate of p. The maximal elevation of $\tau^{-n}(\mathcal{R})$ seems to be approximately $23^\circ/n$.

Because the origin is the unique fixed point of the stretch $(u,v) \mapsto \dfrac{|w|}{u}(u,v)$ in the set $\{(u,v) \mid u \ge v > 0, uv < 1\}$, the only possible fixed point of τ on $\Gamma_\lambda \cap \mathcal{S}$ (besides the endpoints) is the double point of $\Gamma^{\text{bent}}_\lambda$ (Figure 2). But this acute double point is the unique point of $\Gamma^{\text{bent}}_\lambda$ on its parallel to $\mathcal{R}$ and is forced to be its own child. The curve $\mathcal{C}_1$ of the nonisosceles fixed shapes joins I_{60° and I_{90° in $\mathcal{A}$ and lies below $\mathcal{L}^{\text{ac}}$: the shapes $(\alpha,\beta) \in \mathcal{C}_1$ are thus the solutions of the equation $(\alpha,\beta) = \tau(\alpha,\beta) = (\beta_1,\gamma_1)$, *i.e.*, $u = v_1$, which can be transformed into $v^2 + (u-u^3)v + u^2 = 0$. The solution that corresponds to shapes is

$$\mathcal{C}_1: \quad v = \frac{u}{2}\left(u^2-1-\sqrt{(u^2-1)^2-4}\right), \quad u = \tan\alpha,\ v = \tan\beta,\ u \ge \sqrt{3}.$$

When the shape (α,β) of Δ lies on $\mathcal{C}_1$, $T(\Delta)$ is *directly* similar to Δ, since $\Delta = ABC$ and $A_1B_1C_1$ are then equally oriented with $\gamma = \alpha_1 \ge \alpha = \beta_1 \ge \beta = \gamma_1$.

In general, one finds the child of the acute shape (α,β) as follows. Draw the curve Γ_λ through (α,β) by taking $\lambda=\tan\beta/\tan\alpha$. The child is the point where the parallel to $\mathcal{R}$ through (α,β) cuts one of the two bent sections of $\Gamma_\lambda^{\text{bent}}$ (Figure 2).

A proper scalene shape (α,β) on the curve $\mathcal{L}^{\text{obt}}$, which links I_{90° and I_{30°, is the obtuse parent of some $I_{\hat\alpha}$, $45^\circ<\hat\alpha\le 60^\circ$. $I_{\hat\alpha}=(\hat\alpha,\hat\alpha)$ is the bend of $\Gamma_{\hat\lambda}^{\text{bent}}$ on the left roof side. Before bending, $I_{\hat\alpha}$ was the point $(180^\circ-2\hat\alpha,\hat\alpha)$ located on the curve $\Gamma_{\hat\lambda}$, like its parent (α,β). One has thus

$$\hat\lambda=\tan\hat\alpha/\tan(180^\circ-2\hat\alpha)=(\tan^2\hat\alpha-1)/2$$

and $\tan(\alpha,\beta)=(u,v)=(u,\hat\lambda u)$ with $w<0$. The condition $\tau(\alpha,\beta)=I_{\hat\alpha}$ is equivalent to $w_1=v_1$, since the two *smaller* angles of $T(\Delta)$ have to be equal. The equation $w_1=v_1$ can be transformed into $1+u/v=\sqrt{1+w^2}$ and further, since $v=\hat\lambda u$ and $uv<1$, into

$$\hat\lambda\tan\hat\alpha\,u^2+\hat\lambda(\hat\lambda+1)u-\tan\hat\alpha=0.$$

The positive solution u gives

$$\mathcal{L}^{\text{obt}}:\quad u=\tan\alpha=\sqrt{\frac{(\tan\hat\alpha+\cot\hat\alpha)^2}{16}+\frac{2}{\tan^2\hat\alpha-1}}-\frac{\tan\hat\alpha+\cot\hat\alpha}{4},$$
$$\beta=180^\circ-\alpha-2\hat\alpha,\quad 45^\circ<\hat\alpha\le 60^\circ.$$

One obtains the parametric representation of $\mathcal{L}^{\text{ac}}$ by reflection in $\mathcal{R}$ or (since now $uv>1$) from the equation $\hat\lambda\tan\hat\alpha\,u^2-\hat\lambda(\hat\lambda+1)u-\tan\hat\alpha=0$, which leads to

$$\mathcal{L}^{\text{ac}}:\quad \tan\alpha=\sqrt{\frac{(\tan\hat\alpha+\cot\hat\alpha)^2}{16}+\frac{2}{\tan^2\hat\alpha-1}}+\frac{\tan\hat\alpha+\cot\hat\alpha}{4},$$
$$\beta=2\hat\alpha-\alpha,\quad 45^\circ<\hat\alpha\le 60^\circ.$$

The segment $\mathcal{P}_\varphi^{\text{ac}}$ parallel to $\mathcal{R}$ from $I_{45^\circ+\varphi}$ to $I_{90^\circ-2\varphi}$, $0^\circ<\varphi<15^\circ$, has five parent curves (Figure 4): $\mathcal{P}_\varphi^{\text{ac}}$ is doubly covered by itself under τ, once by the portion between the left roof side and $\mathcal{L}^{\text{ac}}$, once by the rest. $\mathcal{P}_\varphi^{\text{ac}}$ is covered twice by $\mathcal{P}_\varphi^{\text{obt}}$ through reflection followed by τ. And the curve

$$\mathcal{K}_\varphi:\quad \tan(\alpha,\beta)=p(\tan(\tilde\alpha,90^\circ+2\varphi-\tilde\alpha)),\quad 45^\circ+\varphi\le\tilde\alpha\le 90^\circ-2\varphi,$$

formed by the parents in $\mathcal{O}_0^{\text{low}}$ of some shape of $\mathcal{P}_\varphi^{\text{ac}}$ is mapped bijectively onto $\mathcal{P}_\varphi^{\text{ac}}$ (along the Γ_λ's). $\mathcal{K}_\varphi$ links $I_{22.5^\circ+\varphi/2}$ and the end shape $(90^\circ-4\varphi,2\varphi)$ of $\mathcal{P}_\varphi^{\text{obt}}$. The region $\mathcal{O}_0\setminus\{I_{30^\circ}\}$ is the disjoint union of the nested pointed arches $\mathcal{K}_\varphi\cup\mathcal{P}_\varphi^{\text{obt}}$, $0^\circ<\varphi<15^\circ$. Such an arch covers $\mathcal{P}_\varphi^{\text{ac}}$ three times under τ: down, and up, and down again.

When restricted to $\mathcal{P}_\varphi^{\text{ac}}$, the transformation τ amounts to the surjective tent map $t\mapsto 2\min(t,1-t)$ of the unit interval and is thus ergodic once the appropriate measure has been defined (see Section 4). $\mathcal{P}_\varphi^{\text{ac}}$ is first folded by τ about the point on $\mathcal{L}^{\text{ac}}$ in such a way that its left endpoint coincides with the fixed right endpoint (Figure 4). Both halves, being held firmly at their common right end, are then stretched to the left until each of them covers simply $\mathcal{P}_\varphi^{\text{ac}}$. The subregions of acute shapes $\mathcal{A}^{\text{left}}$, $\mathcal{A}^{\text{mid}}$, and $\mathcal{A}^{\text{right}}$ delimited in $\mathcal{A}$ by the curves $\mathcal{R}$, $\mathcal{C}_1$, and $\mathcal{L}^{\text{ac}}$ (Figure 2) are thus transformed in the following way along the parallels to $\mathcal{R}$: $\mathcal{A}^{\text{left}}$ is mapped bijectively to $\mathcal{A}^{\text{mid}} \cup \mathcal{A}^{\text{right}}$, $\mathcal{A}^{\text{mid}}$ to $\mathcal{A}^{\text{left}}$, and $\mathcal{A}^{\text{right}}$ to $\mathcal{A}$, giving two acute parents to every acute shape not lying on the left roof side. According as the acute shape is on the right or left of $\mathcal{C}_1$, it is located between its acute parents or on their left, respectively (Figure 4).

Take a unit circle and draw a horizontal chord AB below the diameter at distance $\sin 2\varphi$, $0^\circ < \varphi < 15^\circ$. The acute shapes (α, β) of the segment $\mathcal{P}_\varphi^{\text{ac}}$ are then represented by the inscribed triangles ABC for which C lies between the north pole and the limit position C_{lim} where $\alpha = \gamma = 90^\circ - 2\varphi$. τ moves C on this arc.

The 120 integer-angled shapes I_{n° on the roof and $(2m^\circ, (45 - m)^\circ)$ on $\mathcal{Q}$ have integer-angled children. A systematic search shows that there are only eight further such cases: $\tau(36^\circ, 12^\circ) = (48^\circ, 18^\circ)$, $\tau(42^\circ, 12^\circ) = (54^\circ, 18^\circ)$, $\tau(66^\circ, 18^\circ) = \tau(72^\circ, 24^\circ) = (54^\circ, 42^\circ)$, $\tau(50^\circ, 30^\circ) = \tau(60^\circ, 40^\circ) = (70^\circ, 30^\circ)$, $\tau(70^\circ, 30^\circ) = \tau(60^\circ, 20^\circ) = I_{50^\circ}$. These cases verify the identity

$$\tan(60^\circ - \delta) \tan(60^\circ + \delta) \tan \delta = \tan 3\delta$$

for $\delta = 12^\circ$, 18°, 6°, 20°, and 10° in order: $\tan 12^\circ \tan 48^\circ / \tan 36^\circ = \tan 18^\circ$ and so on.

4. The transformation as symbolic dynamics

We use symbolic dynamics to give a short and elementary proof of the ergodicity of τ on every segment $\mathcal{P}_\varphi^{\text{ac}}$. We identify $\mathcal{P}_\varphi^{\text{ac}}$, $0^\circ < \varphi < 15^\circ$, with the interval $[0, 1]$, where $0 = I_{90^\circ - 2\varphi}$ and $1 = I_{45^\circ + \varphi}$ are the isosceles endpoints on the *right* and left roof sides, respectively (Figure 4). We represent each shape $s \in \mathcal{P}_\varphi^{\text{ac}}$ by its infinite binary address $x = x_1 x_2 x_3 \ldots$ giving the position of s with respect to the fractal subdivision of $\mathcal{P}_\varphi^{\text{ac}}$ induced by the monotonicity intervals of τ and its iterates. The kth digit of the address is 0 or 1 according as τ^k restricted to $\mathcal{P}_\varphi^{\text{ac}}$ is direction-preserving or reversing in a neighborhood of the shape (and addresses of turning points end in a constant sequence of 0s or 1s). If x is eventually periodic, we overline the period's digits. We identify the ends $0\overline{1}$ and $1\overline{0}$. The child $\tau(s)$ is then given by a left shift when $x_1 = 0$ and a left shift with permutation $0 \leftrightarrow 1$ in x when $x_1 = 1$. Note that $\tau^n(x) = x_{n+1} \ldots$ or $\tau^n(x) = (x_{n+1} \ldots)_{0 \leftrightarrow 1}$ according as $x_1 \ldots x_n$ contains an even or odd number of 1s.

The shape on $\mathcal{L}^{\text{ac}}$ is $1/2 = 0\overline{1}$ and the nonisosceles fixed point is $2/3 = \overline{10}$. The parents of x are $0x$ and $1x_{0 \leftrightarrow 1}$ (these are the *acute* parents of the shape). The only ancestors of $\overline{0}$ are the addresses ending in $\overline{0}$ or $\overline{1}$, *i.e.*, integer multiples of some 2^{-n}.

We prove that the orbit of x becomes eventually periodic if and only if the digits of x are eventually periodic. Consider first an eventually periodic sequence $x = x_1 \ldots x_k \overline{p_1 \ldots p_n}$: since $\{\overline{p_1 \ldots p_n}, (\overline{p_1 \cdots p_n})_{0 \leftrightarrow 1}\}$ contains both $\tau^{k+n}(x)$ and $\tau^{k+2n}(x)$, one of them is $\tau^k(x)$, whose orbit is thus periodic. Conversely, if the orbit of $x = x_1 x_2 \ldots$ is eventually periodic with $\tau^k(x) = \tau^{k+n}(x)$, $\tau^k(x)$ is a periodic sequence given by $x_{k+1} x_{k+2} \ldots$ or $(x_{k+1} x_{k+2} \ldots)_{0 \leftrightarrow 1}$, *i.e.*, the sequence x is eventually periodic.

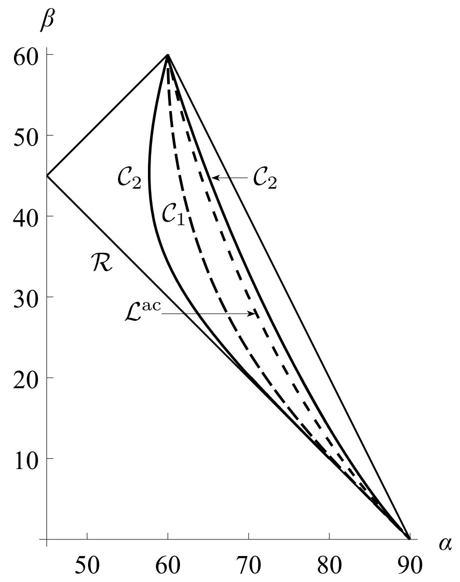

Figure 5. Curves $\mathcal{C}_2$ of the 2-cycles

The only 2-cycle is $2/5 = \overline{0110} \leftrightarrow \overline{1100} = 4/5$. Figure 5 shows the two curves $\mathcal{C}_2$ of these 2-cycles in $\mathcal{A}$. When the shape (α, β) of Δ lies on $\mathcal{C}_2$, $T^2(\Delta)$ is *inversely* similar to Δ, since $\Delta = ABC$ and $A_2B_2C_2$ are then equally oriented with $\gamma = \beta_2 \geq \alpha = \alpha_2 \geq \beta = \gamma_2$ if (α, β) is on the left of $\mathcal{L}^{\mathrm{ac}}$ and $\gamma = \gamma_2 \geq \alpha = \beta_2 \geq \beta = \alpha_2$ if (α, β) is on the right of $\mathcal{L}^{\mathrm{ac}}$. The two 3-cycles are $\overline{010} = 2/7 \mapsto \overline{100} = 4/7 \mapsto \overline{110} = 6/7 \mapsto \overline{010}$ and $00\overline{111000} = 2/9 \mapsto 4/9 \mapsto 8/9 \mapsto 2/9$. The iterated tent map τ^n, $n \geq 1$, has 2^n fixed points in $\mathcal{P}_\varphi^{\mathrm{ac}}$: since $2^n > 2 + 2^2 + \cdots + 2^{n-1}$ for $n \geq 2$, τ has n-cycles (of *fundamental* period n) in $\mathcal{P}_\varphi^{\mathrm{ac}}$ for all integers $n \geq 1$. It is easy to see that the 2^n fixed points are the fractions $2k/(2^n - 1)$, $2k/(2^n + 1)$ in $[0, 1]$ and that for $n \geq 3$ such a fixed point generates an n-cycle if and only if it cannot be written with a smaller denominator $2^m \pm 1$. One can construct addresses with almost any behavior under iteration of τ. We design for example an address x_{dense} whose forward orbit is dense in $\mathcal{P}_\varphi^{\mathrm{ac}}$: concatenate successively all binary words of length 1, 2, 3, and so on to an infinite sequence, and submit if necessary each of them in order to a permutation $0 \leftrightarrow 1$ in such a way that the original word appears as head of the corresponding descendant of x_{dense}.

The measure of an interval of $\mathcal{P}_\varphi^{\mathrm{ac}}$ of kth generation, *i.e.*, with an address of length k, is 2^{-k} by definition. τ is then measure-preserving on $\mathcal{P}_\varphi^{\mathrm{ac}}$. By using the binomial distribution, it is easy to see that almost all shapes of $\mathcal{P}_\varphi^{\mathrm{ac}}$ have a *normal* address [9]. An address x is normal if, for every fixed integer $k \geq 1$, all binary words of k digits appear with the same asymptotic frequency 2^{-k} as heads of the successive $\tau^n(x)$, $n \in \mathbf{N}$. The descendants of a normal address visit thus all intervals of kth generation with equal asymptotic frequency, and this for every k:

the forward orbit of almost every shape of $\mathcal{P}_\varphi^{\mathrm{ac}}$ "goes everywhere in $\mathcal{P}_\varphi^{\mathrm{ac}}$ equally often", τ is an ergodic transformation of $\mathcal{P}_\varphi^{\mathrm{ac}}$.

We conclude by formulating a condensed version of our results.

Theorem 3. *Consider the map that transforms a triangle into the triangle of the tops of its interior perpendicular bisectors.*
(1) *An acute initial triangle generates a sequence with constant largest angle: except for the equilateral case, the transformation is then ergodic in shape and amounts to a surjective tent map of the interval. A proper obtuse initial triangle either becomes acute or degenerates by reaching a right-angled state.*
(2) *A proper triangle is the transform of exactly one, four or five triangles according as it is nonacute, equilateral, or acute but not equilateral, respectively.*

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