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Distances Between the Circumcenter of the Extouch Triangle and the Classical Centers of a Triangle

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Abstract. We compute, in a triangle, the distances between the circumcenter of the extouch triangle and the circumcenter, the incenter, and the orthocenter, respectively. For this calculation, we use the absolute barycentric coordinates and obtain relatively simple formulas which seem unknown. To conclude, we compute the barycentric coordinates of the incenter of the extouch triangle.

1. Introduction

We consider a triangle ABC and we denote by O the circumcenter, I the incenter, H the orthocenter, G the centroid, and N the nine-point center. We denote the side-lengths by a, b, c, the semiperimeter by s, R the circumradius, and r the inradius. The distances between the classical centers of the triangle ABC are well known. We recall that

$$\begin{aligned} OI^2 &= R^2 - 2Rr, \\ OH^2 &= R^2 - 8R^2 \cos A \cos B \cos C, \\ HI^2 &= 2r^2 - 4R^2 \cos A \cos B \cos C. \end{aligned}$$

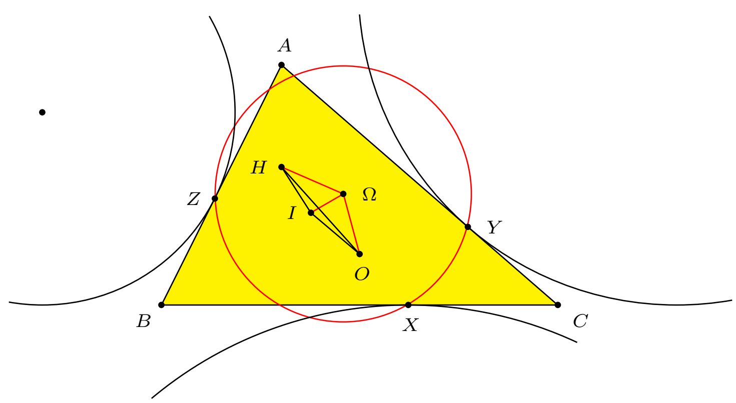

Figure 1.

It is well known that the circle through the excenters of triangle ABC has center I', the reflection of I in O, and that the radii through the excenters are perpendicular to the corresponding sides of ABC. It follows that the extouch triangle XYZ is the pedal triangle of I', and its circumcircle is the common pedal circle of I' and

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its isogonal conjugate I'^{*}. The circumcenter Ω is the midpoint between I' and I'^{*}. In this note we compute the distances between Ω and the above classical triangle centers.

Theorem 1.
(a) $\Omega O^2 = R^2 - \frac{4R^3(R-r)}{r^2} \cos A \cos B \cos C$.
(b) $\Omega I^2 = 2R^2 - 4Rr - \frac{4R^3(R-2r)}{r^2} \cos A \cos B \cos C$.
(c) $\Omega H^2 = 2R^2 - 4Rr - 2r^2 - \frac{4R^2(R-r)(R-3r)}{r^2} \cos A \cos B \cos C$.

We collect a number of useful formulas for cyclic sums of trigonometrical expressions involving the angles of a triangle.

Lemma 2.
(a) $\cos A + \cos B + \cos C = \frac{R+r}{R}$.
(b) $\sum_{\text{cyclic}} \cos B \cos C = \frac{(2R+r)r}{2R^2} + \cos A \cos B \cos C$.
(c) $\sum_{\text{cyclic}} \sin A \cos A = \frac{rs}{R^2}$.
(d) $\sum_{\text{cyclic}}(\cos B + \cos C) \sin A = \sin A + \sin B + \sin C = \frac{s}{R}$.
(e) $\sin A \sin B \sin C = \sum_{\text{cyclic}}(\cos B \cos C) \sin A = \frac{rs}{2R^2}$.

2. Homogeneous barycentric coordinates of some centers

In the *Encyclopedia of triangles centers* [1], henceforth referred to as ETC, Kimberling publishes a list of more than 5600 triangle centers with *homogeneous* trilinear and barycentric coordinates. In this paper we consider barycentric coordinates exclusively. An introduction to barycentric coordinates can be found in [3]. Sometimes it is useful to work with *absolute* barycentric coordinates. For a finite point, the absolute barycentric coordinates can be found from a set of homogeneous barycentric coordinates by dividing by its coordinate sum. If the triangle center $X(n)$ in ETC is a finite point, we denote by $(\alpha_n,\ \beta_n,\ \gamma_n)$ its absolute barycentric coordinates.

n	$X(n)$	α_n
1	I	$\frac{R\sin A}{rs} \cdot r$
3	O	$\frac{R\sin A}{rs}(R\cos A)$
4	H	$\frac{R\sin A}{rs}(2R\cos B\cos C)$
8	N_{a}	$\frac{R\sin A}{rs}(2R(\cos A + \cos B\cos C) - 2r)$
20	L	$\frac{R\sin A}{rs} \cdot 2R(\cos A - \cos B\cos C)$
40	I'	$\frac{R\sin A}{rs}(2R\cos A - r)$

The isogonal conjugate of $I' := X(40)$ is the triangle center $X(84)$.

Proposition 3. $\alpha_{84} = \frac{R\sin A}{rs} \cdot \frac{(2R\cos B - r)(2R\cos C - r)}{r}$.

Proof. Since, in homogeneous barycentric coordinates,

$$X(40) = (\sin A(2R\cos A - r) :\ \sin B(2R\cos B - r) :\ \sin C(2R\cos C - r)),$$

we have

$$X(84) = \left(\frac{\sin A}{2R\cos A - r} : \frac{\sin B}{2R\cos B - r} : \frac{\sin C}{2R\cos C - r} \right).$$

Therefore,

$$\begin{aligned} \alpha_{84} = & \frac{\frac{\sin A}{2R\cos A - r}}{\frac{\sin A}{2R\cos A-r} + \frac{\sin B}{2R\cos B-r} + \frac{\sin C}{2R\cos C-r}} \\ = & \frac{\sin A(2R\cos B - r)(2R\cos C - r)}{\sum_{\text{cyclic}} \sin A(2R\cos B - r)(2R\cos C - r)}. \end{aligned}$$

Using the formulas in Lemma 2, we have

$$\begin{aligned} & \sum_{\text{cyclic}} \sin A(2R\cos B - r)(2R\cos C - r) \\ = & \ 4R^2 \sum_{\text{cyclic}} \sin A \cos B \cos C - 2Rr \sum_{\text{cyclic}} \sin A(\cos B + \cos C) + r^2 \sum_{\text{cyclic}} \sin A \\ = & \ 4R^2 \cdot \frac{rs}{2R^2} - 2Rr \cdot \frac{s}{R} + r^2 \cdot \frac{s}{R} \\ = & \ \frac{r^2 s}{R}. \end{aligned}$$

From this the result follows. □

Lemma 4. *The line joining $X(40)$ and $X(84)$ contains the Nagel point $X(8)$.*

Proof. With $t = \frac{r}{2R}$, we have

$$\begin{aligned} & (1-t)\alpha_{40} + t\alpha_{84} \\ = & \ \frac{R\sin A}{rs} \left(\left(1 - \frac{r}{2R}\right)(2R\cos A - r) + \frac{r}{2R} \cdot \frac{(2R\cos B - r)(2R\cos C - r)}{r} \right) \\ = & \ \frac{R\sin A}{rs} \left(2R\cos A - r\cos A - r + \frac{r^2}{2R} + 2R\cos B\cos C - r(\cos B + \cos C) + \frac{r^2}{2R} \right) \\ = & \ \frac{R\sin A}{rs} \left(2R\cos B\cos C + 2R\cos A - r(\cos A + \cos B + \cos C) - r + \frac{r^2}{R} \right) \\ = & \ \frac{R\sin A}{rs} \left(2R\cos B\cos C + 2R\cos A - r \cdot \frac{R+r}{R} - r + \frac{r^2}{R} \right) \\ = & \ \frac{R\sin A}{rs} (2R\cos B\cos C + 2R\cos A - 2r) \\ = & \ \alpha_8. \end{aligned}$$

□

Proposition 5. *The circumcenter Ω of the extouch triangle lies on the line joining $X(40)$ and $X(8)$. It has first absolute barycentric coordinate*

$$\alpha = \frac{R\sin A}{rs} \left(\frac{2R^2}{r} \cos B\cos C + 2R\cos A - (R+r) \right).$$

Proof. Since Ω is the midpoint of $X(40)$ and $X(84)$, it follows from Lemma 4 that it lies on the line $X(40)X(8)$. Furthermore,

$$
\begin{aligned}
\alpha &= \frac{1}{2}(\alpha_{40}+\alpha_{84})\\
&= \frac{R\sin A}{rs}\left(\frac{2R\cos A - r}{2} + \frac{(2R\cos B - r)(2R\cos C - r)}{2r}\right)\\
&= \frac{R\sin A}{rs}\cdot\frac{(2R\cos A - r)r + (2R\cos B - r)(2R\cos C - r)}{2r}\\
&= \frac{R\sin A}{rs}\cdot\frac{4R^2\cos B\cos C + 4Rr\cos A - 2Rr(\cos A+\cos B+\cos C)}{2r}\\
&= \frac{R\sin A}{rs}\cdot\frac{4R^2\cos B\cos C + 4Rr\cos A - 2(R+r)r}{2r}\\
&= \frac{R\sin A}{rs}\left(\frac{2R^2}{r}\cos B\cos C + 2R\cos A - (R+r)\right).
\end{aligned}
$$

□

Remark. In ETC, Ω is the triangle center $X(1158)$.

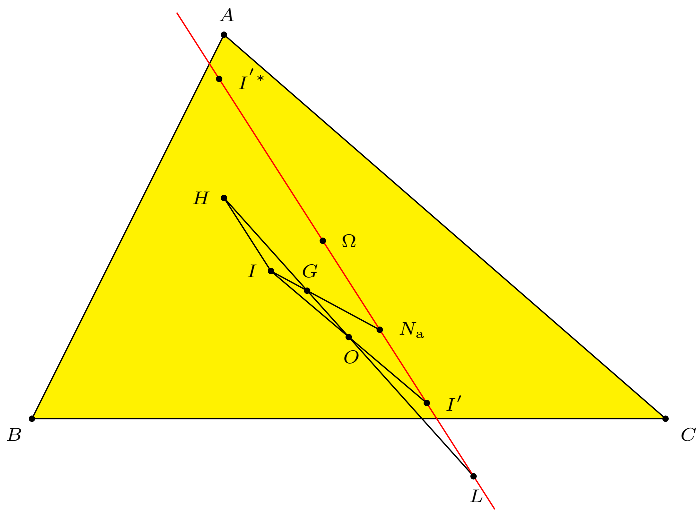

Figure 2.

Figure 2 shows Ω on the line joining I' to N_a. Since the deLongchamps point $L = X(20)$ is the reflection of H in O, O is the common midpoint of II' and HL. From this, IH is parallel to $I'L$. Also, the centroid G divides both segments IN_a and HL in the ratio $1:2$, HI is also parallel to N_a. It follows that L lies on the line $I'N_a$ and I' is the midpoint of LN_a.

Lemma 6. $\Omega I' = \frac{R}{r}\cdot HI$.

Proof. By Proposition 5,

$$\begin{aligned}
\alpha - \alpha_{40} &= \frac{R\sin A}{rs}\left(\frac{2R^2}{r}\cos B\cos C + 2R\cos A - (R+r) - (2R\cos A - r)\right) \\
&= \frac{R\sin A}{rs}\left(\frac{2R^2}{r}\cos B\cos C - R\right) \\
&= \frac{R}{r}\cdot\frac{R\sin A}{rs}\left(2R\cos B\cos C - r\right) \\
&= \frac{R}{r}(\alpha_4 - \alpha_1).
\end{aligned}$$

□

3. Proof of Theorem 1

Lemma 7. (a) $2\Omega O^2 - \Omega I^2 = 4Rr - \frac{4R^4}{r^2}\cos A\cos B\cos C$.
(b) $2\Omega O^2 - \Omega H^2 = 4Rr + 2r^2 - \frac{4R^2}{r^2}\left(R^2 + 2Rr - 3r^2\right)\cos A\cos B\cos C$.

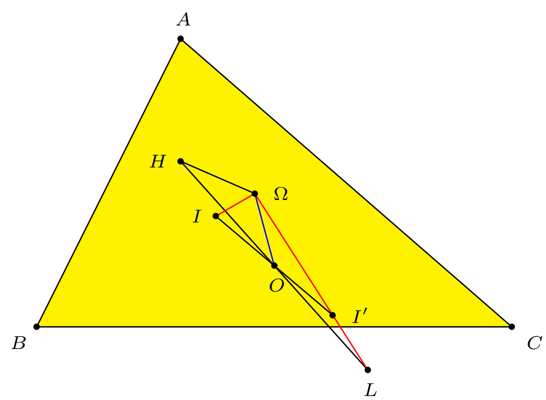

Figure 3.

Proof. (a) Applying Apollonius to the median ΩO of triangle $\Omega II'$, we have

$$\Omega I^2 + \Omega I'^2 = 2(\Omega O^2 + OI^2).$$

From this,

$$\begin{aligned}
2\Omega O^2 - \Omega I^2 &= \Omega I'^2 - 2OI^2 \\
&= \frac{R^2}{r^2}HI^2 - 2OI^2 \\
&= \frac{R^2}{r^2}(2r^2 - 4R^2\cos A\cos B\cos C) - 2R(R-2r) \\
&= 4Rr - \frac{4R^4}{r^2}\cos A\cos B\cos C.
\end{aligned}$$

(b) Applying Apollonius to the median ΩO of triangle ΩHL, we have

$$\Omega H^2 + \Omega L^2 = 2(\Omega O^2 + OH^2).$$

From this,

$$\begin{aligned}2\Omega O^2 - \Omega H^2 &= \Omega L^2 - 2OH^2 \\ &= \frac{(R+r)^2}{r^2} HI^2 - 2OH^2 \\ &= \frac{(R+r)^2}{r^2}(2r^2 - 4R^2\cos A\cos B\cos C) \\ &\quad - 2R^2(1 - 8\cos A\cos B\cos C) \\ &= 4Rr + 2r^2 - \frac{4R^2}{r^2}\left(R^2 + 2Rr - 3r^2\right)\cos A\cos B\cos C.\end{aligned}$$

□

Lemma 8. $\Omega O^2 - \Omega I^2 = -R^2 + 4Rr - \frac{4R^3}{r}\cos A\cos B\cos C$.

Proof. We begin with $AI^2 = \frac{r^2}{\sin^2\frac{A}{2}} = \frac{r^2bc}{(s-b)(s-c)} = \frac{s-a}{s}bc = bc - \frac{abc}{s} = 4R^2\sin B\sin C - 4Rr$; similarly for BI^2 and CI^2. Therefore,

$$\begin{aligned}&\alpha AI^2 + \beta AB^2 + \gamma CI^2 \\ =\ & 4R^2\sin A\sin B\sin C\left(\frac{\alpha}{\sin A} + \frac{\beta}{\sin B} + \frac{\gamma}{\sin C}\right) - 4Rr(\alpha+\beta+\gamma) \\ =\ & 2rs\left(\frac{\alpha}{\sin A} + \frac{\beta}{\sin B} + \frac{\gamma}{\sin C}\right) - 4Rr \\ =\ & 2rs\cdot\frac{R}{rs}\sum_{\text{cyclic}}\left(\frac{2R^2}{r}\cos B\cos C + 2R\cos A - (R+r)\right) - 4Rr \\ =\ & 2R\sum_{\text{cyclic}}\left(\frac{2R^2}{r}\cos B\cos C + 2R\cos A - (R+r)\right) - 4Rr \\ =\ & 2R\left(\frac{2R^2}{r}\left(\frac{(2R+r)r}{2R^2} + \cos A\cos B\cos C\right) + 2R\cdot\frac{R+r}{R} - 3(R+r)\right) - 4Rr \\ =\ & 2R^2 - 4Rr + \frac{4R^3}{r}\cos A\cos B\cos C.\end{aligned}$$

We make use of a formula of Scheer [2]. For an arbitrary point P,

$$\Omega P^2 = \alpha AP^2 + \beta BP^2 + \gamma CP^2 - (\beta\gamma a^2 + \gamma\alpha b^2 + \alpha\beta c^2).$$

Applying this to $P = O$ and $P = I$ respectively, we have

$$\begin{aligned}\Omega O^2 - \Omega I^2 &= (\alpha AO^2 + \beta BO^2 + \gamma CO^2) - (\alpha AI^2 + \beta BI^2 + \gamma CI^2) \\ &= R^2 - (\alpha AI^2 + \beta BI^2 + \gamma CI^2) \\ &= R^2 - (2R^2 - 4Rr + \frac{4R^3}{r}\cos A\cos B\cos C) \\ &= -R^2 + 4Rr - \frac{4R^3}{r}\cos A\cos B\cos C.\end{aligned}$$

□

Now we complete the proof of Theorem 1.
(a) For the distance from Ω to the circumcenter:

$$\begin{aligned}\Omega O^2 =& \ (2\Omega O^2 - \Omega I^2) - (\Omega O^2 - \Omega I^2)\\ =& \left(4Rr - \frac{4R^4}{r^2}\cos A\cos B\cos C\right)\\ &- \left(-R^2 + 4Rr - \frac{4R^3}{r}\cos A\cos B\cos C\right)\\ =& \ R^2 - \frac{4R^3}{r^2}(R-r)\cos A\cos B\cos C.\end{aligned}$$

(b) For the distance from Ω to the incenter:

$$\begin{aligned}\Omega I^2 =& \ \Omega O^2 - (\Omega O^2 - \Omega I^2)\\ =& \left(R^2 - \frac{4R^3}{r^2}(R-r)\cos A\cos B\cos C\right)\\ &- \left(-R^2 + 4Rr - \frac{4R^3}{r}\cos A\cos B\cos C\right)\\ =& \ 2R^2 - 4Rr - \frac{4R^3}{r^2}(R-2r)\cos A\cos B\cos C.\end{aligned}$$

(c) For the distance from Ω to the orthocenter:

$$\begin{aligned}\Omega H^2 =& \ 2\Omega O^2 - (2\Omega O^2 - \Omega H^2)\\ =& \ 2\left(R^2 - \frac{4R^3}{r^2}(R-r)\cos A\cos B\cos C\right)\\ &- \left(4Rr + 2r^2 - \frac{4R^2}{r^2}\left(R^2 + 2Rr - 3r^2\right)\cos A\cos B\cos C\right)\\ =& \ 2R^2 - 4Rr - 2r^2 - \frac{4R^2}{r^2}\left(R^2 - 4Rr + 3r^2\right)\cos A\cos B\cos C.\end{aligned}$$

The proof of Theorem 1 is now complete.

Since the centroid G and the nine-point center N divide the segment OH in the ratio $OG : ON : OH = 2 : 3 : 6$, further applications of the Apollonius theorem yield the distances from Ω to G and N.

Corollary 9.
(a) $\Omega G^2 = \frac{2}{9}(5R^2 - 6Rr - 3r^2) - \frac{4R^2}{9r^2}\left(9R^2 - 18Rr + 5r^2\right)\cos A\cos B\cos C$.
(b) $\Omega N^2 = \frac{5}{4}R^2 - 2Rr - r^2 - \frac{2R^2}{r^2}\left(2R^2 - 5Rr + 2r^2\right)\cos A\cos B\cos C$.

Remarks. (1) Since $4R^2\cos A\cos B\cos C = s^2 - (r+2R)^2$, these distances can all be expressed in terms of R, r, s.

(2) We also note the two simple relations:
(i) $\Omega O^2 + OI^2 = \frac{R(R-r)}{r^2}HI^2$;
(ii) $\Omega I = \frac{OI \times HI}{r}$.

4. The cyclcocevian conjugate of the Nagel point

Since the circumcircle of the extouch triangle is the pedal circle of $I' = X(40)$, it is also the pedal circle of $I'^* = X(84)$. The pedals of $X(84)$ are the vertices of the cyclocevian conjugate of the Nagel point $N_a = X(8)$. It is interesting to note that this is also a point on the line $I'N_a$. In ETC, this is $X(189)$ with homogeneous barycentric coordinates are

$$\left(\frac{1}{\cos B + \cos C - \cos A - 1} : \frac{1}{\cos C + \cos A - \cos B - 1} : \frac{1}{\cos A + \cos B - \cos C - 1}\right).$$

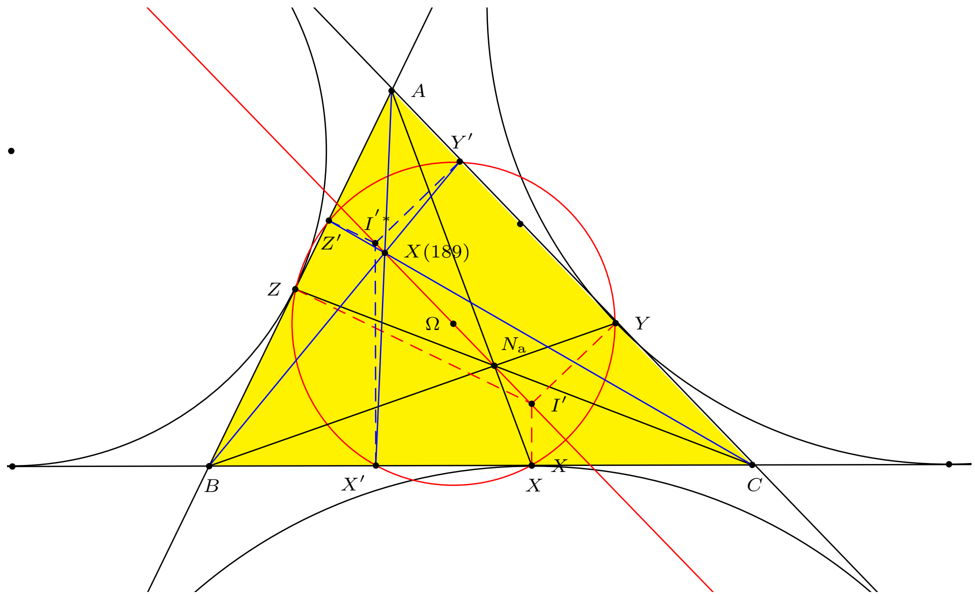

Figure 4

Proposition 10. *The first absolute barycentric coordinate of the cyclocevian conjugate of the Nagel point is*

$$\alpha_{189} = \frac{R \sin A}{rs}\left((2R\cos A - r) + k(2R\cos B\cos C - r)\right),$$

where

$$k = \frac{(4R + r)r + 4R^2 \cos A \cos B \cos C}{r^2 + 4R^2 \cos A \cos B \cos C}.$$

Proof. The point $X(189)$ divides $I'N_a$ in the ratio

$$N_a X(189) : X(189) I' = t : 1 - t$$

for $t = \frac{-4Rr}{r^2+4R^2\cos A\cos B\cos C}$. From this,

$$\begin{aligned}\alpha_{189} =&\; t\alpha_{40} + (1-t)\alpha_8\\ =&\; \frac{R\sin A}{rs}\left(t(2R\cos A - r) + (1-t)(2R\cos B\cos C + 2R\cos A - 2r)\right)\\ =&\; \frac{R\sin A}{rs}\left((2R\cos A - r) + (1-t)(2R\cos B\cos C - r)\right).\end{aligned}$$

The coefficient $1-t$ is k given in the statement of the proposition. □

5. The centroid and orthocenter of the extouch triangle

The centroid of the extouch triangle is very easy to determine: It is the triangle center

$$\begin{aligned}\frac{X+Y+Z}{3} =&\; \frac{1}{3}\left(\frac{(0,c+a-b,a+b-c)}{2a} + \frac{(b+c-a,0,a+b-c)}{2b}\right.\\ &\left. + \frac{(b+c-a,c+a-b,0)}{2c}\right)\\ =&\; \frac{(a(b+c)(b+c-a), b(c+a)(c+a-b), c(a+b)(a+b-c))}{6abc}.\end{aligned}$$

This is the triangle center $X(210)$ in ETC. Clearly,

$$\alpha_{210} = \frac{a(b+c)(b+c-a)}{6abc}.$$

By expressing this in the form

$$\alpha_{210} = \frac{R\sin A}{rs}\left(\frac{R}{3}\cdot(\sin B+\sin C)(\sin B+\sin C-\sin A)\right), \qquad (1)$$

we easy determine also the orthocenter of the extouch triangle:

Proposition 11. *The orthocenter of the extouch triangle has first absolute barycentric coordinate*

$$\begin{aligned}\alpha' =&\; \frac{R\sin A}{rs}\Bigg(R((\sin B+\sin C)(\sin B+\sin C-\sin A) - 4\cos A)\\ &- \frac{4R^2}{r}\cos B\cos C + 2(R+r)\Bigg).\end{aligned}$$

Proof. Since the orthocenter divides the centroid $X(210)$ and the circumcenter Ω in the ratio $3:-2$, we have the first absolute barycentric coordinate equal to $\alpha' = 3\alpha_{210} - 2\alpha$. The result follows from Proposition 5 and (1) above. □

Remark. In terms of a, b, c, the orthocenter of the extouch triangle has homogeneous barycentric coordinates

$$(af(a,b,c)g(a,b,c) : bf(b,c,a)g(b,c,a) : cf(c,a,b)g(c,a,b))$$

where

$$\begin{aligned} f(a,b,c) =&\ a^3(b+c) - a^2(b-c)^2 - a(b+c)(b-c)^2 + (b^2-c^2)^2, \\ g(a,b,c) =&\ a^5 - a^4(b+c) - 2a^3(b-c)^2 + 2a^2(b+c)(b^2+c^2) \\ &+ a(b^4 - 4b^3c - 2b^2c^2 - 4bc^3 + c^4) - (b-c)^2(b+c)^3. \end{aligned}$$

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6. The incenter of the extouch triangle

Lemma 12. *Let a', b', c' be the sidelengths of the extouch triangle XYZ.*

$$a'^2 = a^2(1-\sin B\sin C), \quad b'^2 = b^2(1-\sin C\sin A), \quad c'^2 = c^2(1-\sin A\sin B).$$

Proof. It is enough to establish the expression for a'^2. Since $AY = s-c$ and $AZ = s-b$, applying the law of cosines to triangle AYZ, we have

$$\begin{aligned} a'^2 =&\ (s-b)^2 + (s-c)^2 - 2(s-b)(s-c)\cos A \\ =&\ (s-b)^2 + (s-c)^2 - 2(s-b)(s-c)\left(2\cos^2\frac{A}{2} - 1\right) \\ =&\ (s-b)^2 + (s-c)^2 + 2(s-b)(s-c) - 4(s-b)(s-c)\cdot\frac{s(s-a)}{bc} \\ =&\ ((s-b)+(s-c))^2 - \frac{4\Delta^2}{bc} \\ =&\ a^2 - \frac{4\Delta^2}{bc} \\ =&\ a^2(1-\sin B\sin C) \end{aligned}$$

since $\Delta = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$. □

Proposition 13. *The incenter of the extouch triangle has homogeneous barycentric coordinates*

$$(\sin B + \sin C - \sin A)(\sqrt{1-\sin C\sin A} + \sqrt{1-\sin A\sin B}) : \cdots : \cdots).$$

Proof. With reference to triangle XYZ, this incenter has homogeneous barycentric coordinates $(a' : b' : c')$. The absolute barycentric with reference to ABC is therefore

$$\frac{a'X + b'Y + c'Z}{a'+b'+c'}.$$

In homogeneous coordinates, this can be taken as

$$\begin{aligned} &a'X + b'Y + c'Z \\ =&\ \frac{a'(0, c+a-b, a+b-c)}{2a} + \frac{b'(b+c-a, 0, a+b-c)}{2b} + \frac{c'(b+c-a, c+a-b, 0)}{2c} \\ =&\ \frac{1}{2}\left((b+c-a)\left(\frac{b'}{b}+\frac{c'}{c}\right), (c+a-b)\left(\frac{c'}{c}+\frac{a'}{a}\right), (a+b-c)\left(\frac{a'}{a}+\frac{b'}{b}\right)\right). \end{aligned}$$

The result follows from the law of sines and an application of Lemma 12. □

We conclude by giving the coordinates of the incenter of the extouch triangle in terms of a, b, c. Using the Heron formula

$$\Delta^2 = \frac{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}{16},$$

we have

$$\begin{aligned} a'^2 &= \frac{4a^2bc - 16\Delta^2}{4bc} \\ &= \frac{4a^2bc - 2b^2c^2 - 2c^2a^2 - 2a^2b^2 + a^4 + b^4 + c^4}{4bc} \\ &= \frac{a^4 - 2a^2(b-c)^2 + (b^2-c^2)^2}{4bc}. \end{aligned}$$

Therefore,

$$\frac{a'}{a} = \frac{\sqrt{bc(a^4 - 2a^2(b-c)^2 + (b^2-c^2)^2)}}{2abc};$$

similarly for $\frac{b'}{b}$ and $\frac{c'}{c}$. This leads to

$$\Big((b+c-a)\Big(\sqrt{ca(b^4 - 2b^2(c-a)^2 + (c^2-a^2)^2)} + \sqrt{ab(c^4 - 2c^2(a-b)^2 + (a^2-b^2)^2)}\Big) \\ : \cdots : \cdots\Big).$$

This is a triangle center not in the current edition of ETC. It has $(6-9-13)$-search number $4.66290502201\cdots$.

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