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Semi-Similar Complete Quadrangles

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Abstract. Let $\mathcal{A} = A_1A_2A_3A_4$ and $\mathcal{B} = B_1B_2B_3B_4$ be complete quadrangles and assume that each side A_iA_j is parallel to B_hB_k $(i, j, h, k$ is a permutation of $1, 2, 3, 4$). Then $\mathcal{A}$ and $\mathcal{B}$, in general, are not homothetic; they are linked by another strong geometric relation, which we study in this paper. Our main result states that, modulo similarities, the mapping $A_i \to B_i$ is induced by an involutory affinity (an oblique reflection). $\mathcal{A}$ and $\mathcal{B}$ may have quite different aspects, but they share a great number of geometric features and turn out to be similar when $\mathcal{A}$ belongs to the most popular families of quadrangles: cyclic and trapezoids.

1. Introduction

Two triangles whose sides are parallel in pairs are homothetic. A similar statement, trivially, does not apply to quadrilaterals. How about two complete quadrangles with six pairs of parallel sides? An answer cannot be given unless the question is better posed: let $\mathcal{A} = A_1A_2A_3A_4$, $\mathcal{B} = B_1B_2B_3B_4$ be complete quadrangles and assume that each side A_iA_j is parallel to B_iB_j; then $\mathcal{A}$ and $\mathcal{B}$ are indeed homothetic. In fact, the two triangle homotheties, say $A_1A_2A_3 \to B_1B_2B_3$, and $A_4A_2A_3 \to B_4B_2B_3$, must be the same mappings, as they have the same effect on two points. [1] There is, however, another interesting way to relate the six directions. Assume A_iA_j is parallel to B_hB_k (i, j, h, k will always denote a permutation of $1, 2, 3, 4$). Then $\mathcal{A}$ and $\mathcal{B}$ in general are not homothetic. Given any $\mathcal{A}$, here is an elementary construction (Figure 1) producing such a $\mathcal{B}$. Let B_1B_2 be any segment parallel to A_3A_4. Let B_3 be the intersection of the line through B_1 parallel to A_2A_4 with the line through B_2 parallel to A_1A_4; likewise, let B_4 be the intersection of the line through B_1 parallel to A_2A_3 with the line through B_2 parallel to A_1A_3. We claim that the sixth side B_3B_4 is also parallel to A_1A_2. Consider, in fact, the intersections R of A_1A_3 with A_2A_4, and S of B_1B_3 with B_2B_4. The following pairs of triangles have parallel sides: A_1RA_4 and B_2SB_3, A_4RA_3 and B_1SB_2, A_3RA_2 and B_4SB_1. Hence, they are homothetic in pairs. Since a translation $R \to S$ does not affect our statement, we can assume, for simplicity, $R = S$. Now two pairs of sides are on the the same lines: A_1A_3, B_2B_4 and A_2A_4, B_1B_3. Consider the action of the three homotheties: λ: $A_1 \to B_2$, $A_4 \to B_3$, μ: $A_3 \to B_2$, $A_4 \to B_1$,

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[1] In fact, five pairs of parallel sides A_iA_j, B_iB_j suffice to make the sixth pair parallel.

ν: $A_2 \to B_1$, $A_3 \to B_4$. Here $\nu\mu^{-1}\lambda$: $A_1 \to B_4$ and $\lambda\mu^{-1}\nu$: $A_2 \to B_3$. But these products are the same mapping, as all factors have the same fixed point R. Thus the triangles A_1RA_2 , B_4RB_3 are homothetic and B_3B_4, A_1A_2 are parallel, as we wanted.

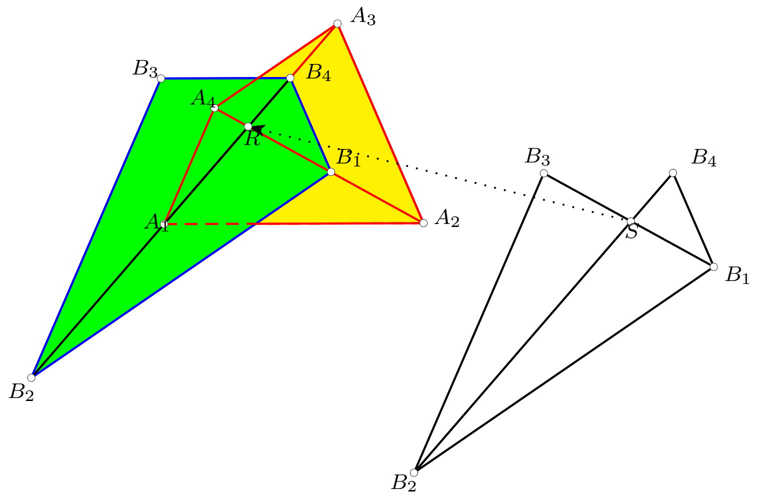

Figure 1. Five pairs of parallel sides imply sixth pair parallel

The purpose of this paper is to examine some geometric relations and invariances connecting $\mathcal{A}$ and $\mathcal{B}$; in particular, $\mathcal{A}$ and $\mathcal{B}$ will have the same pair of *asymptotic directions*, a sort of *central* points at infinity. Some statements, as the somehow unexpected Theorem 1, will be proved by synthetic arguments only; other statements, which involve circumscribed conics, will be proved analytically. Our main result (Theorem 6) states that, modulo similarities, such a mapping $A_i \to B_i$ is induced by an oblique reflection. This involutory affinity depends on $\mathcal{A}$ and can be constructed from $\mathcal{A}$ by ruler and compass. We shall also see that, when $\mathcal{A}$ belongs to the most popular families of quadrangles, namely cyclic quadrangles and trapezoids (including parallelograms), this mapping leaves the shape of the quadrangle unchanged. This may be a reason why this subject, to our knowledge, has not raised previous attention.

2. Notation and terminology

If A, B are points, AB will denote, according to different contexts, the segment or the line through A, B. $|AB|$ is a length. $A^B = C$ means that a half-turn about B maps A onto C; equivalently, we write $B = \frac{1}{2}(A + C)$.

If r, s are lines, $\angle r, s$ denotes the *directed* angle from r to s, to be measured mod π. $\angle ABC$ means $\angle AB, BC$. We shall use the basic properties of directed angles, as described in [3, pp.11–15], for example, $\angle ABC = 0$ is equivalent to A, B, C being collinear, $\angle ABC = \angle ADC$ to A, B, C, D being on a circle. $A - B$ is a vector; $(A-B)\cdot(C-D)$ is a scalar product; $(A-B)\wedge(C-D)$ is a vector product. If two vectors are parallel, we write $\dfrac{A-B}{C-D}=r$ to mean $A-B=r(C-D)$.

In a complete quadrangle $\mathcal{A}=A_1A_2A_3A_4$ the order of the vertices is irrelevant. We shall always assume that three of them are not collinear, so that all the angles $\angle A_iA_jA_h$ are defined and do not vanish. A_iA_j and A_hA_k are a pair of *opposite sides* of $\mathcal{A}$; they meet at the *diagonal point* $A_{ij,hk}$. The diagonal points are vertices of the *diagonal triangle* of $\mathcal{A}$. A quadrangle is a *trapezoid* if there is a pair of parallel opposite sides; then a diagonal point is at infinity. *Parallelograms* have two diagonal points at infinity. $A_{ij}=\frac{1}{2}(A_i+A_j)$ is the *midpoint* of the side A_iA_j. A *bimedian* of $\mathcal{A}$ is the line $A_{ij}A_{hk}$ (or the segment) through the midpoints of a pair of opposite sides. A *complementary triangle* $A_jA_hA_k$ is obtained from $\mathcal{A}$ by ignoring the vertex A_i. A *complementary quadrilateral* is obtained from $\mathcal{A}$ by ignoring a pair of opposite sides. An area is not defined for a complete quadrangle, but its complementary triangles and quadrilaterals do have *oriented areas*, which are mutually related (see §4).

3. Semi-similar and semi-homothetic complete quadrangles

Definition. Two complete quadrangles $\mathcal{A}=A_1A_2A_3A_4$ and $\mathcal{B}=B_1B_2B_3B_4$ will be called *directly* (*inversely*) *semi-similar* if there is a mapping $A_i\to B_i$ such that $\angle A_iA_jA_h=\angle B_hB_kB_i$ ($\angle A_iA_jA_h=\angle B_iB_kB_h$, respectively).

In particular, $\mathcal{A}$ and $\mathcal{B}$ will be called *semi-homothetic* if each side A_iA_j is parallel to B_hB_k.

The property of being semi-similar is obviously symmetric but not reflexive (only special classes of quadrangles will be self-semi-similar). Let $\mathcal{A}$ and $\mathcal{B}$ be semi-similar. If $\mathcal{B}$ is similar to $\mathcal{C}$, then $\mathcal{A}$ is semi-similar to $\mathcal{C}$. On the other hand, if $\mathcal{B}$ is semi-similar to $\mathcal{D}$, then $\mathcal{A}$ and $\mathcal{D}$ are similar. This explains the word *semi* (or *half*). *Direct* and *inverse* also follow the usual product rules. Although the relation of semi-similarity is not explicitly defined in the literature, semi-similar quadrangles do appear in classical textbooks; for example, some statements in [3, §399] regard the following case. Given a complete quadrangle $\mathcal{A}=A_1A_2A_3A_4$, let O_i denote the circum-center of the complementary triangle $A_jA_hA_k$. Then O_iO_j is orthogonal to A_hA_k, and this clearly implies that $\mathcal{A}$ is directly semi-similar to the quadrangle $o(\mathcal{A})=O_1O_2O_3O_4$. [2] It can also be proved that $\mathcal{A}$ is inversely semi-similar to $n(\mathcal{A})=N_1N_2N_3N_4$, where N_i denotes the nine-point center of the triangle $A_jA_hA_k$.

Semi-homotheties are direct semi-similarities. The construction we gave in §1 confirms that, modulo homotheties, there is a unique $\mathcal{B}$ which is semi-homothetic to a given quadrangle $\mathcal{A}$.

A *cyclic* quadrangle is semi-similar to itself. More precisely, the mapping $A_i\to A_i$ defines an inversely semi-similar quadrangle if and only if all the angle equalities $\angle A_iA_jA_h=\angle A_iA_kA_h$ hold, and this is equivalent for the four points A_i to lie on a circle (see §9). We shall see, however, that, if different mappings $A_i\to A_j$

[2] We shall reconsider the mapping $A_i\to O_i$ in the footnote at the end of §10.

are considered, there exist other families of quadrangles for which semi-similarity implies similarity (see §§10 and 12).

Theorem 1. *If $\mathcal{A}$ and $\mathcal{B}$ are semi-homothetic quadrangles, then the bimedians of $\mathcal{A}$ are parallel to the sides of the diagonal triangle of $\mathcal{B}$.*

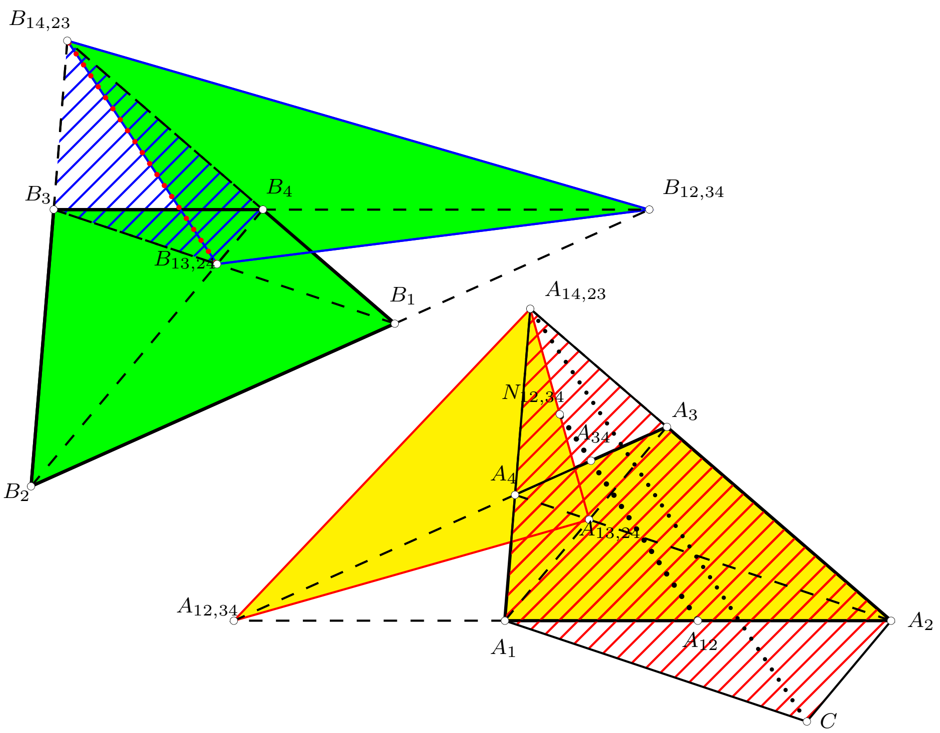

Figure 2. Semi-homothetic quadrangles: the bimedians of $\mathcal{A}$ are parallel to the sides of the diagonal triangle of $\mathcal{B}$

Proof. It is well-known ([3, §91] that a bimedian of $\mathcal{A}$, say $A_{12}A_{34}$, meets a side of the diagonal triangle in its midpoint $N_{12,34} = \frac{1}{2}(A_{13,24} + A_{14,23})$. Let C denote the intersection of the line through A_1 parallel to A_2A_4 with the line through A_2 parallel to A_1A_3. Then $A_{12} = \frac{1}{2}(C + A_{13,24})$, hence $CA_{14,23}$ and $A_{12}N_{12,34}$ are parallel. Now let $\mathcal{B} = B_1B_2B_3B_4$ be semi-homothetic to $\mathcal{A}$ so that each A_iA_j is parallel to B_hB_k. Then in the quadrangles $A_1CA_2A_{14,23}$ and $B_3B_{13,24}B_4B_{14,23}$ (see the striped areas in Figure 2), five pairs of sides are parallel, either by assumption or by construction. Hence the same holds for the sixth pair $A_{14,23}C$, $B_{14,23}B_{13,24}$. But $A_{14,23}C$ is trivially parallel to $A_{34}A_{12}$. Therefore, the side $B_{14,23}B_{13,24}$ of the diagonal triangle of $\mathcal{B}$ is parallel to the bimedian $A_{12}A_{34}$ of $\mathcal{A}$, as we wanted. □

Notice that, by symmetry, the sides of the diagonal triangle of $\mathcal{A}$ are parallel to the bimedians of $\mathcal{B}$.

4. Semi-isometric quadrangles

We shall now introduce a relationship between semi-similar quadrangles which replaces isometry. This notion is based on the following

Theorem 2. *Let $\mathcal{A} = A_1A_2A_3A_4$ and $\mathcal{B} = B_1B_2B_3B_4$ be semi-homothetic quadrangles. Then the product $\mu = \dfrac{B_i - B_j}{A_h - A_k} \cdot \dfrac{B_h - B_k}{A_i - A_j}$ is invariant under all permutations of indices.*

Proof. Notice that the factors in the definition of μ are ratios of parallel vectors, hence scalars with their own sign. Now consider, for example, the following triangles: $A_1A_4A_{12,34}$, $A_2A_3A_{12,34}$, $A_1A_3A_{12,34}$, $A_2A_4A_{12,34}$ which are homothetic, respectively, to the triangles $B_3B_2B_{12,34}$, $B_4B_1B_{12,34}$, $B_4B_2B_{12,34}$, $B_3B_1B_{12,34}$. Each of these homotheties implies an equal ratio of parallel vectors:

$$\begin{aligned}
\frac{B_3 - B_2}{A_1 - A_4} &= \frac{B_2 - B_{12,34}}{A_4 - A_{12,34}}, \\
\frac{B_4 - B_1}{A_2 - A_3} &= \frac{B_1 - B_{12,34}}{A_3 - A_{12,34}}, \\
\frac{B_4 - B_2}{A_1 - A_3} &= \frac{B_2 - B_{12,34}}{A_3 - A_{12,34}}, \\
\frac{B_3 - B_1}{A_2 - A_4} &= \frac{B_1 - B_{12,34}}{A_4 - A_{12,34}}.
\end{aligned}$$

By multiplication one finds

$$\begin{aligned}
\mu = {} & \frac{B_3 - B_2}{A_1 - A_4} \cdot \frac{B_4 - B_1}{A_2 - A_3} \\
= {} & \frac{B_2 - B_{12,34}}{A_4 - A_{12,34}} \cdot \frac{B_1 - B_{12,34}}{A_3 - A_{12,34}} \\
= {} & \frac{B_1 - B_{12,34}}{A_4 - A_{12,34}} \cdot \frac{B_2 - B_{12,34}}{A_3 - A_{12,34}} \\
= {} & \frac{B_3 - B_1}{A_2 - A_4} \cdot \frac{B_4 - B_2}{A_1 - A_3}.
\end{aligned}$$

Likewise, $\mu = \dfrac{B_3 - B_4}{A_1 - A_2} \cdot \dfrac{B_1 - B_2}{A_3 - A_4}$, as we wanted. □

Thus a pair $\mathcal{A}, B$ of semi-similar quadrangles defines a scale factor

$$|\mu| = \frac{|B_1B_2||B_3B_4|}{|A_1A_2||A_3A_4|} = \frac{|B_1B_3||B_2B_4|}{|A_1A_3||A_2A_4|} = \frac{|B_1B_4||B_2B_3|}{|A_1A_4||A_2A_3|}.$$

A geometric meaning for the sign of μ will be seen later (§7).

Corollary 3. *Let $\mathcal{A} = A_1A_2A_3A_4$ and $\mathcal{B} = B_1B_2B_3B_4$ be semi-similar quadrangles. Then the products of the lengths of the pairs of opposite sides are proportional:*[3]

$$\begin{aligned}&|A_1A_2||A_3A_4| : |A_1A_3||A_2A_4| : |A_1A_4||A_2A_3|\\ &= |B_1B_2||B_3B_4| : |B_1B_3||B_2B_4| : |B_1B_4||B_2B_3|.\end{aligned}$$

A sort of isometry takes place when $|\mu|$=1:

Definition. Two quadrangles $\mathcal{A}$ and $\mathcal{B}$ will be called semi-isometric if the they are semi-similar and the lengths of two corresponding opposite sides have the same product: $|A_iA_j||A_hA_k| = |B_iB_j||B_hB_k|$.

We have just seen that if this equality holds for a pair of opposite sides of semi-similar quadrangles, then $|\mu| = 1$ and therefore the same happens to the other two pairs.

We shall now derive a number of further relations between semi-isometric quadrangles which are almost immediate consequences of the definition. Some of them are better described if referred to the three complementary quadrilaterals. The oriented areas of these quadrilaterals are given by one half of the cross products $(A_1 - A_2) \wedge (A_3 - A_4), (A_1 - A_4) \wedge (A_2 - A_3), (A_1 - A_3) \wedge (A_2 - A_4)$. Therefore the defining equalities $|A_iA_j||A_hA_k| = |B_iB_j||B_hB_k|$, if combined with the angle equalities $\angle A_iA_j, A_hA_k = -\angle B_iB_j, B_hB_k$, imply that the three areas only change sign when passing from $\mathcal{A}$ to $\mathcal{B}$. On the other hand, it is well-known that the three cross products above, if added or subtracted in the four essentially different ways, produce 4 times the oriented area of the complementary triangles $A_jA_hA_k$. For example, by applying standard properties of vector calculus, one finds

$$\begin{aligned}&(A_1 - A_2) \wedge (A_3 - A_4) + (A_1 - A_4) \wedge (A_2 - A_3) + (A_1 - A_3) \wedge (A_2 - A_4)\\ &= 2(A_1 - A_4) \wedge (A_1 - A_2),\\ &(A_1 - A_2) \wedge (A_3 - A_4) + (A_1 - A_4) \wedge (A_2 - A_3) - (A_1 - A_3) \wedge (A_2 - A_4)\\ &= 2(A_3 - A_4) \wedge (A_2 - A_3),\end{aligned}$$

etc. Therefore we have

Theorem 4. *If $\mathcal{A}$ and $\mathcal{B}$ are semi-isometric quadrangles, the four pairs of corresponding complementary triangles $A_iA_jA_h$ and $B_iB_jB_h$ have the same* (absolute) *areas.*

This insures, incidentally, that, when semi-similarity implies similarity, then semi-isometry implies isometry.

Other invariants can be written in terms of perimeters: if one first adds, then subtracts the lengths of the four contiguous sides in a complementary quadrilateral,

[3] The following example shows that the inverse statement does not hold: let A_1A_2 be the diameter of a circle; then, for any choice of a chord A_3A_4 orthogonal to A_1A_2, the three products above are proportional to $2 : 1 : 1$.

then the product is invariant; for example, the product

$$(|A_1A_2| + |A_2A_3| + |A_3A_4| + |A_4A_1|)(|A_1A_2| - |A_2A_3| + |A_3A_4| - |A_4A_1|)$$

is the same if all A_i are changed into B_i. In particular,

$$|A_1A_2| - |A_2A_3| + |A_3A_4| - |A_4A_1| = 0$$

if and only if

$$|B_1B_2| - |B_2B_3| + |B_3B_4| - |B_4B_1| = 0.$$

Since these equalities are well-known to be equivalent to the fact that two pairs of opposite sides are tangent to a same circle, we conclude that semi-similarity of complete quadrangles preserves *inscribability* for a complementary quadrilateral. Another invariance takes place if we subtract the squares of the lengths of two bimedians; for example,

$$|A_{12}A_{34}|^2 - |A_{14}A_{23}|^2 = |B_{12}B_{34}|^2 - |B_{14}B_{23}|^2.$$

These and other similar equalities can be derived by applying the classical formulas for tha area of a quadrilateral (Bretschneider's formula etc., see [5]).

Something more intriguing happens if one considers the circumcircles of the complementary triangles.

Theorem 5. *If two quadrangles are semi-similar, the circumradii of corresponding complementary triangles are inversely proportional .*

Proof. Let R_i and S_i be, respectively, the circumradii of $A_jA_hA_k$ and $B_jB_hB_k$. We claim that R_1, R_2, R_3, R_4 are inversely proportional to S_1, S_2, S_3, S_4. In fact, by the law of sines, we can write, for example, the product $|A_3A_4||B_3B_4|$ in two ways:

$$(2R_1 \sin A_3A_2A_4)(2S_1 \sin B_3B_2B_4) = (2R_2 \sin A_4A_1A_3)(2S_2 \sin B_4B_1B_3).$$

Since $\angle A_3A_2A_4 = \angle B_4B_1B_3$ and $\angle A_4A_1A_3 = \angle B_3B_2B_4$, all the sines can be canceled to conclude $R_1S_1 = R_2S_2$. Thus the product R_iS_i is the same for all indices i. □

5. The principal reference of a quadrangle

The invariants we met in the last section suggest the possible presence of an affinity. In fact, our main result (Theorem 6) will state that for each quadrangle $\mathcal{A}$ there exists an involutory affinity (depending on $\mathcal{A}$) that maps $\mathcal{A}$ into a semi-homothetic, semi-isometric quadrangle $\mathcal{B}$. It will then appear that any semi-similarity of quadrangles $\mathcal{A} \to \mathcal{C}$ is a product of an *oblique reflection* $\mathcal{A} \to \mathcal{B}$ by a similarity $\mathcal{B} \to \mathcal{C}$. Since an oblique reflection is determined, modulo translations, by a pair of directions, in order to identify which reflection properly works for $\mathcal{A}$, we shall describe in the next section how to associate to a quadrangle $\mathcal{A}$ a pair of characteristic directions, that we shall call the *asymptotic directions* of $\mathcal{A}$. As we shall see, these directions also play a basic role in connection with some conics that are canonically defined by four points.[4]

It is well-known that a quadrangle $\mathcal{A} = A_1A_2A_3A_4$ has a unique circumscribed rectangular hyperbola $\Psi = \Psi_A$ [5]. The center of Ψ is a central (synonym: *notable*) point of $\mathcal{A}$ that we denote by $H = H_A$. Several properties and various geometric constructions of H from the points A_i are described in [4] (but this point is defined also in [3, §396-8], and [2, Problem 46]). For example, H is the intersection of the nine-point circles of the four complementary triangles $A_jA_hA_k$. The directions of the asymptotes of $\Psi = \Psi_A$ will be called the *principal directions* of $\mathcal{A}$. The hyperbola Ψ essentially defines what we call the *principal reference* of $\mathcal{A}$, namely an orthogonal Cartesian xy-frame such that the equation for Ψ is $xy = 1$, the ambiguity between x and y not creating substantial difficulties (Figures 3 and 4). Our next proofs will be based on this reference (an approach which was first used by Wood in [6]). The principal reference is not defined when two opposite sides of $\mathcal{A}$ are perpendicular. This class of quadrangles (*orthogonal* quadrangles) requires a different analytic treatment and will be discussed separately in §11.

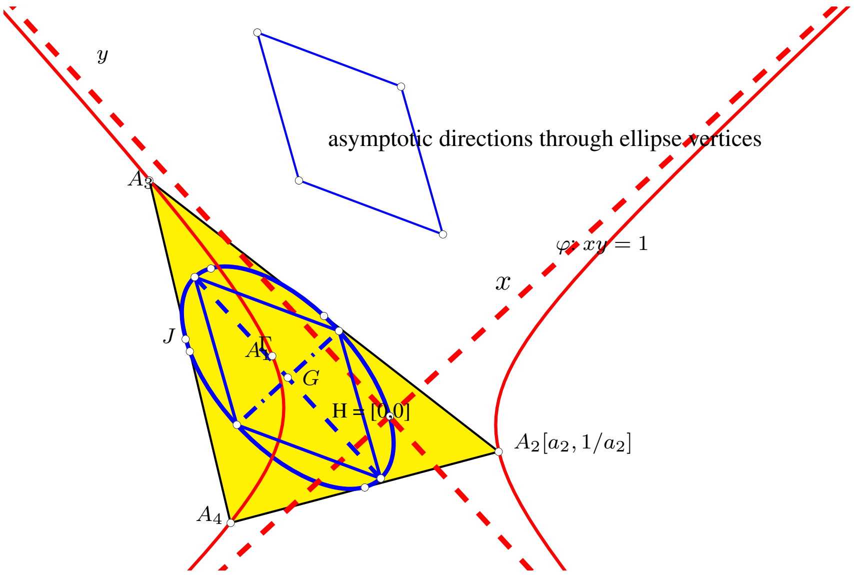

Figure 3. A concave quadrangle and its principal reference. Asymptotic directions derived from the medial ellipse Γ

Within the principal reference of $\mathcal{A}$, the origin is the central point $H_A = [0, 0]$ and the vertices will be denoted by $A_i = [a_i, \frac{1}{a_i}],\ i = 1, 2, 3, 4$. In view of the next

[4]These directions may be thought of as a pair of *central points* of $\mathcal{A}$ at infinity. In this respect, the present paper may be considered a complement of [4]. Our treatment, however, will be self-contained, not requiring the knowledge of [4].

[5]The only exceptions will be considered in §12.

calculations, it is convenient to introduce the elementary symmetric polynomials:

$$\begin{aligned} s_1 &= a_1 + a_2 + a_3 + a_4, \\ s_2 &= a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4, \\ s_3 &= a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4, \\ s_4 &= a_1a_2a_3a_4. \end{aligned}$$

Notice that the restriction for $\mathcal{A}$ not to be orthogonal not only implies $s_4 \neq 0$ but also excludes $s_4 = -1$, as the scalar product of two opposite sides turns out to be

$$A_iA_j \cdot A_hA_k = (a_i - a_j)(a_h - a_k)\left(1 + \frac{1}{s_4}\right).$$

The sign of $s_4 = a_1a_2a_3a_4$ has the following relevant geometric meaning: s_4 is positive if and only if the number of vertices A_i which lie on a branch of Ψ is even: $4, 2$ or 0. By applying standard arguments to the real convex function $f(x) = \frac{1}{x}$, this condition is found to be equivalent to $\mathcal{A}$ being *convex*. On the other hand, if the branches of Ψ contain 1 and 3 vertices, then $s_4 < 0$ and $\mathcal{A}$ is *concave*, namely, there is a vertex A_i which lies inside the complementary triangle $A_jA_hA_k$.

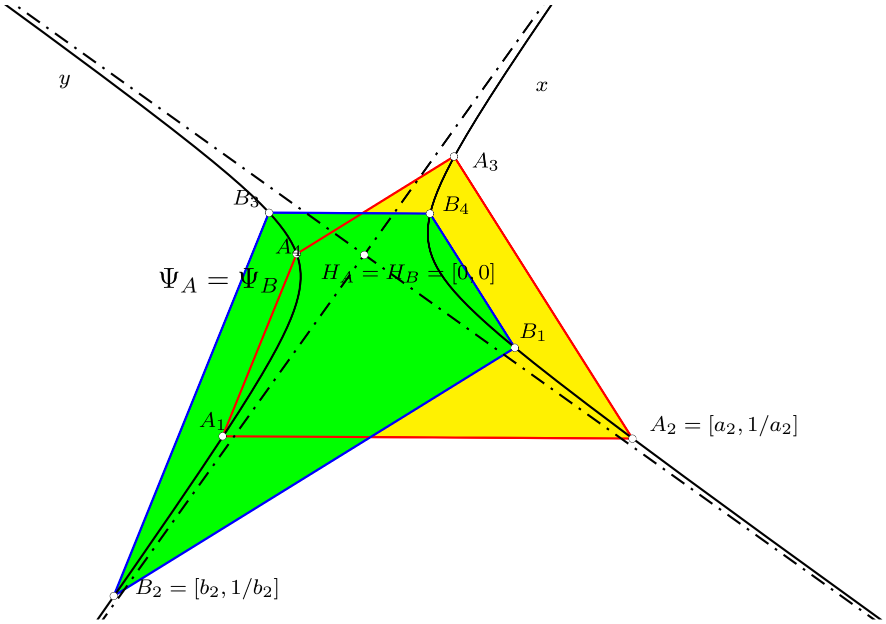

Figure 4. Semi-homothetic semi-isometric quadrangles with the same principal reference

6. Central conics and asymptotic directions

It is well-known that, within the family of the conic sections circumscribed to a given quadrangle $\mathcal{A}$, the locus of the centers is itself a conic $\Gamma = \Gamma_A$. We call it the *medial* or the nine-points conic of $\mathcal{A}$, as Γ contains the six midpoints A_{ij} and the three diagonal points $A_{ij,hk}$ ([1, §16.7.5]). The equation for Γ is calculated to be

$$x^2 - s_4 y^2 - \frac{1}{2}s_1 x + \frac{1}{2}s_3 y = 0,$$

or

$$(x - x_G)^2 - s_4(y - y_G)^2 = \frac{1}{16 s_4}(s_4 s_1^2 - s_3^2),$$

which confirms that $H = [0, 0]$, the center of Ψ, lies on Γ. On the other hand, the center of Γ is

$$G = G_A = \frac{1}{4}[s_1, \frac{s_3}{s_4}] = \frac{1}{4}(A_1 + A_2 + A_3 + A_4).$$

This is clearly the *centroid* or center of gravity, another central point of $\mathcal{A}$. Two cases must be now distinguished:.

(i) $\mathcal{A}$ is convex: $s_4 > 0$ and Γ is a hyperbola. By definition, the points at infinity of Γ will be called the *asymptotic directions* of $\mathcal{A}$; it appears from the equation that the slopes of the asymptotes are $\pm\frac{1}{\sqrt{s_4}}$ (Figure 5). This proves that the principal directions bisect the asymptotic directions. In particular, when $s_4 = 1$, Γ is a rectangular hyperbola; as we shall soon see, this happens if and only if $\mathcal{A}$ is *cyclic*.

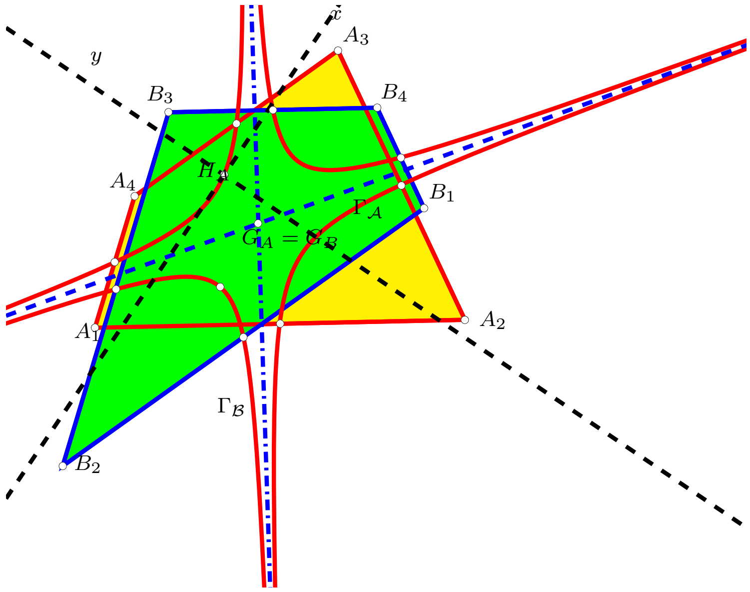

Figure 5. Semi-homothetic semi-isometric convex quadrangles with conjugate medial hyperbolas Γ_A, Γ_B. Asymptotic directions from Γ asymptotes

(ii) $\mathcal{A}$ is concave: $s_4 < 0$ and Γ is an ellipse. The asymptotic directions of $\mathcal{A}$ will be defined by connecting contiguous vertices of the ellipse Γ. Equivalently, we can inscribe Γ in a minimal rectangle and consider the directions of its diagonals. [6]

[6] They can also be defined as the directions of the only pair of *conjugate diameters* of the ellipse Γ_A which have equal lengths; see [2, Problem 54].

Their slopes turn out to be $\pm\dfrac{1}{\sqrt{-s_4}}$ (Figure 3) . Again, the principal directions bisect the asymptotic directions. Notice that Γ cannot be a parabola, as $s_4 \neq 0$. We have also seen in §5 that $s_4 \neq -1$, so that Γ cannot even be a circle. [7]

Next we want to introduce a new central point $J = J_A$, defined as the reflection of H in G: $J = H^G = \frac{1}{2}\left[s_1, \dfrac{s_3}{s_4}\right]$. Since G is the center of Γ and H lies on Γ, the point J also lies on Γ. Therefore, there exists a conic $\Theta = \Theta_A$ circumscribed to $\mathcal{A}$ and centered at J. The equation for Θ is found to be

$$x^2 + s_4 y^2 - s_1 x - s_3 y + s_2 = 0$$

or

$$(x - x_J)^2 + s_4(y - y_J)^2 = \frac{1}{4s_4}(s_4 s_1^2 + s_3^2) - s_2.$$

Looking at the roles of s_4 and $-s_4$ in the equations for Γ and Θ, it appears that Γ is an ellipse when Θ is a hyperbola and conversely. [8]

Moreover, the hyperbola asymptotes are parallel to the ellipse diagonals. We have thus produced two alternative ways for defining the asymptotic directions of any quadrangle $\mathcal{A}$:
if $\mathcal{A}$ is concave, by the asymptotes of Θ or by the vertices of Γ (Figure 3);
if $\mathcal{A}$ is convex, by the asymptotes of Γ or by the vertices of Θ (Figure 5).

A third equivalent definition only applies to the latter case: it is well-known (see, for example, [2, Problem 45]) that a convex quadrangle has two circumscribed parabolas, say Π_+ and Π_-. Their equations are [9]

$$\Pi_+ : \qquad (x + \sqrt{s_4}y)^2 - s_1 x - s_3 y + s_2 - 2\sqrt{s_4} = 0,$$

$$\Pi_- : \qquad (x - \sqrt{s_4}y)^2 - s_1 x - s_3 y + s_2 + 2\sqrt{s_4} = 0.$$

Therefore the asymptotic directions of a convex quadrangle may be also defined by the axes of symmetry of the two circumscribed parabolas.

If $s_4 = 1$ then Θ is a circle and $\mathcal{A}$ is *cyclic*. In this case the diagonals of Θ are not defined; but Γ, Π_+ , Π_- define the asymptotic directions, which are just the bisectors $y = \pm x$ of the principal directions. [10]

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[7] See §10 for exceptions.

[8] When $\mathcal{A}$ is convex, Θ_A turns out to be the ellipse that *deviates least from a circle* among all the ellipses circumscribed to $\mathcal{A}$, this meaning that the ratio between the major and the minor axis attains its minimum value. This problem was studied by J. Steiner. One can also prove that each ellipse circumscribed to $\mathcal{A}$ has a pair of conjugate diameters that have the asymptotic directions of $\mathcal{A}$; see [2, Problem 45].

[9] These equations are easily derived from the equation of the generic circumscribed conic, which can be written as $\lambda\Psi + \mu\Theta = 0$.

[10] For a different definition, not involving conics, see §9.

Notice that, whatever choice one makes among the definitions above, there exist classical methods which produce by straight-edge and compass the asymptotic directions of a quadrangle, starting from its vertices[11].

7. Oblique reflections

We are now ready to introduce oblique reflections. We recall this notion by introducing the following

Definition. Given an ordered pair (r, s) of non parallel lines, an (r, s)-reflection is the plane transformation $\phi : P \to P'$, such that $P - P'$ is parallel to r and the midpoint $\frac{1}{2}(P + P')$ lies on s.

An (r, s)-reflection is an involutory affine transformation. Among the well-known properties of affinities, we shall use the fact that they map lines into lines, midpoints into midpoints, conics into conics. Like in a standard reflection (a particular case, when r, s are orthogonal) the line s is the locus of fixed points, the other fixed lines being parallel to r. Replacing r with a parallel line r' does not affect ϕ ; replacing s with a parallel line s' only affects P' by the translation $s \to s'$ parallel to r. Interchanging r with s amounts to letting P' undergo a half turn around the intersection of r and s. An rs-reflection ϕ *preserves* many features of quadrangles; for example, if $\phi(A_i) = B_i$, then the diagonal triangle of $\mathcal{B} = B_1B_2B_3B_4$ is the ϕ-image of the diagonal triangle of $\mathcal{A} = A_1A_2A_3A_4$. Other corresponding elements are the bi-median lines, the centroid, the medial conic, the circumscribed parabolas. As for analytic representations, if, for example, $r : y = rx + p$, $s : y = sx + q$, then ϕ is the bilinear mapping

$$[x, y] \to \frac{1}{(r - s)}[(r + s)x - 2y, 2rsx - (r + s)y] + [x_0, y_0]$$

where $[x_0, y_0]$ is the image of $[0, 0]$. The transformation matrix of ϕ has determinant

$$\frac{1}{(r - s)^2}(-(r + s)^2 + 4rs) = -1.$$

Therefore all oriented areas undergo a change of sign.

Theorem 6. *Let $\mathcal{A}$ be a complete quadrangle. Let ϕ be an (r, s)-reflection, where r, s are parallel to the asymptotic directions of $\mathcal{A}$. Let χ be an* (orthogonal) *reflection in a line p parallel to a principal direction of $\mathcal{A}$. Define a mapping ψ as follows:*

$$\psi = \begin{cases} \phi, & \text{if } \mathcal{A} \text{ is convex,} \\ \phi\chi, & \text{if } \mathcal{A} \text{ is concave.} \end{cases}$$

Then $\mathcal{A}$ and $\mathcal{B} = \psi(\mathcal{A})$ are semi-homothetic, semi-isometric quadrangles.

[11] For example, a celebrated page of Newton describes how to construct the axes of a parabola if four of its points are given.

Proof. Notice that $\mathcal{B}$ is uniquely defined by $\mathcal{A}$, modulo translations and midturns: in fact, a translation parallel to r takes place when s is translated; a midturn takes place if the principal directions or the lines r and s are interchanged.

Along the proof we can assume, without loss of generality, that both s and p pass through $H_A = [0, 0]$, so that $\psi(H_A) = H_A$.

First case: $\mathcal{A}$ is convex ($s_4 > 0$). Then the lines r, s have equations, say r : $y = \dfrac{-x}{\sqrt{s_4}}$ and $s : y = \dfrac{x}{\sqrt{s_4}}$. The rs-reflection maps the point $P[x, y]$ into $\phi(P) = [y\sqrt{s}_4, \frac{x}{\sqrt{s_4}}]$, so that the vertex $A_i = [a_i, \frac{1}{a_i}]$ is mapped into $B_i = \phi(A_i) = [\frac{\sqrt{s}_4}{a_i}, \frac{a_i}{\sqrt{s_4}}]$. Substituting in $xy = 1$ proves that B_i lies on Ψ_A. Since a quadrangle has a unique circumscribed rectangular hyperbola [12], we have Ψ_B= $\Psi_{\phi(A)}$ = Ψ_A. In particular, $\mathcal{A}$ and $\mathcal{B}$ have the same principal directions and $H_B = \phi(H_A) = H_{\phi(A)} = H_A$. Now consider the sides $B_i - B_j = [\sqrt{s}_4(\frac{1}{a_i} - \frac{1}{a_j}), \frac{a_i - a_j}{\sqrt{s_4}}]$ and $A_h - A_k = [a_h - a_k, \frac{1}{a_h} - \frac{1}{a_k}]$. If we take into account that $\frac{\sqrt{s}_4}{a_i a_j} = \frac{a_h a_k}{\sqrt{s_4}}$ we find that these vectors are parallel, their ratio being $\dfrac{B_i - B_j}{A_h - A_k} = -\dfrac{\sqrt{s}_4(a_i - a_j)}{(a_h - a_k)a_i a_j} = \dfrac{A_i - A_j}{B_h - B_k}$. This proves that $\mathcal{A}$ and $\mathcal{B}$ are semi-homothetic (Figure 6). Moreover, the following scalar products turn out to be the same $(A_i - A_j).(A_h - A_k) = (a_i - a_j)(a_h - a_k)(1 + \frac{1}{s_4}) = (B_i - B_j).(B_h - B_k)$. Thus $\mathcal{A}$ and $\mathcal{B}$ are also semi-isometric : $|A_i A_j||A_h A_k| = |B_i B_j||B_h B_k|$, as we wanted. Since the matrix for $\phi = \psi$ has determinant -1, the oriented areas of the corresponding complementary triangles and quadrilaterals, as expected, undergo a sign change.

Second case: $\mathcal{A}$ is concave ($s_4 < 0$). The argument is similar: let $r : y = \dfrac{-x}{\sqrt{-s_4}}$ and $s : y = \dfrac{x}{\sqrt{-s_4}}$. Then the reflection χ, for example in the x-axes, takes $[x, y]$ into $[x, -y]$ and $\psi : [x, y] \to [y\sqrt{-s_4}, \frac{x}{\sqrt{-s_4}}]$. Thus $B_i = \psi(A_i) = [\frac{-\sqrt{-s_4}}{a_i}, \frac{a_i}{\sqrt{-s_4}}]$. If we take into account that $\frac{\sqrt{-s_4}}{a_i a_j} = \frac{-a_h a_k}{\sqrt{-s_4}}$, we find that the vectors $B_i - B_j, A_h - A_k$ are parallel. For their ratio we find the equality $\dfrac{B_i - B_j}{A_h - A_k} = \dfrac{\sqrt{-s_4}(a_i - a_j)}{(a_h - a_k)a_i a_j} = -\dfrac{A_i - A_j}{B_h - B_k}$. The matrix for ψ has now determinant 1 so that the oriented areas are conserved. □

Incidentally, the foregoing argument also shows that in the semi-homothety of Theorem 2 in §5 the sign of the scalar $\mu = \dfrac{B_i - B_j}{A_h - A_k}.\dfrac{B_h - B_k}{A_i - A_j}$ is respectively 1 or -1 for convex and concave quadrangles. This proves that semisimilarities preserve convexity.

Since the mapping of Theorem 6 is involutory and the principal references for $\mathcal{A}$ and $\mathcal{B}$ have been shown to be the same, we have substantially proved that

[12] The only exceptions will be considered in §12

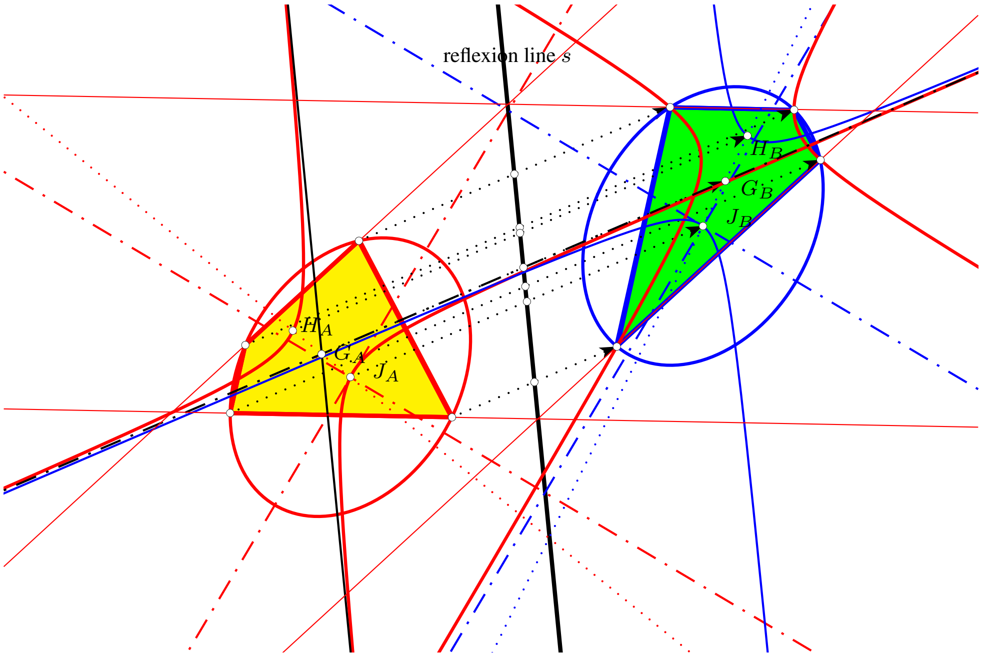

Figure 6. An oblique reflection producing semi-homothetic semi-isometric quadrangles. Pairs of corresponding sides, central lines, central conics etc meet on line s

Theorem 7. *Two semi-homothetic quadrangles have the same asymptotic directions.*

This statement will be confirmed in the next section.

8. Behaviour of central conics

We want now to examine how the central conics Ψ, Γ, Θ (see §4) of two semi-similar quadrangles are related to each other. We claim that, modulo similarities, these conics are either identical ellipses or conjugate hyperbolas.[13]

The problem can obviously be reduced to a pair of semi-homothetic, semi-isometric quadrangles.

Theorem 8. *Let $\mathcal{A}$ and $\mathcal{B}$ be semi-homothetic, semi-isometric quadrangles.*
(1) *Assume Ψ_A and Ψ_B have the same center: $H_A = H_B$. Then either $\Psi_A = \Psi_B$ ($\mathcal{A}$ convex) or Ψ_A and Ψ_B are conjugate ($\mathcal{A}$ concave).*
(2) *Assume Γ_A and Γ_B have the same center: $G_A = G_B$. Then either Γ_A=Γ_B ($\mathcal{A}$ concave) or Γ_A and Γ_B are conjugate ($\mathcal{A}$ convex). In the latter case, the two circumscribed parabolas $\Pi\pm_A$and $\Pi\pm_B$ are either equal or symmetric with respect to G.*

[13]Two hyperbolas are said to be conjugate if their equations, in a convenient orthogonal frame, can be written as $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = \pm 1$. Conjugate hyperbolas have the same asymptotes and their foci form a square.

(3) *Assume Θ_A and Θ_B have the same center: $J_A = J_B$. Then either Θ_A=Θ_B ($\mathcal{A}$ convex) or Θ_A and Θ_B are conjugate ($\mathcal{A}$ concave).*

Proof. Without loss of generality, we can assume that $\mathcal{A}$ and $\mathcal{B}$ are linked by a mapping ψ as in Theorem 6. According to the various statements (1), (2), (3), it will be convenient to choose ψ in such a way that a specific point F is fixed. We shall denote by ψ_F this particular mapping: $\psi_F(F)$ =F. For example, in the proof of Theorem 6 we had $\psi = \psi_H$. To obtain ψ_F from ψ_H one may just apply an additional translation $H \to F$.

(1) First assume that $\mathcal{A}$ is convex: $s_4 > 0$. While proving Theorem 6 we have already noticed that the point $B_i = \psi_H(A_i) = \psi_H([a_i, \frac{1}{a_i}]) = [\frac{\sqrt{s_4}}{a_i}, \frac{a_i}{\sqrt{s_4}}]$ lies on $xy = 1$, hence $H_A = H_B, \Psi_A = \Psi_B$. Now assume $\mathcal{A}$ concave: $s_4 < 0$. A $principal$ reflection, say in the x-axes, takes $[a_i, \frac{1}{a_i}]$ into $[a_i, -\frac{1}{a_i}]$. Then $B_i = \psi_H(A_i) = \psi_H([a_i, -\frac{1}{a_i}]) = [-\frac{\sqrt{-s_4}}{a_i}, \frac{a_i}{\sqrt{-s_4}}]$, clearly a point of the hyperbola $xy = -1$, the conjugate of Ψ_A, as we wanted.

(2) Since affinities preserve midpoints and conics, for any choice of ψ we have $\Gamma_B = \psi(\Gamma_A)$, and $G_B = \psi(G_A)$. The assumption $G_B = G_A$ suggests the choice $\psi = \psi_G$ in Theorem 6. Assume $\mathcal{A}$ is convex: $s_4 > 0$. Then B_i = $\psi_G(A_i)$ is obtained by applying the translation $\psi_H(G_A) \to G_A$ to the point $\psi_H([a_i, \frac{1}{a_i}]) = [\frac{\sqrt{s_4}}{a_i}, \frac{a_i}{\sqrt{s_4}}]$. Here $G_A = \frac{1}{4}[s_1, \frac{s_3}{s_4}], \psi_H(G_A) = \frac{1}{4\sqrt{s_4}}[s_3, s_1]$. Therefore

$$B_i = [\frac{\sqrt{s_4}}{a_i} + \frac{s_1}{4} - \frac{s_3}{4\sqrt{s_4}}, \frac{a_i}{\sqrt{s_4}} + \frac{s_3}{4s_4} - \frac{s_1}{4\sqrt{s_4}}].$$

Straightforward calculations prove that the midpoints $\frac{1}{2}(B_i + B_j)$ satisfy the equation

$$(x - x_G)^2 - s_4(y - y_G)^2 = -\frac{1}{16s_4}(s_3^2 + s_4 s_1^2),$$

which is the conjugate hyperbola of Γ_A, as we wanted.

As for the circumscribed parabolas, we already know, again by the general properties of affinities, that $\psi(\Pi_{+A})$ and $\psi(\Pi_{-A})$ will be parabolas circumscribed to $\mathcal{B}$. More precisely, one can verify that the above points $B_i = \psi_H(A_i)+G_A-\psi_H(G_A)$ satisfy the equation for

$$\Pi_{+A}: \qquad (x - y\sqrt{s_4})^2 - s_1x - s_3y + s_2 + 2\sqrt{s_4} = 0.$$

A midturn around G_A maps B_i into $2G_A - B_i$. The new points are $-\psi_H(A_i) + G_A + \psi_H(G_A)$ and they are checked to satisfy the equation

$$(x + y\sqrt{s_4})^2 - s_1x - s_3y + s_2 - 2\sqrt{s_4} = 0.$$

Therefore $\Pi_{-B} = \Pi_{-A}, \Pi_{+B} = (\Pi_{+A})^G$.

Now assume $\mathcal{A}$ concave: $s_4 < 0$. As before, $\psi_H(A_i) = [-\frac{\sqrt{-s_4}}{a_i}, \frac{a_i}{\sqrt{-s_4}}]$. The mapping ψ_G is again obtained by applying the translation $\psi_H(G_A) \to G_A$, but here $\psi_H(G_A) = \frac{1}{4\sqrt{-s_4}}[s_3, s_1]$. Therefore,

$$B_i = \psi_G(A_i) = [-\frac{\sqrt{-s_4}}{a_i} + \frac{s_1}{4} - \frac{s_3}{4\sqrt{-s_4}}, \frac{a_i}{\sqrt{-s_4}} + \frac{s_3}{4s_4} - \frac{s_1}{4\sqrt{-s_4}}].$$

By direct calculation, one checks that the midpoints $\frac{1}{2}(B_i + B_j)$ lie on

$$\Gamma_A : \qquad (x - x_G)^2 - s_4(y - y_G)^2 = \frac{1}{16}(s_4 s_1^2 - s_3^2).$$

Thus the medial ellipses are the same: $\Gamma_B = \psi_G(\Gamma_A) = \Gamma_A$, as we wanted.

(3) The proof is as above, except that we want $\psi = \psi_J$ and the translation is $\psi_H(J_A) \to J_A$. When $\mathcal{A}$ is convex, one finds that the points

$$B_i = [\frac{\sqrt{s_4}}{a_i} + \frac{s_1}{2} - \frac{s_3}{2\sqrt{s_4}}, \frac{a_i}{\sqrt{s_4}} + \frac{s_3}{2s_4} - \frac{s_1}{2\sqrt{s_4}}]$$

lie on

$$\Theta_A : \qquad (x - x_J)^2 + s_4(y - y_J)^2 = \frac{1}{4s_4}(s_3^2 + s_4 s_1^2) - s_2.$$

Hence $\Theta_B = \psi_J(\Theta_A) = \Theta_A$.

When $\mathcal{A}$ is concave, similar arguments lead to the points

$$B_i = \psi_G(A_i) = [-\frac{\sqrt{-s_4}}{a_i} + \frac{s_1}{2} - \frac{s_3}{2\sqrt{-s_4}}, \frac{a_i}{\sqrt{-s_4}} + \frac{s_3}{2s_4} - \frac{s_1}{2\sqrt{-s_4}}]$$

which satisfy the equation

$$(x - x_J)^2 + s_4(y - y_J)^2 = -\frac{1}{4s_4}(s_3^2 + s_4 s_1^2) + s_2,$$

the conjugate hyperbola of Θ_A. This completes the proof. □

Proof of Theorem 7. By applying proper homotheties, we can reduce the proof to the case that $\mathcal{A}$ and $\mathcal{B}$ are semi-isometric; by further translations, we can even assume that $\psi : \mathcal{A} \to \mathcal{B}$ as in Theorem 6 and the conics centers are the same. Then, according to Theorem 8, the circumscribed conics Γ and Θ are either equal or conjugate. In any case $\mathcal{A}$ and $\mathcal{B}$ have the same asymptotic directions.

9. A special case: cyclic quadrangles

A cyclic (or *circumscriptible*) quadrangle $\mathcal{A}$ is convex and corresponds to $s_4 = 1$. In this case Θ_A is the circumcircle of equation $x^2+y^2-s_1x-s_3y+s_2 = 0$. The center of Θ_A is $J = J_A$ and its radius is $\rho = \frac{1}{2}\sqrt{s_1^2 + s_3^2 - 4s_2}$. Incidentally, since $\rho^2 = |JH|^2 - s_2$, we have discovered for s_2 a geometric interpretation, namely the $power$ of H with respect to the circumcircle Θ. The medial conic Γ is the rectangular hyperbola:

$$x^2 - y^2 - \frac{1}{2}s_1 x + \frac{1}{2}s_3 y = 0$$

and the circumscribed parabolas are

$$(x \pm y)^2 - s_1 x - s_3 y + s_2 = \pm 2.$$

Therefore the lines r and s defining ψ $(= \phi)$ in Theorem 6 are perpendicular with slope ± 1 and ψ is just the orthogonal reflection in the line s . For the asymptotic directions of cyclic quadrangles we have a simple geometric interpretation at finite, not involving conics: they merely bisect the angle formed by any pair of opposite sides of $\mathcal{A}$. In fact, the reflection ψ maps the line A_iA_j into the line B_iB_j, which is parallel to A_hA_k, because of the semi-homothety. We may also think of the asymptotic directions as the mean values of the directions of the radii JA_i, because ψ_J maps the perpendicular bisectors of A_iA_j into the perpendicular bisector of A_hA_k, namely a bisector of $\angle A_iJA_j$ into a bisector of $\angle A_hJA_k$.

For cyclic quadrangles, Theorem 1 states that the oriented angles formed by the sides of the diagonal triangle of $\mathcal{A}$ are just the opposite (as a result of the reflection ψ) of those formed by the bimedians. Using the symbols of Theorem 1 this can be written as $\angle N_{14,23}GN_{12,34} = \angle N_{14,23}G, GN_{12,34} = -\angle A_{12,34}A_{13,24}, A_{13,24}A_{14,23}$. Since a triangle and its medial are obviously homothetic, we have $\angle N_{14,23}GN_{12,34} = \angle N_{14,23}N_{13,24}N_{12,34}$. Hence the four points $N_{ij,hk}$ and G lie on a circle. This suggests the following statement, that we have been unable to find in the literature:

Corollary 9. *A quadrangle is cyclic if and only if its centroid lies on the nine-point circle of its diagonal triangle.*

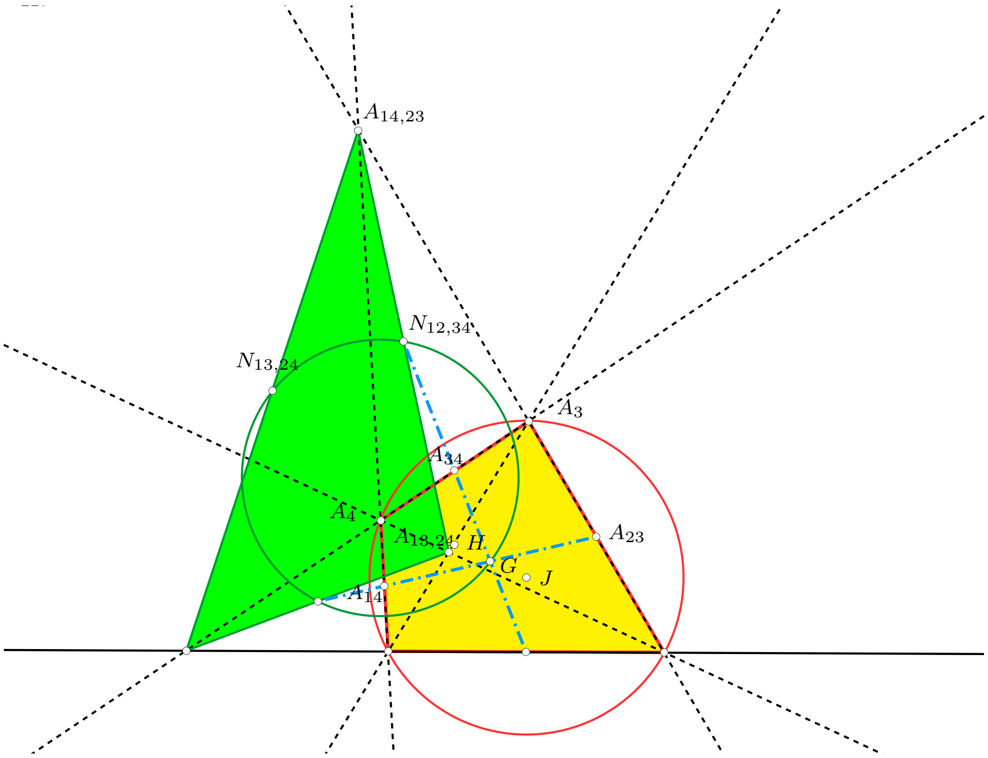

Figure 7. The centroid of a cyclic quadrangle lies on the ninepoint circle of its diagonal triangle

Proof. (Figure 7) It is well-known (see [2, Problem 46], or [1, §17.5.4]) that a conic circumscribed to a triangle $\mathcal{D}$ is a rectangular hyperbola if and only if its center lies on the ninepoint circle of $\mathcal{D}$. Let $\mathcal{A}$ be a quadrangle whose centroid G lies on the nine-point circle of its diagonal triangle $\mathcal{D}$. Since the medial conic

Γ_A is circumscribed to $\mathcal{D}$ and its center is the centroid G_A, we know that Γ_A is a rectangular hyperbola. Then, by previous theorems, we have $s_4 = 1$, Θ_A is a circle and $\mathcal{A}$ is cyclic. The converse argument is similar. □

10. Another special case: trapezoids

Trapezoids form another popular family of convex quadrangles, corresponding to the case $s_4 = \frac{s_3^2}{s_1^2}$. In fact, two opposite sides, say A_1A_2, A_3A_4 are parallel if and only if $\frac{a_1 - a_2}{\frac{1}{a_1} - \frac{1}{a_2}} = \frac{a_3 - a_4}{\frac{1}{a_3} - \frac{1}{a_4}}$, hence $a_1a_2 = a_3a_4$; and a straightforward calculation gives $(a_1a_2 - a_3a_4)(a_1a_3 - a_2a_4)(a_1a_4 - a_2a_3) = s_4s_1^2 - s_3^2$.

For trapezoids the medial conic Γ degenerates into two lines: $(s_1x + s_3y - \frac{1}{2}s_1^2)(s_1x - s_3y) = 0$ which are bimedians for $\mathcal{A}$; their slopes are simply $\pm\frac{s_1}{s_3}$: the asymptotic direction $-\frac{s_1}{s_3}$ is shared by the parallel sides of $\mathcal{A}$; the other is the direction of the line $s_1x - s_3y = 0$, on which one finds H, G, J, plus the two diagonal points at finite $A_{ih,jk}, A_{ik,jh}$. Π_+ also degenerates into the pair of parallel opposite sides. If two of the differences $a_ia_j - a_ha_k$ vanish, then $\mathcal{A}$ is a parallelogram, $H = J$ is its center and the asymptotic directions are parallel to the sides. For a cyclic trapezoid we have the additional condition $s_1^2 = s_3^2$ and HG is a symmetry line for $\mathcal{A}$. As for the oblique reflection ψ of Theorem 6, if, for example, the parallel sides are A_1A_2 and A_3A_4 , then the lines r and s are the bimedians $A_{12}A_{34}$, $A_{14}A_{23}$ and the oblique reflection interchanges two pairs of vertices: $\psi : A_1 \to A_2, A_2 \to A_1, A_3 \to A_4, A_4 \to A_3$. Thus $\mathcal{A}$ and $\mathcal{B} = \psi(\mathcal{A}\,)$ are the same quadrangle, but ψ is not the identity!

The fact that for both cyclic quadrangles and trapezoids semi-similarities leave the quadrangle shape invariant may perhaps explain why the relation of semi-similarity, to our knowledge, has not been studied. [14]

11. Orthogonal quadrangles

We still have to consider the family of quadrangles which have a pair of orthogonal opposite sides, because in this case the foregoing analytical geometry does not work. We call these quadrangles *orthogonal*. We shall first assume that the other pairs of sides are not perpendicular, leaving still out the subfamily of the so-called *orthocentric* quadrangles, which will be considered as very last. For orthogonal (non orthocentric) quadrangles, the statements of the Theorems of §§3 to 10 remain exactly the same, but a principal reference cannot be defined as before and different analytic proofs must be provided. First notice that for this family the hyperbola

[14] Going back to the semi-similar quadrangles $\mathcal{A}$ and $o(\mathcal{A}) = O_1O_2O_3O_4$ which we mentioned in the introduction and appear in Johnson′s textbook [3], it follows from our previous arguments that $A_i \to O_i$ is induced by an affine transformation which can be thought of as the product of four factors: a rotation of a straight angle, an oblique reflection, a homothety (the three of them fixing H) and a final translation $H \to J$. It can be proved that $A_i \to O_i$ is also induced, modulo an isometry, by a circle inversion centered at the so-called *isoptic* point of $\mathcal{A}$, see [6, 4].

Ψ degenerates into a pair of orthogonal lines. We can represent these lines by the equation $xy = 0$ (replacing $xy = 1$) and use them as xy-axes of a new principal reference (the unit length is arbitrary). Within this frame we can assume without loss of generality $A_1 = [x_1, 0], A_2 = [0, y_2], A_3 = [x_3, 0], A_4 = [0, y_4]$. Notice that the product $x_1y_2x_3y_4$ cannot vanish, as we have excluded quadrangles with three collinear vertices. The role of the elementary symmetric polynomials can be played here by other polynomials, as $s_x = x_1+x_3, s_y = y_2+y_4, p_x = x_1x_3, p_y = y_2y_4$. We have $H = [0, 0], G = \frac{1}{4}[s_x, s_y], J = \frac{1}{2}[s_x, s_y]$. One of the diagonal points is H; the remaining two are $\dfrac{1}{x_1y_4 - x_3y_2}[x_1x_3(y_2 - y_4), y_2y_4(x_1 - x_3)]$ and $\dfrac{1}{x_1y_2 - x_3y_4}[-x_1x_3(y_2 - y_4), y_2y_4(x_1 - x_3)]$. This shows that the xy-axes bisect an angle of the diagonal triangle. The fraction $\dfrac{p_x}{p_y}$ (or the product p_xp_y) plays the role of s_4 . More precisely: convexity and concavity are represented by $p_xp_y > 0$ or $p_xp_y < 0$ respectively ($p_xp_y = 0$ has been already excluded); $\mathcal{A}$ is cyclic if and only if $p_x = p_y$; $\mathcal{A}$ is a non-cyclic trapezoid when $p_xs_y^2 = p_ys_x^2$. Similar conditions can be established for s_x, s_y, p_x, p_y to characterize the various families of quadrangles ($skites$, $diamonds$, $squares$). The equations for the central conics are

$$\Gamma : \qquad p_yx^2 - p_xy^2 - \frac{1}{2}p_ys_xx + \frac{1}{2}p_xs_yy = 0,$$

and

$$\Theta : \qquad p_yx^2 + p_xy^2 - p_ys_xx - p_xs_yy + p_xp_y = 0.$$

The asymptotic directions have slope $\pm\sqrt{\dfrac{p_y}{p_x}}$ and $\pm\sqrt{-\dfrac{p_y}{p_x}}$ for the convex or concave case, respectively; the corresponding affinity of Theorem 6 is $[x, y] \to [y\sqrt{\frac{p_x}{p_y}}, x\sqrt{\frac{p_y}{p_x}}]$ for convex $\mathcal{A}$ etc. Not surprisingly, all statements and proofs of the foregoing theorems remain substantially the same and do not deserve special attention.

12. An extreme case: orthocentric quadrangles

If two pairs of opposite sides of $\mathcal{A}$ are orthogonal, then the same holds for the third pair. Such a concave quadrangle is called *orthocentric*, as each vertex A_i is the orthocenter of the complementary triangle $A_jA_hA_k$. For these quadrangles all the circumscribed conics are rectangular hyperbolas, so that Ψ, H, J, Θ are not defined. On the other hand, the medial conic Γ is defined, being merely the common nine-point circle for all the complementary triangles $A_jA_hA_k$. The asymptotic directions of $\mathcal{A}$ cannot be defined, but any pair of orthogonal directions can be used for defining the affinity of §5, and semi-similar orthocentric quadrangles turn out to be just directly similar. As an example, the elementary construction we gave in the introduction, when applied to an orthocentric quadrangle $\mathcal{A}$, modulo homotheties, just rotates $\mathcal{A}$ by a straight angle. We may also notice that some statements regarding orthocentric quadrangles can be obtained from the general case, as limits for s_4 tending to the value -1.

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