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Inversions in an Ellipse

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Abstract. In this paper we study the inversion in an ellipse which generalizes the classical inversion with respect to a circle and some properties. We also study the inversive images of lines, ellipses and other curves. Finally, we generalize the Pappus chain theorem to ellipses.

1. Introduction

In this paper we study inversions in an ellipse, which was introduced in [2], and some related properties to the distance of inverse points, cross ratio, harmonic conjugates and the images of various curves. This notion generalizes the classical inversion, which has a lot of properties and applications, see [1, 3, 4].

Definition. Let $\mathcal{E}$ be an ellipse centered at a point O with foci F_1 and F_2 in $\mathbb{R}^2$, the inversion in $\mathcal{E}$ is the mapping $\psi : \mathbb{R}^2 \setminus \{O\} \to \mathbb{R}^2 \setminus \{O\}$ defined by $\psi(P) = P'$, where P' lies on the ray $\overrightarrow{OP}$ and $OP \cdot OP' = OQ^2$, where Q is the intersection of the ray OP with the ellipse.

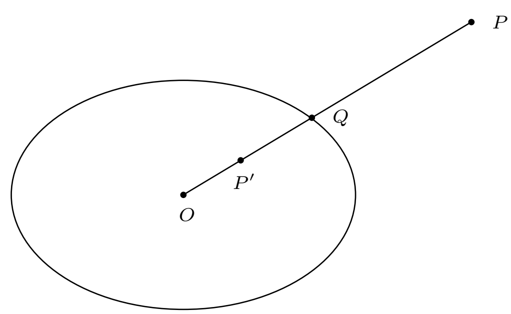

Figure 1

The point P' is said to be the inverse of P in the ellipse $\mathcal{E}$. We call $\mathcal{E}$ the ellipse of inversion, O the center of inversion, and the number $w := OQ$ the radius of inversion (see Figure 1). Unlike the classical case, the radius of inversion is not constant. Clearly, ψ is an involution, i.e., $\psi(\psi(P)) = P$ for every $P \neq O$. The fixed points are the points on the ellipse $\mathcal{E}$. Indeed, P is in the exterior of $\mathcal{E}$ if and only if P' is in the interior of $\mathcal{E}$. By introducing a point at infinity O_∞ as the inversive image of O, we regard ψ as an involution on the extended inversive plane $\mathbb{R}^2_\infty$.

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2. Basic properties

Proposition 1. *The inverse of P in an ellipse $\mathcal{E}$ is the intersection of the line OP and the polar of P with respect to $\mathcal{E}$. More precisely, if $\mathcal{E}$ is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the inverse of the point (u,v) in the ellipse is the point*

$$\left(\frac{a^2b^2u}{b^2u^2+a^2v^2},\ \frac{a^2b^2v}{b^2u^2+a^2v^2}\right).$$

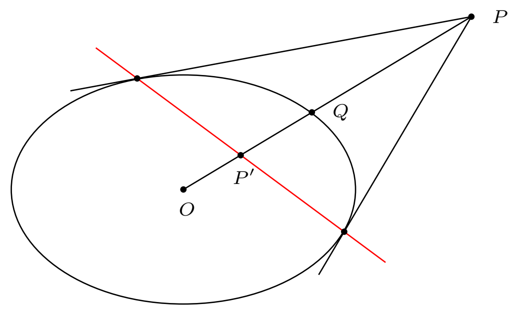

Figure 2

Proof. If $P=(u,v)$, the ray $\overrightarrow{OP}$ intersects $\mathcal{E}$ at $Q=(tu,\ tv)$ for $t>0$ satisfying $t^2\left(\frac{u^2}{a^2}+\frac{v^2}{b^2}\right)=1$. Now, the polar of P is the line $\frac{ux}{a^2}+\frac{vy}{b^2}=1$. This intersects the line OP (with equation $vx-uy=0$) at the point $(u',v')=(ku,kv)$ for k satisfying $k\left(\frac{u^2}{a^2}+\frac{v^2}{b^2}\right)=1$. Comparison gives $k=t^2$. Hence $OP\cdot OP'=OQ^2$, and (u',v') is the inverse of P in $\mathcal{E}$. Explicitly, $u'=\frac{a^2b^2u}{b^2u^2+a^2v^2}$ and $v'=\frac{a^2b^2v}{b^2u^2+a^2v^2}$. □

Theorem 2. *Let P and T be distinct points with inversion radii w and u with respect to $\mathcal{E}$. If P' and T' are the inverses of P and T in $\mathcal{E}$,*

$$P'T'=\begin{cases}\frac{w^2\cdot TP}{OP\cdot OT}, & \text{if } O,\ P,\ T \text{ are collinear},\\[2ex] \frac{\sqrt{(w^2-u^2)(w^2\cdot OT^2-u^2\cdot OP^2)+w^2u^2\cdot PT^2}}{OP\cdot OT}, & \text{otherwise}.\end{cases}$$

Proof. If O, P, T are collinear, the line containing them also contains Q, P' and T'. Clearly,

$$P'T'=OT'-OP'=\frac{OQ^2}{OT}-\frac{OQ^2}{OP}=\frac{w^2(OP-OT)}{OP\cdot OT}=\frac{w^2\cdot TP}{OP\cdot OT}.$$

Now suppose O, P, T are not collinear. Then neither are O, P', T' (see Figure 3). Let α be the measure of angle $P'OT'$. By the law of cosines, we have, in triangle POT,

$$\cos\alpha=\frac{OP^2+OT^2-PT^2}{2\cdot OP\cdot OT}.$$

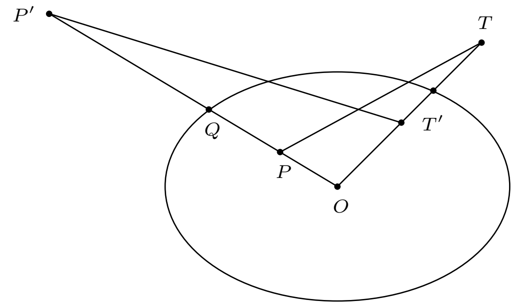

Figure 3.

Also, in triangle $P'OT'$,

$$\begin{aligned} P'T'^2 &= OP'^2 + OT'^2 - 2 \cdot OP' \cdot OT' \cdot \cos\alpha \\ &= \left(\frac{w^2}{OP}\right)^2 + \left(\frac{u^2}{OT}\right)^2 - 2 \cdot \frac{w^2}{OP} \cdot \frac{u^2}{OT} \cdot \frac{OP^2 + OT^2 - PT^2}{2 \cdot OP \cdot OT} \\ &= \frac{w^4 \cdot OT^2 + u^4 \cdot OP^2 - w^2u^2(OP^2 + OT^2 - PT^2)}{OP^2 \cdot OT^2} \\ &= \frac{(w^2 - u^2)(w^2 \cdot OT^2 - u^2 \cdot OP^2) + w^2u^2 \cdot PT^2}{OP^2 \cdot OT^2}. \end{aligned}$$

From this the result follows. □

3. Cross ratios and harmonic conjugates

Let A, B, C and D be four distinct points on a line ℓ. We define the cross ratio

$$(AB, CD) := \frac{AC \cdot BD}{AD \cdot BC},$$

where AB denotes the signed distance from A to B. We say that C, D divide A, B harmonically if the cross ratio $(AB, CD) = -1$. In this case we say that C and D are harmonic conjugates with respect to A and B. The cross ratio is an invariant under inversion in a circle whose center is not any of the four points A, B, C, D (see [1]). However, the inversion in an ellipse does not preserve the cross ratio. Nevertheless, in the case of harmonic conjugates, we have the following theorem.

Theorem 3. *Let $\mathcal{E}$ be an ellipse with center O, and Q_1Q_2 a diameter of $\mathscr{E}$. Two points on the line Q_1Q_2 are harmonic conjugates with respect to Q_1 and Q_2 if and only if they are inverse to each other with respect to $\mathcal{E}$.*

Proof. Let P and P' be two points on a diameter Q_1Q_2. Since

$$\begin{aligned} Q_1P \cdot Q_2P' &= (Q_1O + OP) \cdot (Q_2O + OP') \\ &= (w + OP)(-w + OP') \\ &= -w^2 - w(OP - OP') + OP \cdot OP', \\ Q_1P' \cdot Q_2P &= -w^2 + w(OP - OP') + OP \cdot OP', \end{aligned}$$

the points P and P' are harmonic conjugates with respect to Q_1 and Q_2 if and only if $OP \cdot OP' = w^2$, i.e., P and P' are inverse with respect to $\mathcal{E}$. □

4. Images of curves under an inversion in an ellipse

Theorem 4. *Consider the inversion ψ in an ellipse $\mathcal{E}$ with center O.*

(a) *Every line containing O is invariant under the inversion.*

(b) *The image of a line not containing O is an ellipse containing O and homothetic to $\mathcal{E}$.*

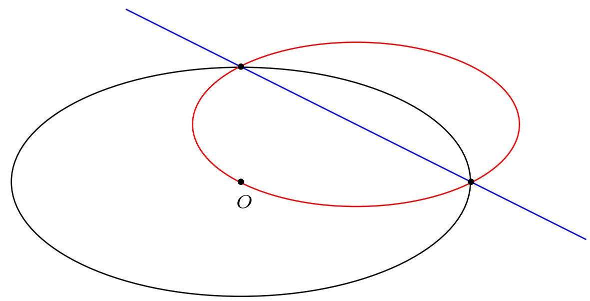

Figure 4

Proof. (a) This is clear from definition.

(b) Consider a line ℓ not containing O, with equation $px + qy + r = 0$ with $r \neq 0$. (x, y) is the inversive image of a point on ℓ, then the image of (x, y) lies on ℓ. In other words,

$$p \cdot \frac{a^2b^2x}{b^2x^2 + a^2y^2} + q \cdot \frac{a^2b^2y}{b^2x^2 + a^2y^2} + r = 0.$$

$$a^2b^2(px + qy) + r(b^2x^2 + a^2y^2) = 0. \tag{1}$$

This is clearly an ellipse containing $O(0, 0)$. Indeed, by rearranging its equation as

$$\frac{\left(x + \frac{a^2p}{2r}\right)^2}{a^2} + \frac{\left(y + \frac{b^2q}{2r}\right)^2}{b^2} = \frac{a^2p^2 + b^2q^2}{4r^2},$$

we note that this is the ellipse with center $\left(-\frac{a^2p}{2r}, -\frac{b^2q}{2r}\right)$, and homothetic to $\mathcal{E}$ with ratio $\frac{2r}{\sqrt{a^2p^2+b^2q^2}}$. □

Corollary 5. *Let ℓ_1 and ℓ_2 be perpendicular lines intersecting at a point P.*

(a) *If $P = O$, then $\psi(\ell_1)$ and $\psi(\ell_2)$ are perpendicular lines.*

(b) *If ℓ_1 does not contain O but ℓ_2 does, then $\psi(\ell_1)$ is an ellipse through O orthogonal to $\psi(\ell_2) = \ell_2$ at O.*

(c) *If none of the lines contains O, then $\psi(\ell_1)$ and $\psi(\ell_2)$ are ellipses containing P' and O, and are orthogonal at O.*

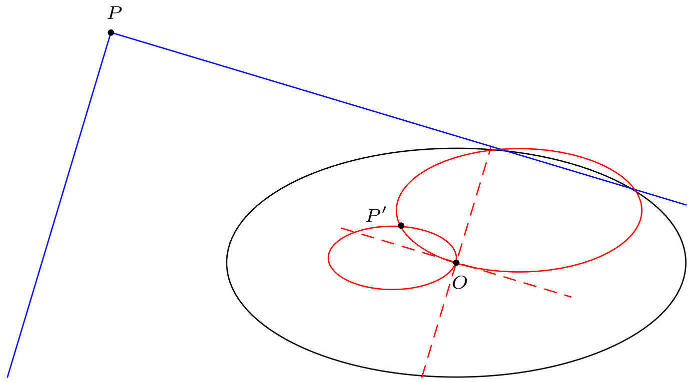

Figure 5

Proof. (a) The lines ℓ_1 and ℓ_2 are invariant.

(b) Let ℓ_1 be the line $px + qy + r = 0$ (with $r \neq 0$). Its image in $\mathcal{E}$ is the ellipse given by (1). The tangent at O is the line whose equation is obtained by suppressing the x^2 and y^2 terms, and replacing x and y by $\frac{1}{2}x$ and $\frac{1}{2}y$. This results in the line $\frac{1}{2}a^2b^2(px + qy) = 0$, or simply $px + qy = 0$, parallel to ℓ_1 and orthogonal to ℓ_2 at O.

(c) Let ℓ_1 and ℓ_2 be the orthogonal lines $p(x - h) + q(y - k) = 0$ and $q(x - h) - p(y - k) = 0$ intersecting at $P = (h, k) \neq O$. Their inverse images in $\mathcal{E}$ are ellipses intersecting at O and P'. By (b) above, the tangents at O are the orthogonal lines $px + qy = 0$ and $qx - py = 0$. □

Remark. In (c), the images are not necessarily orthogonal at P'.

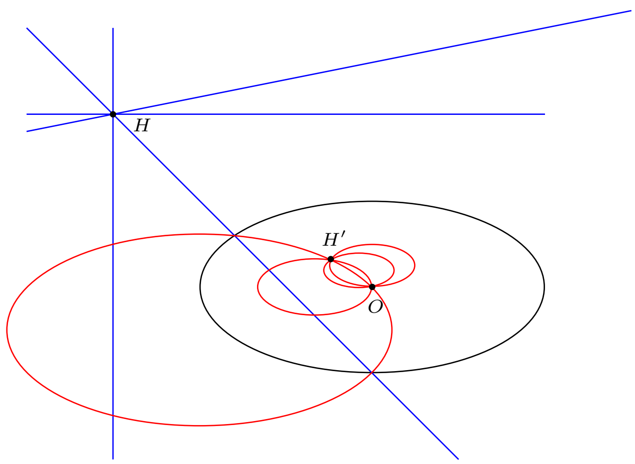

Figure 6.

Corollary 6. (a) *If $P \neq O$, the inverse images of the pencil of lines through P are coaxial ellipses through O and P'* (see Figure 6).

(b) *The inverse images of a system of straight lines parallel to ℓ_0 through O are ellipses homothetic to $\mathcal{E}$ tangent to ℓ_0 at O* (see Figure 7).

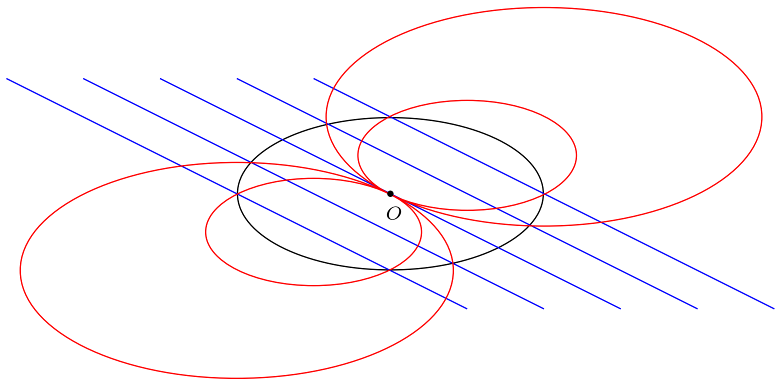

Figure 7.

Theorem 7. *Let $\mathcal{E}$ be the ellipse of inversion with center O, and $\mathcal{E}'$ an ellipse homothetic to $\mathcal{E}$. The image of $\mathcal{E}'$ is*
(a) *an ellipse homothetic to $\mathcal{E}$ if $\mathcal{E}'$ does not pass through O,*
(b) *a line if $\mathcal{E}'$ passes through O.*

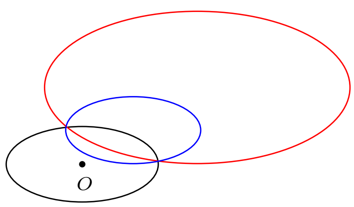

Figure 8

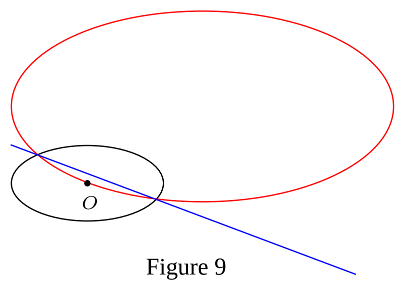

Figure 9

Proof. An ellipse $\mathcal{E}'$ homothetic to $\mathcal{E}$ has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + px + qy + r = 0.$$

The ellipse $\mathcal{E}'$ passes through O if and only if $r = 0$.

(a) If $\mathcal{E}$ does not pass through O, then $r \neq 0$. The inversive image consists of points $P(x, y)$ for which P' lies on the ellipse, i.e.,

$$\frac{\left(\frac{a^2b^2x}{b^2x^2+a^2y^2}\right)^2}{a^2} + \frac{\left(\frac{a^2b^2y}{b^2x^2+a^2y^2}\right)^2}{b^2} + p\left(\frac{a^2b^2x}{b^2x^2+a^2y^2}\right) + q\left(\frac{a^2b^2y}{b^2x^2+a^2y^2}\right) + r = 0. \tag{2}$$

Simplifying, we obtain

$$(b^2x^2 + a^2y^2)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{p}{r}\cdot x + \frac{q}{r}\cdot y + \frac{1}{r}\right) = 0.$$

Since $b^2x^2 + a^2y^2 \neq 0$, we must have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{p}{r}\cdot x + \frac{q}{r}\cdot y + \frac{1}{r} = 0.$$

This is an ellipse homothetic to $\mathcal{E}$ (see Figure 8).

(b) If $\mathcal{E}'$ passes through O, then $r = 0$. Equation (2) reduces to $px + qy + 1 = 0$ (see Figure 9). □

Corollary 8. *Let $\mathcal{E}'$ be an ellipse with center O' homothetic to $\mathcal{E}$ with center O. If $\mathcal{E}'$ is invariant under inversion in $\mathcal{E}$, and P is a common point of the ellipses, then $O'P$ and OP are tangent to $\mathcal{E}$ and $\mathcal{E}'$ respectively.*

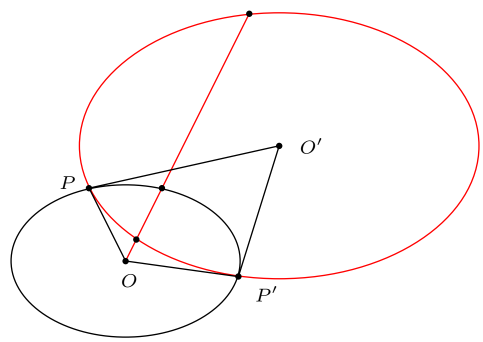

Figure 10

Proof. Comparing the equations of $\mathcal{E}'$ and its image under inversion in $\mathcal{E}$ in the proof of Theorem 7 above, we conclude that the ellipse $\mathcal{E}'$ is invariant if and only if its equation is of the form

$$(\mathcal{E}') : \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} + px + qy + 1 = 0.$$

Note that the center O' of $\mathcal{E}'$ has coordinates $\left(-\frac{pa^2}{2}, -\frac{qb^2}{2}\right)$.

Let $P = (x_0, y_0)$ be a common point of the two ellipses. Clearly,

$$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1, \qquad (3)$$

$$px_0 + qy_0 + 2 = 0. \qquad (4)$$

The tangents to $\mathcal{E}$ and $\mathcal{E}'$ at (x_0, y_0) are the lines

$$(t) : \qquad \frac{x_0x}{a^2} + \frac{y_0y}{b^2} - 1 = 0,$$

and

$$(t'): \qquad \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{1}{2}p(x + x_0) + \frac{1}{2}q(y + y_0) + 1 = 0.$$

Substitution of (x, y) by the coordinates O' into (t) and $(0, 0)$ into (t') lead to $\mp\left(\frac{px_0}{2} + \frac{qy_0}{2} + 1\right)$ respectively. By (4), this is zero in both cases. This shows that $O'P$ is tangent to $\mathcal{E}$ and OP is tangent to $\mathcal{E}'$. □

Theorem 9. *Given an ellipse $\mathcal{E}$ with center O, the image of a conic $\mathcal{C}$ not homothetic to $\mathcal{E}$ is*
(i) *a cubic curve if $\mathcal{C}$ passes through O,*
(ii) *a quartic curve if $\mathcal{C}$ does not pass through O.*

In Figures 11, 12, 13 below, we show the inversive images of a circle, a parabola, and a hyperbola in an ellipse.

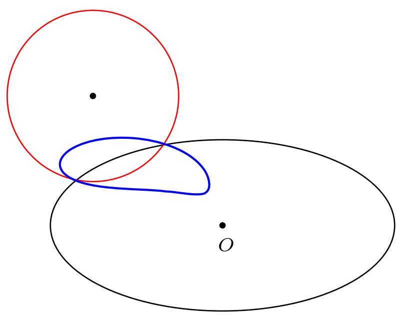

Figure 11

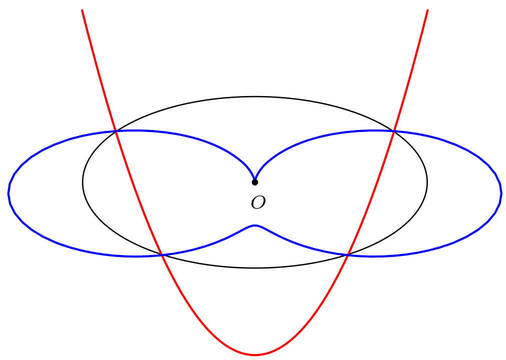

Figure 12

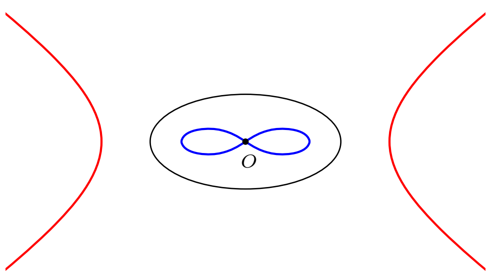

Figure 13

Note that the inversion in an ellipse is not conformal.

5. Pappus chain of ellipses

The classical inversion has a lot of applications, such as the Pappus chain theorem, Feuerbach's Theorem, Steiner Porism, the problem of Apollonius, among others [1, 3, 4]. We conclude this note with a generalization of the Pappus chain theorem to ellipses.

Theorem 10. *Let $\mathcal{E}$ be a semiellipse with principal diameter AB, and $\mathcal{E}'$, $\mathcal{E}_0$ semiellipses on the same side of AB with principal diameters AC and CB respectively, both homothetic to $\mathcal{E}$* (see Figure 14). *Let $\mathcal{E}_1$, $\mathcal{E}_2$, $\dots$, be a sequence of ellipses tangent to $\mathcal{E}$ and $\mathcal{E}'$, such that $\mathcal{E}_n$ is tangent to $\mathcal{E}_{n-1}$ and $\mathcal{E}_{n+1}$ for all $n \geq 1$. If r_n is the semi-minor axis of $\mathcal{E}_n$ and h_n the distance of the center of $\mathcal{E}_n$ from AB, then $h_n = 2nr_n$.*

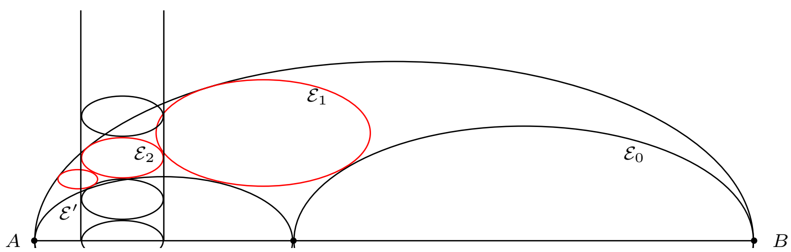

Figure 14.

Proof. Let ψ_i be the inversion in the ellipse $\mathcal{E}_i$. (In Figure 14 we select $i = 2$).

By Theorem 7, $\psi_i(\mathcal{E})$ and $\psi_i(\mathcal{E}_0)$ are lines perpendicular to AB and tangent to the ellipse $\mathcal{E}_i$. Hence, the ellipses $\psi_i(\mathcal{E}_1)$, $\psi_i(\mathcal{E}_2)$, $\dots$will be inverted to tangent ellipses to parallel lines to $\psi_i(\mathcal{E})$ and $\psi_i(\mathcal{E}_0)$. Hence, $h_i = 2ir_i$. □

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