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A Simple Proof of Gibert's Generalization of the Lester Circle Theorem

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Abstract. We give a simple proof of Gibert's generalization of the Lester circle theorem.

The famous Lester circle theorem states that for a triangle, the two Fermat points, the nine point center and the circumcenter lie on a circle, the Lester circle of the triangle. Here is Gibert's generalization of the Lester circle theorem, given in [2] and [4, Theorem 6]: *Every circle whose diameter is a chord of the Kiepert hyperbola perpendicular to the Euler line passes through the Fermat points*. In this note we show that this follows from a property of rectangular hyperbolas.

Lemma 1. *Let F_+ and F_- be two antipodal points on a rectangular hyperbola. For every point H on the hyperbola, the tangent to the circle (F_+F_-H) at H is parallel to the tangents of the hyperbola at F_+ and F_-.*

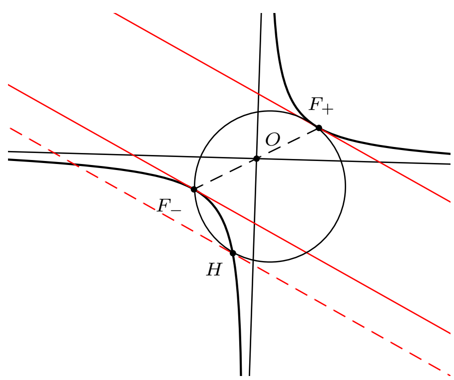

Figure 1

Proof. In a Cartesian coordinate system, let the rectangular hyperbola be represented by $xy = a$, and $F_+\left(x_0, \frac{a}{x_0}\right)$ and $F_-\left(-x_0, \frac{-a}{x_0}\right)$ two antipodal points. The slope of the tangents at $F_\pm$ is $-\frac{a}{x_0^2}$. Let $H\left(x_H, \frac{a}{x_H}\right)$ be a point on the hyperbola. Consider the circle through $F_\pm$ and H. Writing its equation in the form

$$x^2 + y^2 + Ax + By + C = 0,$$

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and substituting the coordinates of $F_\pm$ and H above, we obtain

$$\begin{aligned} x_0^2+y_0^2+Ax_0+By_0+C=&\ 0,\\ x_0^2+y_0^2-Ax_0-By_0+C=&\ 0,\\ x_H^2+y_H^2+Ax_H+By_H+C=&\ 0. \end{aligned}$$

Solving these equations we have

$$A=-x_H+y_H\cdot\frac{a}{x_0^2},\qquad B=-\frac{Ax_0^2}{a},\qquad C=-(x_0^2+y_0^2).$$

The tangent of the circle at H is the line

$$2x_Hx+2y_Hy+A(x+x_H)+B(y+y_H)+2C=0.$$

It has slope

$$-\frac{2x_H+A}{2y_H+B}=-\frac{x_H+y_H\cdot\frac{a}{x_0^2}}{y_H+x_H\cdot\frac{x_0^2}{a}}=-\frac{x_H+\frac{a}{x_H}\cdot\frac{a}{x_0^2}}{\frac{a}{x_H}+x_H\cdot\frac{x_0^2}{a}}=-\frac{x_H^2+\frac{a^2}{x_0^2}}{a+x_H^2\cdot\frac{x_0^2}{a}}=-\frac{a}{x_0^2}.$$

This tangent is parallel to the tangents of the hyperbola at $F_\pm$. □

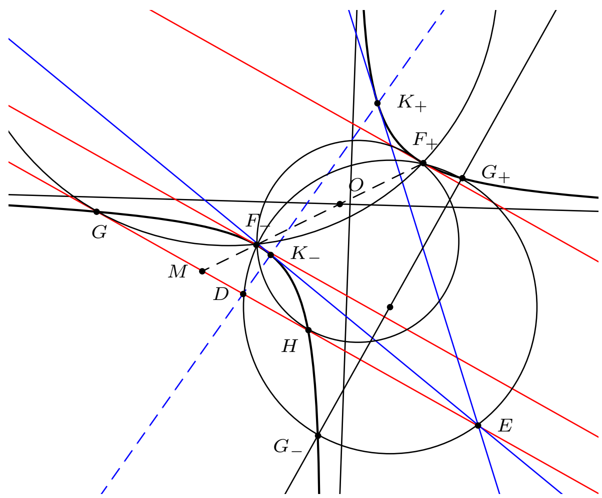

Figure 2

Theorem 2 ([1])**.** *Let H and G lie on one branch of a rectangular hyperbola, and*
(i) *F_+ and F_- antipodal points on the hyperbola the tangents at which are parallel to the line HG,*
(ii) *K_+ and K_- two points on the hyperbola the tangents at which intersect at a point E on the line HG.*
If the line K_+K_- intersects HG at D, and the perpendicular bisector of DE intersects the hyperbola at G_+ and G_-, then the six points F_+, F_-, D, E, G_+, G_- lie on a circle.

Proof. By Lemma 1, the circle (F_+F_-H) is tangent to HG at H. Similarly, the circle (F_+F_-G) is tangent to the same line HG at G.

Let M be the intersection of F_+F_- and HG. It lies on the radical axis of the circles (F_+F_-H) and (F_+F_-G), and satisfies $MG^2 = MF_+ \cdot MF_- = MH^2$. Therefore, M is the midpoint of HG.

Since the tangents of the hyperbola at K_+ and K_- intersect at E, the line K_+K_- is the polar of E. If it intersects the line HG at D, then $(G, D; H, E)$ is a harmonic range. Since M is the midpoint of HG, by a famous property of harmonic range, we have $MG^2 = MD \cdot ME$. Therefore, $MF_+ \cdot MF_- = MD \cdot ME$, and the four points F_+, F_-, D, E lie on a circle.

Now let the circle (F_+F_-DE) intersect the rectangular hyperbola at two points G_+ and G_-. By Lemma 1, the tangents of the circle at G_+, G_- are parallel to those of the hyperbola at F_+ and F_-, and therefore also to HG. It follows that G_+G_- is a diameter of the circle perpendicular to HG, and G_+, G_- lie on the perpendicular bisector of the chord DE of the circle. The proof of the theorem is complete. □

References

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