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A Note on the Fermat-Torricelli Point of a Class of Polygons

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Abstract. The aim of this note is to prove a result related to the Fermat-Torricelli point for a class of polygons.

The French mathematician Pierre Fermat (1601-1665) proposed at the end of his book *Treatise on Minima and Maxima* the search for a point T in the plane of a triangle ΔABC for which the sum $TA + TB + TC$ of the distances from T to the vertices is minimum. As the problem was first proved by the Italian scientist Evangelista Torricelli (1608-1647), the point T is sometimes called the Fermat-Torricelli point. The geometric construction of the Fermat-Torricelli point can be found in many textbooks, the most well known being that which uses the equilateral triangles constructed on the sides of the given triangle. If all angles of the given triangles are smaller than or equal to $2\pi/3$, then

$$\angle ATB = \angle BTC = \angle CTA = \frac{2\pi}{3}.$$

The problem has been studied by Fejes Tóth [2], Kazarinoff [3], and other specialists in geometric inequalities.

A history of Fermat's problem can be find in Boltyanski et al. [1].

We propose here the following new result that can be considered as an extention of Fermat's problem for a particular class of polygons. Remarks on this form can be found in [4]. Moreover, the proof provided is quite elementary.

Let there be given in plane $n + 1$ points T, A_1, A_2, $\ldots$, A_n. We say that the figure consisting of the union of the segments TA_1, TA_2, $\ldots$, TA_n is a *star* if $A_1A_2...A_n$ is a n-sided polygon and

$$\angle A_1TA_2 = \angle A_2TA_3 = \cdots = \angle A_nTA_1 = \frac{2\pi}{n}.$$

Let us denote such a star by $[T; A_1, A_2, \ldots, A_n]$. Now we are in the position to give our result.

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Theorem 1. *Let $[T; A_1, A_2, \ldots, A_n]$ be a star. Then for every point M, we have*

$$TA_1 + TA_2 + \cdots + TA_n \leq MA_1 + MA_2 + \cdots + MA_n.$$

Proof. Let us consider the complex plane with the origin T and the positive real axis TA_1. Let $r_k = TA_k$, and assume that $r_k\omega^{k-1}$ is the complex number associated with the point A_k, for every $1 \leq k \leq n$, where

$$\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}.$$

For any point M associated with a complex number z, we have

$$\begin{aligned}
& MA_1 + MA_2 + \cdots + MA_n \\
&= |z - r_1| + |z - r_2\omega| + \cdots + \left|z - r_n\omega^{n-1}\right| \\
&= |z - r_1| + \left|\frac{z}{\omega} - r_2\right| + \cdots + \left|\frac{z}{\omega^{n-1}} - r_n\right| \\
&\geq \left|(z - r_1) + \left(\frac{z}{\omega} - r_2\right) + \cdots + \left(\frac{z}{\omega^{n-1}} - r_n\right)\right| \\
&= \left|z\left(1 + \frac{1}{\omega} + \cdots + \frac{1}{\omega^{n-1}}\right) - (r_1 + r_2 + \cdots + r_n)\right| \\
&= |-(r_1 + r_2 + \cdots + r_n)| \\
&= r_1 + r_2 + \cdots + r_n \\
&= TA_1 + TA_2 + \cdots + TA_n,
\end{aligned}$$

and we are done. □

References

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[4] C. Mortici, *600 Problems (600 de probleme)*, Gil Publishing House, Zalău, 2000 (in Romanian).

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