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The Miquel Points, Pseudocircumcenter, and Euler-Poncelet Point of a Complete Quadrilateral

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Abstract. We prove over 40 similarities in the configuration of a complete quadrilateral and the Miquel points. Then we introduce a generalized circumcenter and prove a theorem on the Euler-Poncelet point.

1. Introduction

In this paper we will study, using purely synthetical methods, the configuration concerning complete quadrilateral $ABCD$ denoting $P = AC \cap BD$, $Q = AB \cap CD$, and $R = AD \cap BC$. We begin with revealing a large number of similarities within the configuration concerning the associated Miquel points. Then we proceed to introduce a quadrilateral center which generalizes the circumcenter of a cyclic quadrilateral, and finally we will prove a result regarding the Euler-Poncelet Point.

The Euler-Poncelet point X is the common point of the nine-point circles of triangles ABC, BCD, CDA, DAB. Also, it is known to lie on the pedal circles of A with respect to triangle BCD, and the cyclic variants. More on the Euler-Poncelet point can be found in [5].

Our main result is the following:

Theorem 1. *The Euler-Poncelet point X lies on the circumcircle of the triangle PQR.*

It is of particular interest that our result is a strong generalization of the celebrated result by Emelyanov and Emelyanova (see [2]).

Theorem 2 (Emelyanov, Emelyanova)**.** *Let ABC be a triangle with incenter I. Let $AI \cap BC = P$, $BI \cap CA = Q$, $CI \cap AB = R$, then the Feuerbach point F_e lies on the circumcircle of triangle PQR.*

Proof. If we take the complete quadrilateral to be $ABCI$, then the Euler-Poncelet point lies on both the incircle (pedal of I with respect to triangle ABC) and the nine-point circle of ABC, thus it coincides with their point of contact, i.e., the Feuerbach point. Theorem 1 now implies the result. $\square$

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2. Similarities on the Miquel points

In this section we define Miquel points (which were previously studied in [3]) as spiral similarity centers and uncover a surprising amount of similarities within the configuration.

The following two results on spiral similarities are well-known. Their proofs can be found for example in [4].

Proposition 3. *Let A, B, A', B' be points in plane such that no three of them are collinear. Assume that the lines AB and $A'B'$ intersect at P. Then there exists a unique spiral similarity that sends A to A' and B to B'. The center of this spiral similarity is the second intersection of the circles $(AA'P)$ and $(BB'P)$.*

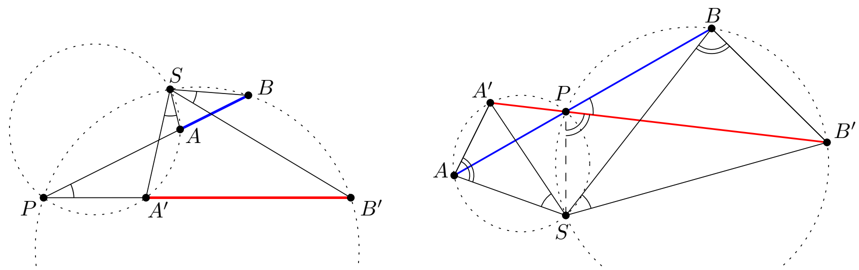

Figure 1.

Proposition 4. *Let $\mathcal{S}(S, k, \varphi)$ be the spiral similarity that maps A to A' and B to B'.*
(a) $\triangle SAB \sim \triangle SA'B'$.
(b) $\triangle SAA' \sim \triangle SBB'$.
(c) *There is a spiral similarity $\mathcal{S}'(S, k', \varphi')$ that maps A to B and A' to B' for suitable choice of k' and φ'.*

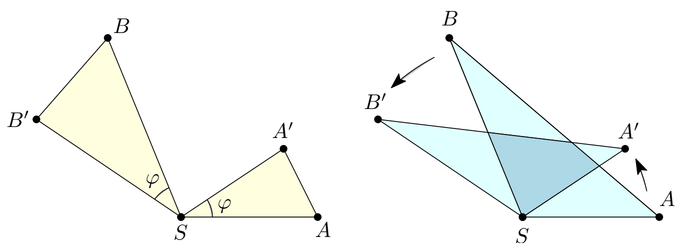

Figure 2.

From these propositions we immediately deduce that there is a unique point which is the center of two spiral similarities.

With this notion we can define three different Miquel points associated to a complete quadrilateral.

Definition. We define the Miquel points M_p, M_q, and M_r as the following spiral similarity centers (segments are considered directed):

Point	Center taking	and (at the same time)
M_p	$AB \mapsto DC$	$AD \mapsto BC$
M_q	$AC \mapsto DB$	$AD \mapsto CB$
M_r	$AC \mapsto BD$	$AB \mapsto CD$

As expected, Proposition 3 gives us many circles passing through the Miquel points. Also let us use the notation M_{XY} for the midpoint of the segment XY.

Proposition 5. *The following sets of points are concyclic:*
(a) $(M_pRAB), (M_pRDC), (M_pQAD), (M_pQBC)$,
(b) $(M_qRAC), (M_qRDB), (M_qPAD), (M_qPCB)$,
(c) $(M_rQAC), (M_rQBD), (M_rPAB), (M_rPCD)$,
(d) $(M_pM_qRM_{AD}M_{BC}), (M_qM_rPM_{AC}M_{DB}), (M_rM_pQM_{AB}M_{CD})$.

Proof. Parts (a), (b), and (c) follow from the definition of the Miquel points and Proposition 3 as each of them is a center of two different spiral similarities.

For part (d) observe that from $\triangle M_pAD \sim \triangle M_pBC$ it follows that

$$\triangle M_pM_{AD}A \sim \triangle M_pM_{BC}B$$

(directly) and hence M_p is the spiral similarity center which takes M_{AD} to A and M_{BC} to B. Proposition 3 now implies that M_p lies on the circumcircle of $\triangle RM_{AD}M_{BC}$. Point M_q lies on the same circle for analogous reasons. The other circles are established in the same way. □

Proposition 6. *The following sets of triangles are directly similar:*
(a) $\triangle M_pAB \sim \triangle M_pDC \sim \triangle M_{BA}M_{BD}M_{AC} \sim \triangle M_{CD}M_{AC}M_{DB}$,
$\triangle M_pAD \sim \triangle M_pBC \sim \triangle M_{DA}M_{DB}M_{AC} \sim \triangle M_{CB}M_{AC}M_{BD}$;
(b) $\triangle M_qAD \sim \triangle M_qCB \sim \triangle M_{DA}M_{DC}M_{AB} \sim \triangle M_{BC}M_{AB}M_{CD}$,
$\triangle M_qAC \sim \triangle M_qDB \sim \triangle M_{CA}M_{CD}M_{AB} \sim \triangle M_{BD}M_{AB}M_{DC}$;
(c) $\triangle M_rAB \sim \triangle M_rCD \sim \triangle M_{BA}M_{BC}M_{AD} \sim \triangle M_{DC}M_{AD}M_{CB}$,
$\triangle M_rAC \sim \triangle M_rBD \sim \triangle M_{CA}M_{CB}M_{AD} \sim \triangle M_{DB}M_{AD}M_{BC}$.

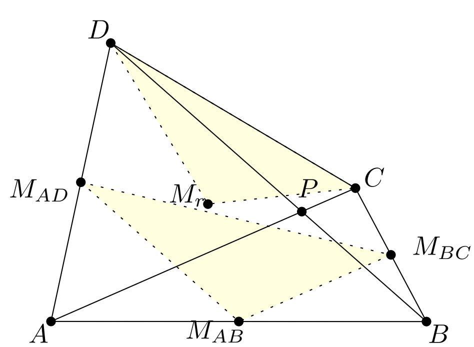

Figure 3.

Proof. Again, from symmetry of the complete quadrilateral it suffices to prove only one part and this time we choose (thinking of a nice diagram) to prove part (c). In the first chain of similarities the first and the third one are immediate from the definition of the Miquel point and the fact that the points M_{BA}, M_{AD}, M_{DC}, M_{CB} form a Varignon parallelogram, respectively.

Note that the lines $M_{AB}M_{BC}$ and $M_{AB}M_{AD}$ are midlines in triangles ABC and ABD, respectively. Angle-chasing (with the use of Proposition 5(c)) now gives

$$\angle(M_{BC}M_{AB}, M_{AD}M_{AB}) = \angle(AC, BD) = \angle(CP, PD) = \angle(CM_r, M_rD)$$

and after the ratio calculation (using $\triangle M_rAC \sim \triangle M_rBD$)

$$\frac{M_{BC}M_{AB}}{M_{AD}M_{AB}} = \frac{AC}{BD} = \frac{CM_r}{DM_r},$$

we have the desired similarity by SAS. The second part is proved likewise. □

Lemma 7. *If $A'B'C'D'$ is the image of $ABCD$ in the inversion with respect to M_p (M_q, M_r, respectively), then quadrilateral $ABCD$ is indirectly similar to $C'D'A'B'$ ($B'A'D'C'$, $D'C'B'A'$, respectively).*

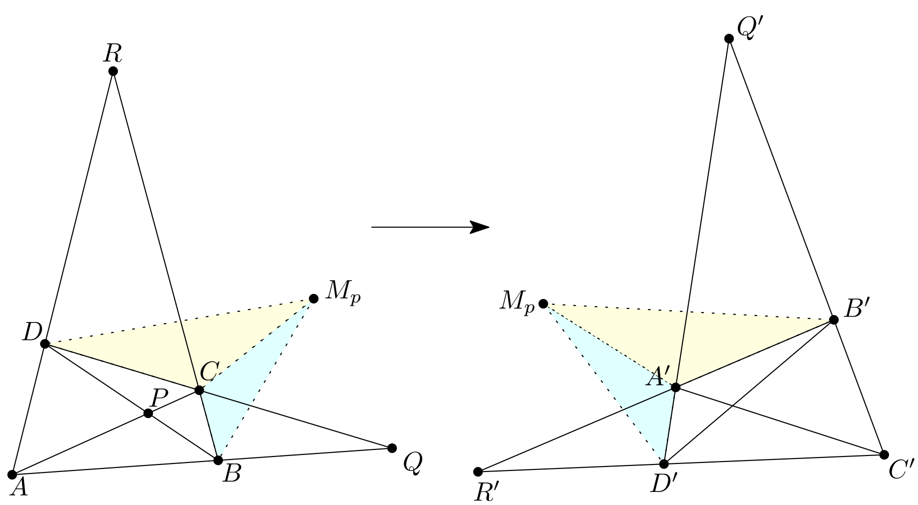

Figure 4.

Proof. From the definition of the Miquel point and inversion, respectively, we have $\triangle M_pBC \sim \triangle M_pAD \sim \triangle M_pD'A'$, where the first similarity is direct and the second one indirect. Similarly, we get $\triangle M_pDC \sim \triangle M_pAB \sim \triangle M_pB'A'$ (again first directly and then indirectly). But this implies indirect similarity of the quadrilaterals M_pDCB and $M_pB'A'D'$, from which we obtain indirect similarity of $\triangle DCB$ and $\triangle B'A'D'$. In the same vein, we find that $\triangle ABD \sim \triangle C'D'B'$ (indirectly) and the desired similarity of quadrilaterals now follows. The proof for M_q and M_r goes along the same lines. □

Proposition 8.
(a) M_p *is the center of spiral similarity which sends* M_rR *to* QM_q.
(b) M_q *is the center of spiral similarity which sends* M_pP *to* RM_r.
(c) M_r *is the center of spiral similarity which sends* M_qQ *to* PM_p.

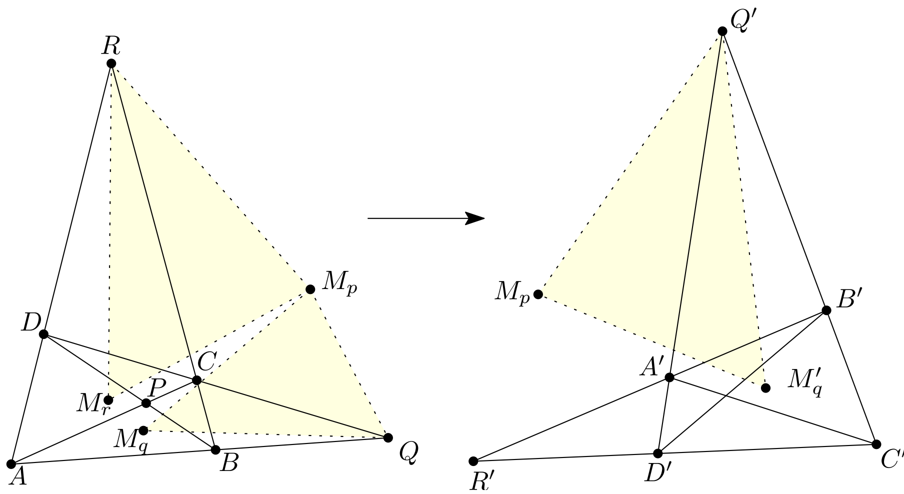

Figure 5.

Proof. From symmetry it is enough to prove part (a). Consider inversion with respect to point M_p and use standard notation for images.

Further, according to Lemma 7 $ABCD$ is indirectly similar to $C'D'A'B'$. In this similarity Q (intersection of AB and CD) corresponds to R' (intersection of $C'D'$ and $A'B'$). Analogously R corresponds to Q'. Also, M_r (second intersection of (QBD) and (QAC)) corresponds to M'_q (second intersection of $(R'D'B')$ and $(R'C'A')$) and M_p (second intersection of (RAB) and (RCD)) corresponds to M_p (second intersection of $(Q'C'D')$ and $(Q'A'B')$).

From these observations we can deduce that $\triangle M_pM_rR \sim \triangle M_pM'_qQ'$ (indirectly). From inversion we also have $\triangle M_pM'_qQ' \sim \triangle M_pM_qQ$ (indirectly). Altogether we have the direct similarity of $\triangle M_pM_rR$ and $\triangle M_pM_qQ$, which implies the result. □

3. The pseudocircumcenter

It is well-known (see e.g. [6]) that in the case of a cyclic quadrilateral $ABCD$ inscribed in a circle centered at O the triangle PQR has M_p, M_q, and M_r as its feet of altitudes and point O as its orthocenter. Also, the Euler-Poncelet point X is symmetric to O with respect to the centroid G of $ABCD$.

We introduce a point O associated to a (not necessarily cyclic) complete quadrilateral which inherits most of these properties.

Theorem 9. *The circles* (PM_qM_r), (QM_rM_p), *and* (RM_pM_q) *meet in one point.*

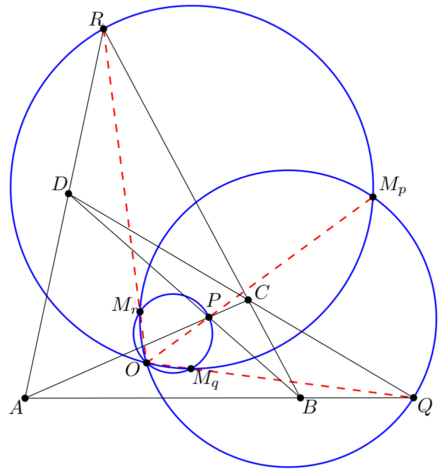

Figure 6.

Proof. Define O as the second intersection of (QM_rM_p) and (RM_pM_q). Then

$$\begin{aligned}\angle(M_qO, OM_r) &= \angle(M_qO, OM_p) + \angle(M_pO, OM_r)\\ &= \angle(M_qR, RM_p) + \angle(M_pQ, QM_r).\end{aligned}$$

Now using the circles from Proposition 5 we obtain

$$\angle(M_qR, RM_p) = \angle(M_qR, RA) + \angle(AR, RM_p) = \angle(M_qC, CA) + \angle(AB, BM_p)$$

and likewise

$$\angle(M_pQ, QM_r) = \angle(M_pQ, QC) + \angle(CQ, QM_r) = \angle(M_pB, BC) + \angle(CA, AM_r),$$
$$\angle(M_rP, PM_q) = \angle(M_rP, PB) + \angle(BP, PM_q) = \angle(M_rA, AB) + \angle(BC, CM_q)$$

After careful inspection, the three right-hand sides add up to 0, hence

$$\angle(M_qO, OM_r) - \angle(M_qP, PM_r) = 0$$

and the conclusion follows. □

We call this point O the *pseudocircumcenter* of $ABCD$.

Theorem 10. *The lines PM_p, RM_r, QM_q meet in O.*

Proof. It suffices to prove O, R, and M_r are collinear. With the use of Proposition 8, M_q is the center of spiral similarity which takes M_r to R and P to M_p, therefore $\triangle M_qM_rP \sim \triangle M_qRM_p$ (directly) and so $\angle(M_rP, PM_q) = \angle(RM_p, M_pM_q)$. We can now conclude after using the circles (M_qM_rPO) and (M_pM_qRO) as follows:

$$\angle(M_rO, OM_q) = \angle(M_rP, PM_q) = \angle(RM_p, M_pM_q) = \angle(RO, OM_q)$$

which immediately implies collinearity of O, R, M_r. □

Theorem 11.
(a) *The circles* $(M_{AB}M_{AC}M_{AD})$, $(M_{BA}M_{BC}M_{BD})$, $(M_{CA}M_{CB}M_{CD})$, *and* $(M_{DA}M_{DB}M_{DC})$ *all pass through* O.
(b) O *and* X *are symmetric with respect to* G.

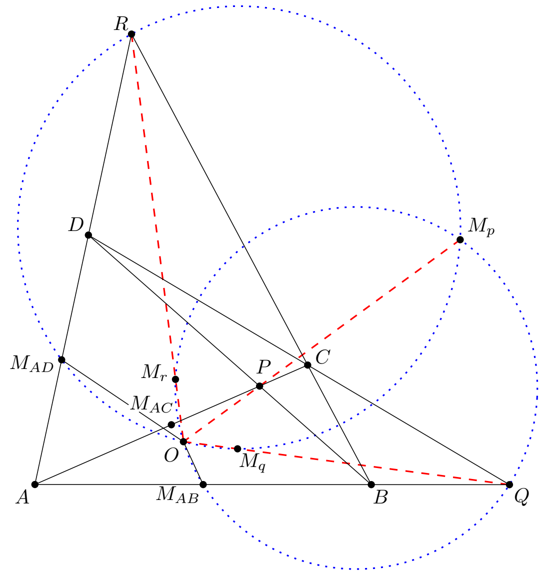

Figure 7.

Proof. From symmetry, it suffices to prove part (a) for one of the mentioned circles, for example $(M_{AB}M_{AC}M_{AD})$. Using midlines in triangles ACD and ABC, we obtain

$$\angle(M_{AD}M_{AC}, M_{AC}M_{AB}) = \angle(CD, CB).$$

Further, point O lies on (RM_pM_{AD}) and (QM_pM_{AB}) (consult Proposition 5(d) and Theorem 9), hence (using also the basic circles from Proposition 5(c))

$$\begin{aligned}\angle(M_{AD}O, OM_{AB}) &= \angle(M_{AD}O, OM_p) + \angle(M_pO, OM_{AB})\\ &= \angle(DR, RM_p) + \angle(M_pQ, QB)\\ &= \angle(DC, CM_p) + \angle(M_pC, CB) = \angle(DC, CB).\end{aligned}$$

Hence O indeed lies on $(M_{AB}M_{AC}M_{AD})$.

For part (b) apply the symmetry with respect to G. It is well-known that M_{AB}, M_{AC}, M_{AD}, M_{BC}, M_{BD}, M_{CD} are sent to M_{CD}, M_{BD}, M_{BC}, M_{AD}, M_{AC}, M_{AB}, respectively. Therefore the circles $(M_{AB}M_{AC}M_{AD})$, $(M_{BA}M_{BC}M_{BD})$, $(M_{CA}M_{CB}M_{CD})$, $(M_{DA}M_{DB}M_{DC})$ are sent to the nine-point circles of the triangles BCD, ACD, ABD, ABC, respectively and these circles have X as their common point. Hence X is the image of O. □

4. Proof of Theorem 1

Lemma 12. *Reflect a triangle ABC about a point P in its plane to triangle $A'B'C'$. Then the circles $(AB'C')$, $(BC'A')$, $(CA'B')$ are concurrent on (ABC).*

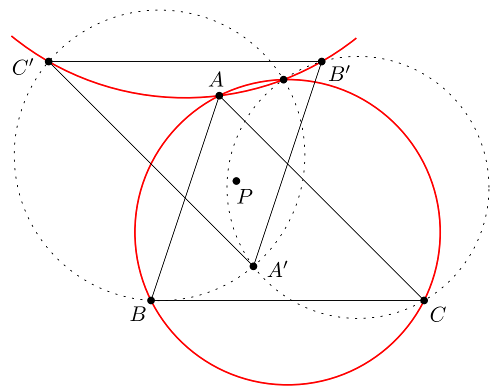

Figure 8.

Proof. Intersect (ABC) and $(AB'C')$ at X. Then angle-chase using the circles and parallel lines

$$\begin{aligned}\angle(BX, XC') &= \angle(BX, XA) + \angle(AX, XC') \\ &= \angle(BC, CA) + \angle(AB', B'C') \\ &= \angle(BC, A'C') + \angle(A'B, BC) \\ &= \angle(BA', A'C'),\end{aligned}$$

which proves that X lies on $(BC'A')$. Analogously, we prove it lies on $(CA'B')$. $\square$

Theorem 1. *Point X lies on (PQR).*

Proof. Let us denote by P', Q', R' the reflections of P, Q, R, respectively, about the centroid G. From Theorem 11, it suffices to prove that O lies on the circle $(P'Q'R')$. We will prove that O lies on the circles $(P'QR)$, $(Q'RP)$, $(R'PQ)$ and then the result will follow from Lemma 12 applied on triangle $P'Q'R'$. In fact, we only need (by symmetry) to prove the circle $(P'QRO)$.

The line $M_{AB}M_{CD}$ is the Newton-Gauss line of $ABCD$ so it passes through the midpoint of PR. As it also passes through G, it is the midline in $\triangle RPP'$. Similarly, we prove that $M_{AD}M_{BC}$ is the midline in $\triangle QPP'$. It follows that

$$\angle(RP', P'Q) = \angle(M_{AB}M_{CD}, M_{AD}M_{BC}).$$

At the same time using Theorems 9 and 10 we obtain

$$\angle(RO, OQ) = \angle(M_rO, OQ) = \angle(M_rM_{CD}, M_{CD}Q) = \angle(M_rM_{CD}, CD).$$

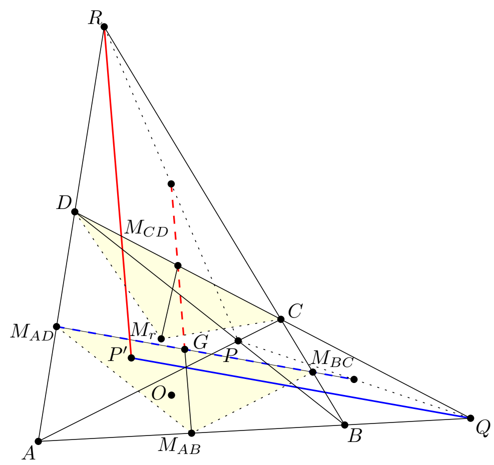

Figure 9.

But in Proposition 6 we proved the direct similarity

$$\triangle M_r CD \sim \triangle M_{BA} M_{BC} M_{AD}.$$

As the angles $\angle(M_r M_{CD}, CD)$ and $\angle(M_{AB} M_{CD}, M_{AD} M_{BC})$ correspond in this similarity (angles by medians), they are equal. It follows that

$$\angle(RP', P'Q) = \angle(RO, OQ),$$

which concludes the entire proof.

□

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