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A Note on Reflections

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Abstract. We prove some simple results associated with the triangle formed by the reflections of a point in the midpoints of the sides of a given triangle.

Let ABC be a given triangle with midpoints M_a, M_b, M_c of the sides BC, CA, AB respectively. Consider a point P and its reflections X, Y, Z in M_a, M_b, M_c respectively.

Proposition 1. *Triangle XYZ is oppositely congruent to ABC at the complement* (inferior) *of P.*

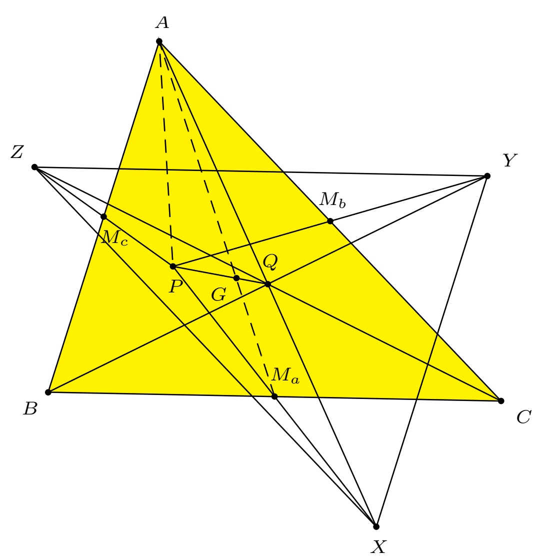

Figure 1

Proof. Let G be the centroid of triangle ABC. Consider triangle APX. It has the segment AM_a as a median, and so has centroid G. If Q is the midpoint of AX, then PQ is another median of triangle APX. Therefore, G divides PQ in the ratio $PG : GQ = 2 : 1$, and Q is the complement of P.

Similarly, the same point Q is the midpoint of BY and CZ. It follows that XYZ is oppositely congruent to ABC at Q. $\square$

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Let X^*, Y^*, Z^* be the reflections of X, Y, Z in the sidelines BC, CA, AB respectively.

Proposition 2. *The circle through X^*, Y^*, Z^* has center O and contains the P and its reflection in O.*

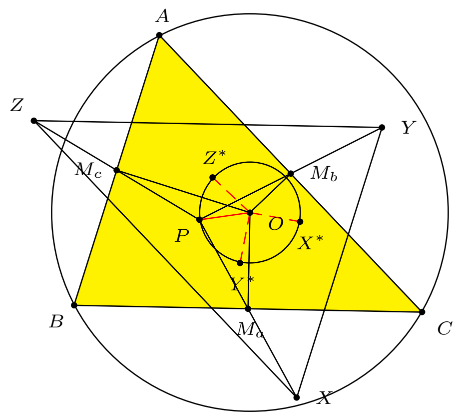

Figure 2

Proof. X^*, Y^*, Z^* are the reflections of P in the perpendicular bisectors of BC, CA, AB respectively. Each of these points is equidistant with P from the circumcenter O of triangle ABC. Therefore the circle through P, center O, also contains the reflection of P in O. □

Theorem 3. *If the Euler lines of PBC, PCA, PAB are concurrent at S, then the Euler lines of AZY, BXZ, CYX are concurrent at the superior* (anticomplement) *of S.*

Proof. Triangle AZY is a translation of triangle PBC.

$$Y - C = (C + A - P) - C = A - P = (A + B - P) - B = Z - B.$$

Clearly the two Euler lines of the two triangles are parallel. Since the centroid of AZY is the superior of the centroid of PBC, every point on the Euler line of AYZ is the superior of a point on the Euler line of PBC.

The same is true for the pairs BXZ, PCA and CYX, PAB. It follows that if the Euler lines of PBC, PCA, PAB are concurrent at S, then those of AZY, BXZ, CYX are concurrent at the superior (anticomplement) of S. □

Remark. It is well known that for $P = I$, the incenter, the Euler lines of IBC, ICA, IAB are concurrent at the Schiffler point $X(21)$ on the Euler line of ABC (see [4]). It follows that the Euler lines of AYZ, BZX, CXY are concurrent at the superior (anticomplement) of $X(21)$ (see Figure 3). This is the triangle center $X(2475)$ of [5], also on the Euler line of ABC.

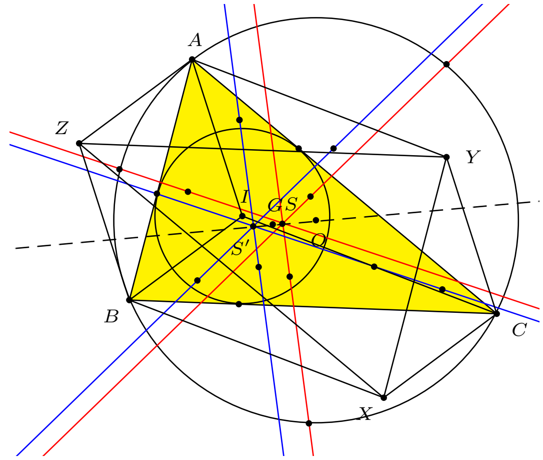

Figure 3

On the other hand, Yiu [8] has noted that, for $P = I$, the Euler lines of XBC, YCA, ZAB are concurrent at the cevian quotient Q/I, where Q is the Spieker center, the inferior (complement) of I.

Lemma 4. *For $P = I$, the incenter, the line AX intersects the Euler line of triangle XBC on the side BC.*

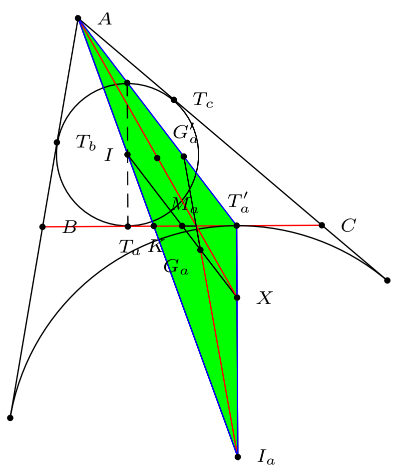

Figure 4

Proof. Let I_a be the center of the excircle tangent to BC at T'_a. Denote by r and r_a the inradius and radius of the A-excircle. We shall make use of the formulas $r = \frac{\Delta}{s}$ and $r_a = \frac{\Delta}{s-a}$, where Δ and s are the area and semiperimeter of triangle ABC.

Since IX and BC have a common midpoint M_a, $IBXC$ is a parallelogram. Therefore, BX and CX are perpendicular to CI_a and BI_a respectively. From this we note that X is the orthocenter of triangle I_aBC. Consequently, I_a is the orthocenter of triangle XBC.

Let G_a be the point dividing XM_a in the ratio $XG_a : G_aM_a = 2 : 1$. This is the centroid of triangle XBC, and I_aG_a is the Euler line.

Extend I_aG_a to intersect AT'_a at G'_a. Since the line AT'_a contains the antipode of T_a on the incircle, it is parallel to IX. Therefore, $AG'_a : G'_aT'_a = TG_a : G_aX = 2 : 1$.

Let AI_a intersect BC at K. Consider triangle AT'_aI_a with X on T'_aI_a, K on I_aA, and G'_a on AT'_a. We have

$$\frac{AG'_a}{G'_aT'_a} \cdot \frac{T'_aX}{XI_a} \cdot \frac{I_aK}{KA} = \frac{2}{1} \cdot \frac{r}{r_a - r} \cdot \frac{r_a}{\frac{2\Delta}{a}} = \frac{arr_a}{(r_a - r)\Delta} = \frac{a \cdot \frac{1}{s} \cdot \frac{1}{s-a}}{\frac{1}{s-a} - \frac{1}{s}} = 1.$$

By Ceva's theorem, AX, $I_aG'_a$ and T'_aK are concurrent. This means that AX and the Euler line I_aG_a of triangle XBC intersect on BC. □

Theorem 5. *For $P = I$, the incenter, the Euler lines of triangles XBC, YCA, ZAB are concurrent at the cevian quotient Q/I, where Q is the Spieker center.*

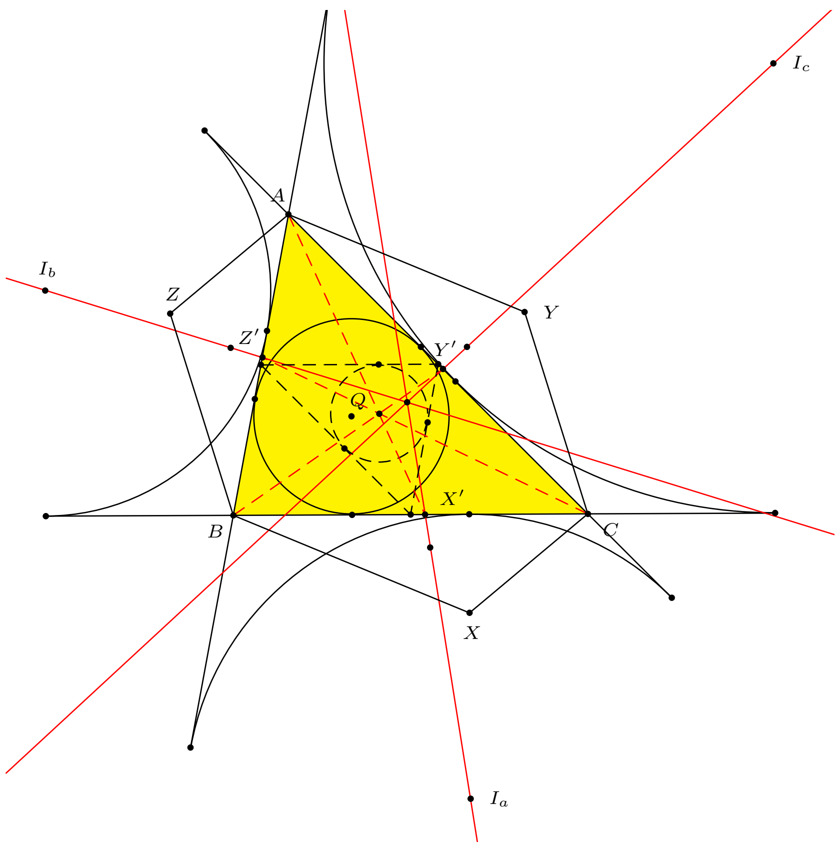

Figure 5

Proof. The line AX contains the Spieker center Q as its midpoint. Denote by $X'Y'Z'$ the cevian triangle of Q. By Lemma 4 above, the Euler line of triangle XBC is I_aX'. Similarly, the Euler lines of YCA and ZAB are the lines I_bY' and I_cZ'. Now, $I_aI_bI_c$ is the anticevian triangle of I, and XYZ is the cevian triangle of S. These lines are concurrent at the cevian quotient Q/I (see [7, §8.3]). □

Remark. For $Q =$ the Spieker center, the cevian quotient Q/I is the triangle center $X(191)$ of [5]. It is the reflection of I in the Schiffler point $X(21)$ (see Remark after Theorem 3 above).

Theorem 6 (Collings)**.** *The circles* (AYZ), (BZX), (CXY) *are concurrent at a point on the circumcircle of* ABC*, which is the superior of the center of the rectangular hyperbola through* A, B, C*, and* P*.*

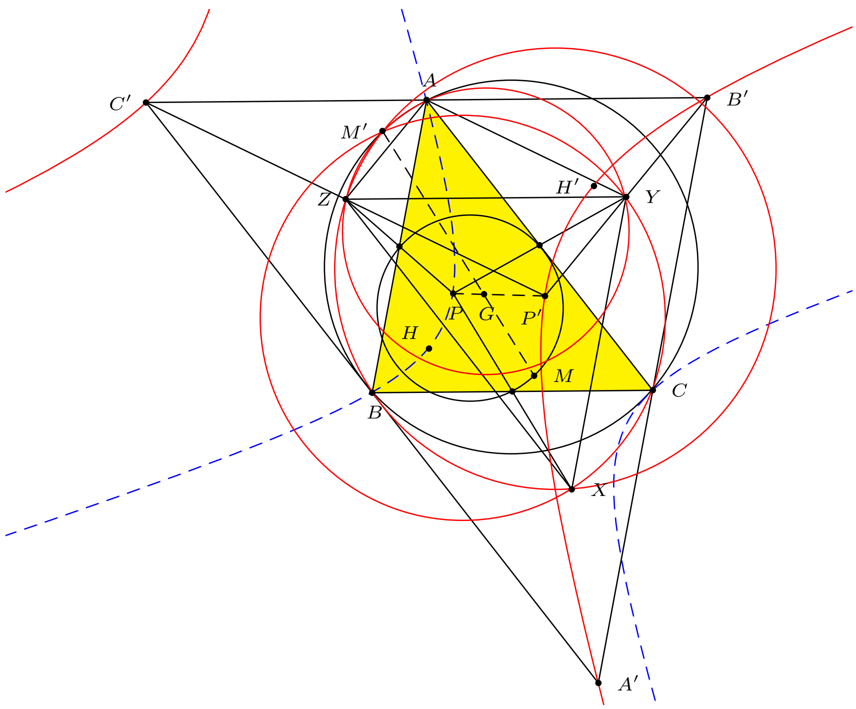

Figure 6

Proof. Collings [1] actually shows that if $A'B'C'$ is the superior (anticomplementary) triangle of ABC, and P' is the superior of P, then the circles (AYZ), (BZX), $CXY)$ and (ABC) intersect at the center M' of the rectangular hyperbola through A', B', C', P' (see Figure 3). Since A', B', C', P' are the superiors of A, B, C, P respectively, M' is the superior of the center M of the rectangular hyperbola through A, B, C, P. This is a point on the nine-point circle of ABC. It follows that M' is a point on the circumcircle. □

For example, if $P = I$, the incenter, the rectangular hyperbola through A, B, C, I (and H) has center the Feuerbach point F. The common point of the circles (AZY), (BXZ), (CYX) is the inferior of F. This is the point $X(100)$.

Peter Moses has informed us [2], among other things, that triangle XYZ is perspective to the mid-arc triangle at $X(100)$ (see Figure 7). The vertices of the mid-arc triangle are the intersections of the angle bisectors with the circumcircle. Since the inferiors (complements) of X, Y, Z are the midpoints X'', Y'', Z'' of IA, IB, IC respectively, it is enough to prove that $X''Y''Z''$ is perspective with the mid-arc triangle of the medial triangle $M_aM_bM_c$.

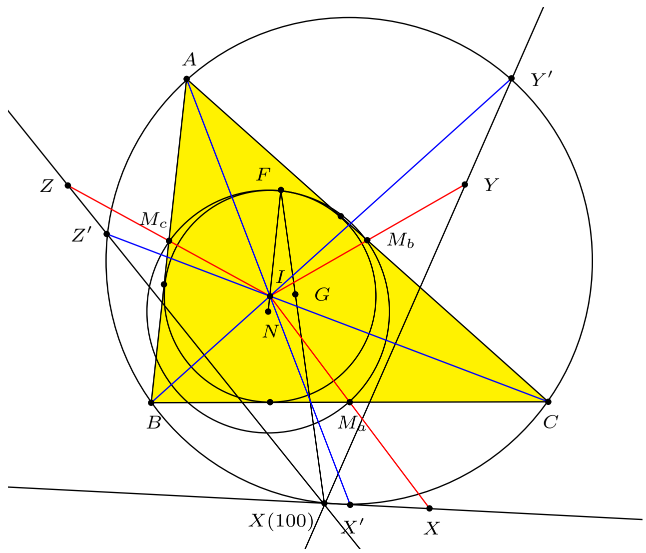

Figure 7

Proposition 7. *Let X'', Y'', Z'' be the midpoints of IA, IB, IC respectively, and M'_a, M'_b, M'_c the midpoints of the the arcs M_bM_c, M_cM_a, M_aM_b of the nine-point circle of triangle ABC not containing M_a, M_b, M_c. The triangles $X''Y''Z''$ and $M'_aM'_bM'_c$ are perspective at the Feuerbach point of triangle ABC.*

Proof. The nine-point circle of triangle ABC is tangent to the incircle at the Feuerbach point F. Let M'_a be the midpoint of the arc M_bM_c of the nine-point circle (not containing M_a).

$$\angle M'_aFM_c = \frac{1}{2}\angle M_bFM_c = \frac{1}{2}\angle M_bM_aM_c = \frac{1}{2}\angle BAC. \tag{1}$$

On the other hand, if X'' is the midpoint of IA, then the circle through X'', M_c and T_c is the nine-point circle of triangle IAB, and it passes through the Feuerbach point F as well (see Remark below), and

$$\angle X''FM_b = \angle X''T_bA = \angle X''AT_b = \frac{1}{2}\angle BAC. \tag{2}$$

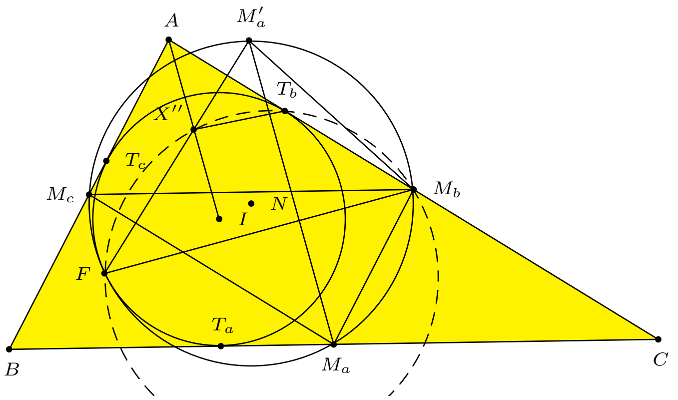

Figure 8

It follows from (1) and (2) that M'_a, X'' and F are collinear.

The same reasoning shows that M'_b, Y'', and F are collinear, so are M'_c, Z'' and F. Therefore, the triangles $X''Y''Z''$ and $M'_aM'_bM'_c$ are perspective at the Feuerbach point. □

Remark. The nine-point circles of IBC, ICA, IAB, and ABC are concurrent at the center of the rectangular hyperbola through A, B, C, I. This is the Feuerbach point F. See Theorem 6 above. A synthetic proof of this fact can be found in [6].

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