Two Pairs of Archimedean Circles in the Arbelos

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Abstract. We construct four circles congruent to the Archimedean twin circles in the arbelos.

Consider an arbelos formed by semicircles \((O_1), (O_2),\) and \((O)\) of radii \(a, b,\) and \(a + b.\) The famous Archimedean twin circles associated in the arbelos have equal radii \(\frac{ab}{a+b}\) (see [2, 3]).

Let \(CD\) be the dividing line of the smaller semicircles, and extend their common tangent \(PQ\) to intersect \((O)\) at \(T_a\) and \(T_b.\)

**Theorem 1.** Let \(A'\) and \(B'\) be the orthogonal projections of \(D\) on the tangents to \((O)\) at \(T_a\) and \(T_b\) respectively. The circles with diameters \(DA'\) and \(DB'\) are congruent to the Archimedean twin circles.

**Proof.** Let the tangents at \(T_a\) and \(T_b\) intersect at \(T.\) Since \(OT\) is the perpendicular bisector of \(T_aT_b,\) it intersects the semicircle \((O)\) at the midpoint \(D\) of the arc \(T_aT_b\) (see [3, §5.2.1]). Since \(O_1P, OM\) and \(O_2Q\) are parallel, and \(O_1P = OO_2 = a,\) \(O_2Q = O_1O = b,\)

\[
OM = \frac{a}{a+b} \cdot O_1P + \frac{b}{a+b} \cdot O_2Q = \frac{a^2 + b^2}{a+b} \quad \Rightarrow \quad DM = OD - OM = \frac{2ab}{a+b}.
\]
Now, \( \angle DT_aT = \angle DT_bT_a = \angle DT_bT_b \). Therefore, \( T_aD \) bisects angle \( TT_aT_b \). Similarly, \( T_bD \) bisects angle \( TT_bT_a \), and \( D \) is the incenter of triangle \( TT_aT_b \). It follows that \( DA' = DB' = DM \), and the circles with \( DA' \) and \( DB' \) are congruent to the Archimedean twin circles.

**Remark.** The circle with \( DM \) as diameter is the Archimedean circle \( (A_3) \) in [2] (or \( (W_4) \) in [1]).

**Theorem 2.** Let \( A_1A_2 \) and \( B_1B_2 \) be tangents to the smaller semicircles with \( A_1 \), \( B_1 \) on the line \( AB \) and \( A_1A_2 = a, B_1B_2 = b \). If \( H \) and \( K \) are the midpoints of the semicircles \( (O_1) \) and \( (O_2) \) respectively, and \( A'' = CH \cap A_1B_2, B'' = CK \cap B_1A_2 \), then the circles through \( C \) with centers \( A'' \) and \( B'' \) are congruent to the Archimedean twin circles.

**Proof.** Clearly, \( \angle A''CA_1 = \angle HCO_1 = 45^\circ \). Since \( B_1B_2 = O_2B_2 = b\), \( \angle B_2B_1O_2 = 45^\circ \), the lines \( CA'' \) and \( B_1B_2 \) are parallel. Also, \( B_1O_2 = \sqrt{2}b \). Similarly, \( A_1O_1 = \sqrt{2}a \), and \( A_1B_1 = (\sqrt{2} + 1)(a + b) \). Therefore,

\[
CA'' = B_1B_2 \cdot \frac{A_1C}{A_1B_1} = b \cdot \frac{(\sqrt{2} + 1)a}{(\sqrt{2} + 1)(a + b)} = \frac{ab}{a + b}.
\]

Similarly, \( CB'' = \frac{ab}{a + b} \). Therefore, the circles through \( C \) with centers \( A'' \) and \( B'' \) are congruent to the Archimedean twin circles.

**References**


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