Forum Geometricorum
Volume 14 (2014) 201–202.

FORUM GEOM
ISSN 1534-1178

Two Pairs of Archimedean Circles in the Arbelos

Dao Thanh Oai

Abstract. We construct four circles congruent to the Archimedean twin circles in the arbelos.

Consider an arbelos formed by semicircles (O_1), (O_2), and (O) of radii a, b, and $a+b$. The famous Archimedean twin circles associated in the arbelos have equal radii $\frac{ab}{a+b}$ (see [2, 3]).

Let CD be the dividing line of the smaller semicircles, and extend their common tangent PQ to intersect (O) at T_a and T_b.

Theorem 1. *Let A' and B' be the orthogonal projections of D on the tangents to (O) at T_a and T_b respectively. The circles with diameters DA' and DB' are congruent to the Archimedean twin circles.*

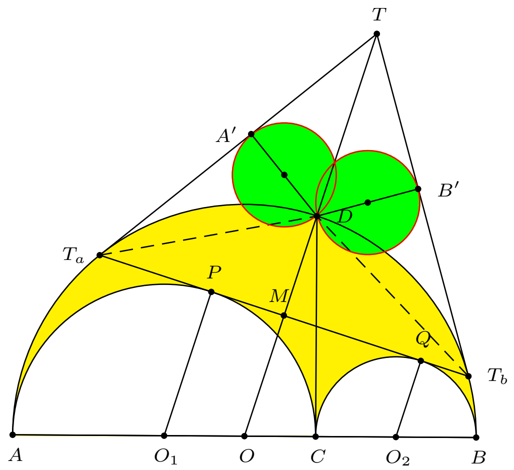

Figure 1

Proof. Let the tangents at T_a and T_b intersect at T. Since OT is the perpendicular bisector of T_aT_b, it intersects the semicircle (O) at the midpoint D of the arc T_aT_b (see [3, §5.2.1]). Since O_1P, OM and O_2Q are parallel, and $O_1P = OO_2 = a$, $O_2Q = O_1O = b$,

$$OM = \frac{a}{a+b}\cdot O_1P + \frac{b}{a+b}\cdot O_2Q = \frac{a^2+b^2}{a+b} \implies DM = OD - OM = \frac{2ab}{a+b}.$$

Publication Date: September 2, 2014. Communicating Editor: Floor van Lamoen.

Now, $\angle DT_aT = \angle DT_bT_a = \angle DT_aT_b$. Therefore, T_aD bisects angle TT_aT_b. Similarly, T_bD bisects angle TT_bT_a, and D is the incenter of triangle TT_aT_b. It follows that $DA' = DB' = DM$, and the circles with DA' and DB' are congruent to the Archimedean twin circles. □

Remark. The circle with DM as diameter is the Archimedean circle (A_3) in [2] (or (W_4) in [1]).

Theorem 2. *Let A_1A_2 and B_1B_2 be tangents to the smaller semicircles with A_1, B_1 on the line AB and $A_1A_2 = a$, $B_1B_2 = b$. If H and K are the midpoints of the semicircles (O_1) and (O_2) respectively, and $A'' = CH \cap A_1B_2$, $B'' = CK \cap B_1A_2$, then the circles through C with centers A'' and B'' are congruent to the Archimedean twin circles.*

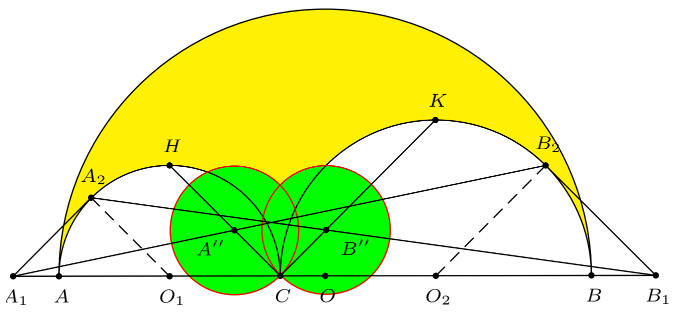

Figure 2

Proof. Clearly, $\angle A''CA_1 = \angle HCO_1 = 45°$. Since $B_1B_2 = O_2B_2 = b$, $\angle B_2B_1O_2 = 45°$, the lines CA'' and B_1B_2 are parallel. Also, $B_1O_2 = \sqrt{2}b$. Similarly, $A_1O_1 = \sqrt{2}a$, and $A_1B_1 = (\sqrt{2}+1)(a+b)$. Therefore,

$$CA'' = B_1B_2 \cdot \frac{A_1C}{A_1B_1} = b \cdot \frac{(\sqrt{2}+1)a}{(\sqrt{2}+1)(a+b)} = \frac{ab}{a+b}.$$

Similarly, $CB'' = \frac{ab}{a+b}$. Therefore, the circles through C with centers A'' and B'' are congruent to the Archimedean twin circles. □

References

[1] C. W. Dodge, T. Schoch, P. Y. Woo and P. Yiu, Those ubiquitous Archimedean circles, *Math. Mag.*, 72 (1999) 202–213.
[2] F. M. van Lamoen, *Online catalogue of Archimedean circles*, `http://home.kpn.nl/lamoen/wiskunde/Arbelos/Catalogue.htm`
[3] P. Yiu, *Euclidean Geometry*, Florida Atlantic University Lecture Notes, 1998, available at `http://math.fau.edu/Yiu/Geometry.html`

Dao Thanh Oai: Cao Mai Doai, Quang Trung, Kien Xuong, Thai Binh, Viet Nam
E-mail address: daothanhoai@hotmail.com