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A Simple Property of Isosceles Triangles with Applications

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Abstract. In this paper we prove a simple property of isosceles triangles and give two applications: construction of third proportional line segments and construction of the inverse point with respect to a circle.

1. Introduction

Here we give an interesting property of isosceles triangles. It is known that if a and b are two given line segments, then their third proportional line segment c can be constructed geometrically; see [2, VI.11]. We can more easily construct the third proportional line segment by using the simple geometric property of the isosceles triangles. Two points A and B are inverse points with respect to the inversion circle with center O and radius r, if $OA \cdot OB = r^2$; see [3]. We can construct the inverse point with respect to a circle by using the same property of the isosceles triangle.

2. The property

Lemma 1. *Let ABC be an isosceles triangle with $AB = BC$. Let D be a point on the ray BC and let h be the ray obtained by reflecting the ray AD in the line AC. Then the ray h cuts the ray BC in a point E which lies outside the segment BC if the point D lies inside this segment* (see Figure 1) *and inside the segment BC if the point D lies outside* (see Figure 2).

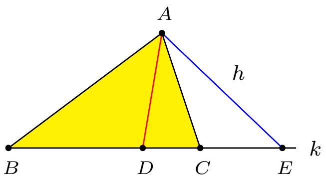

Figure 1

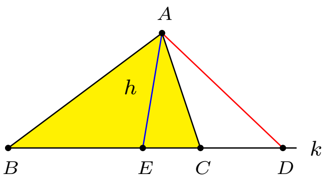

Figure 2

Proof. For Figure 1, let k denote the ray which is the part of the ray BC outside the segment BC. Then,

$$\begin{aligned}\angle(k, CA) + \angle(AC, h) &= \pi - \angle ACB + \angle(AC, h)\\ &= \pi - \angle BAC + \angle(AC, h)\\ &= \pi - \angle BAC + \angle DAC\\ &< \pi.\end{aligned}$$

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The second equality holds since the triangle ABC is isosceles, the third by reflection, and the fourth since D is an interior point of the segment BC. Now according to Euclid's fifth postulate the rays k and h meet in a point E. By the properties of reflection it is obvious that this intersection point E must lie on the ray BC outside the segment BC.

For Figure 2, $\angle(h, AC) = \angle CAD < \angle ACB = \angle BAC$. The first equality holds by reflection, the inequality by the Exterior Angle Theorem, the second equality since the triangle ABC is isosceles. Thus, the ray h runs first within the triangle ABC and meets the side BC in an interior point E. $\square$

Theorem 2. *If ABC is an isosceles triangle and points D, E are given as in* Lemma 1*, then $BC^2 = BD \cdot BE$, i.e., BC is the geometric mean of BD and BE.*

Proof. Firstly, we assume the point D inside the segment BC. The triangles ABD and EBA share the angle at vertex B. Now consider the angle sums of the triangles ABC and ABD.

$$\angle ABC + \angle BCA + \angle CAB = \pi,$$
$$\angle ABD + \angle BDA + \angle DAB = \pi.$$

Note, that $\angle CAB = \angle CAD + \angle DAB$. Thus, comparison of the two angle sums yields $\angle BDA = \angle BCA + \angle CAD$. But $\angle BCA = \angle CAB$ since the triangle ABC is isosceles and $\angle CAD = \angle EAC$ by reflection. Thus,

$$\angle BDA = \angle CAB + \angle EAC = \angle EAB.$$

Therefore the triangles ABD and EBA are inversely similar, and

$$BC : BD = AB : BD = BE : BA = BE : BC.$$

From this, $BC^2 = BD \cdot BE$.

Secondly, if the point D lies outside the segment BC then interchanging the roles of the points D and E in the previous argument yields the same result. $\square$

3. Applications

3.1. *Construction of third proportional line segments.* Let a and b be the lengths of two line segments, and we have to draw a line segment of length c such that the square of b equals the product of a and c.

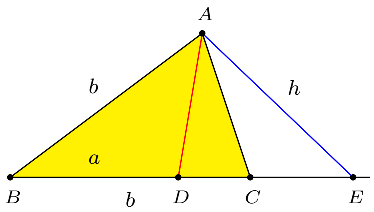

Figure 3

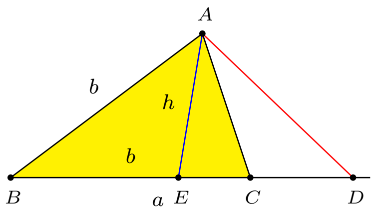

Figure 4

For this, construct an isosceles triangle ABC with $AB = BC = b$ (see Figures 3 and 4). Let D be a point on the ray BC such that $BD = a$ and let h be the line obtained by reflecting the ray AD in the line AC. By Lemma 1 the ray h cuts the ray BC in a point E. Theorem 2 implies $BE = c$.

3.2. *Construction of the inverse point with respect to a circle.* Consider a circle $\mathcal{C}$ with center B and D a point which may lies inside or outside of the circle $\mathcal{C}$. In both cases we can follow the same steps to construct the inverse point of D with respect to the circle $\mathcal{C}$, a case distinction as in the usual treatments, see for example [1, pp.108–109], is not needed.

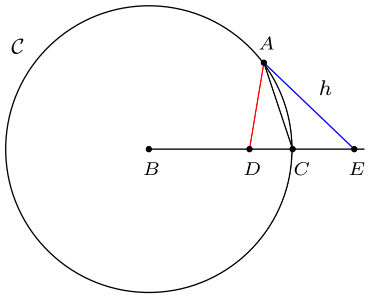

Figure 5

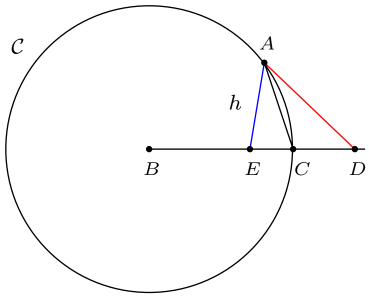

Figure 6

Take the intersection point C of the ray BD with the circle $\mathcal{C}$, see Figures 5 and 6. Connect the point C with an arbitrary point A on the circle $\mathcal{C}$ (different from C) and let h be the ray obtained by reflecting the ray AD in the line AC. The ray h cuts the ray BC in a point E by Lemma 1 which is the inverse point of D with respect to the circle $\mathcal{C}$ in view of Theorem 2.

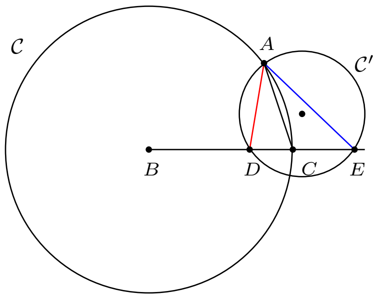

Figure 7

Note that the circumcircle $\mathcal{C}'$ of the triangle ADE is orthogonal to the circle $\mathcal{C}$, since it is invariant under the inversion at the circle $\mathcal{C}$ (see Figure 7).

References

[1] H. S. M. Coxeter and S. L. Greitzer, *Geometry revisited*, Random House, New York, 1967.
[2] T. L. Heath, *The Thirteen Books of Euclid's Elements*, 3 volumes, Dover reprint, 1956.
[3] E. W. Weisstein, *Inverse Points*, MathWorld – A Wolfram Web Resource, `http://mathworld.wolfram.com/InversePoints.html`

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