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Two Tangent Circles from Jigsawing Quadrangle

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Abstract. We establish the tangency of two circles associated with the jigsawing quadrangle of Floor van Lamoen.

Floor van Lamoen [1] has given a construction, for an acute angled triangle ABC, a pair of isotomic points QP, Q on BC such that when perpendiculars at P, Q to BC are constructed to intersect AB and AC at P' and Q' respectively, the quadrangle $PP'Q'Q$ satisfies $P'Q' = PP' + QQ'$ (see Figure 1). If the triangles BPP' and CQQ' are rotated about P' and Q' respectively, so that the images of P and Q coincide at a point on $P'Q'$, then the images of B and C coincide at a point A' such that the quadrangle $AP'A'Q'$ is cyclic. van Lamoen also showed that AA' passes through the circumcenter O of triangle ABC. Denote by S the center of the circle through the four points. We show that this circle is tangent at A' to another circle naturally associated with the triangle. Let the tangents at B and C to the circumcircle of ABC intersect at T.

Theorem. *The circles through A, P', A', Q' is tangent at A' to the circle, center T, passing through B and C.*

Proof. (1) We first show that A' also lies on the circle, center T, passing through B and C. Note that triangles $P'BA'$ and $Q'CA'$ are isosceles (see Figure 1). Therefore,

$$\begin{aligned}
\angle BA'C &= 360^\circ - \angle BA'P' - \angle P'A'Q' - \angle Q'A'C \\
&= 360^\circ - \frac{180^\circ - \angle BP'A'}{2} - (180^\circ - \angle BAC) - \frac{180^\circ - \angle CQ'A'}{2} \\
&= \frac{\angle BP'A' + \angle CQ'A'}{2} + \angle BAC \\
&= \frac{\angle AQ'A' + \angle CQ'A'}{2} + \angle BAC \\
&= 90^\circ + \angle BAC \\
&= 180^\circ - \frac{1}{2}\angle BTC.
\end{aligned}$$

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This shows that A' lies on the circle, center T, passing through B and C.

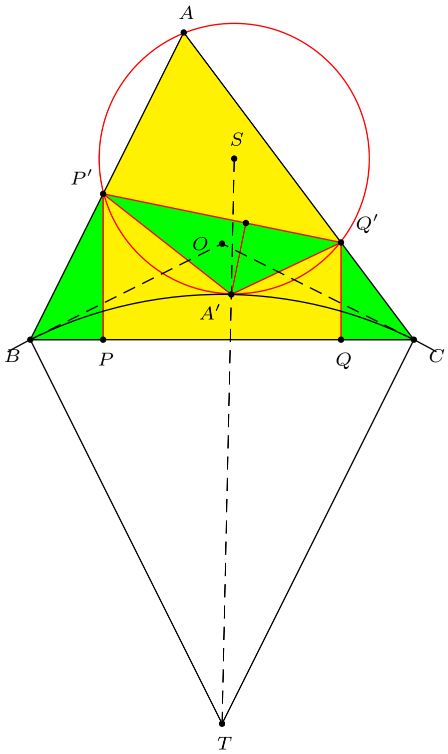

Figure 1.

(2) We show that the points S, A', and T' are collinear.

$$\begin{aligned} & \angle SA'Q' + \angle Q'A'C + \angle CA'T \\ = \; & (90^\circ - \angle A'P'Q') + \angle Q'CA' + (90^\circ - \angle A'BC) \\ = \; & 180^\circ - \angle BP'P + \angle ACB - (180^\circ - \angle BA'C) \\ = \; & 180^\circ - (90^\circ - \angle ABC) + \angle ACB - (90^\circ - \angle BAC) \\ = \; & \angle ABC + \angle ACB + \angle BAC \\ = \; & 180^\circ. \end{aligned}$$

Therefore, A' lies on the segment ST.

It follows that the two circles are tangent externally at A'. □

Reference

[1] F. M. van Lamoen, Jigsawing a quadrangle from a triangle, *Forum Geom.*, 13 (2013) 149–152.

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