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Two More Pairs of Archimedean Circles in the Arbelos

Tran Quang Hung

Abstract. We construct two more pairs of Archimedean circles in the arbelos. One of them is a pair constructed by Floor van Lamoen in another way.

In addition to the two pairs of Archimedean circles associated with the arbelos constructed by Dao Thanh Oai [1], we construct two more pairs. Given a segment AB with an interior point C, consider the semicircles (O), (O_1), (O_2) with diameters AB, AC, and CB, all on the same side of AB. The perpendicular to AB at C intersects (O) at D. Let a and b be the radii of the semicircles (O_1) and (O_2) respectively. The Archimedean circles have radii $\frac{ab}{a+b}$.

Theorem 1. *Let the perpendiculars to AB at O_1 and O_2 intersect (O) at E and F respectively. If AF intersects (O_1) at H and BE intersects (O_2) at K, then the circles tangent to CD with centers H and K are Archimedean circles.*

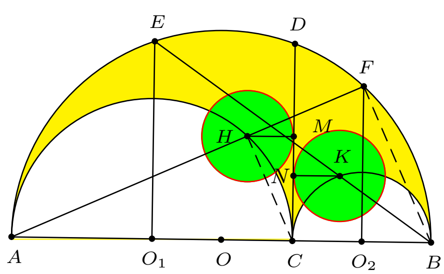

Figure 1

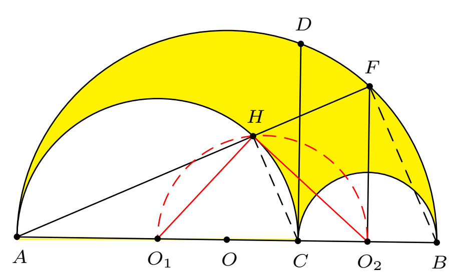

Figure 2

Proof. Let M and N be the orthogonal projections of H and K on CD respectively. Since CH and BF are both perpendicular to AF, the right triangles CHM and FBO_2 are similar (see Figure 1).

$$\frac{HM}{BO_2} = \frac{CH}{FB} = \frac{AC}{AB} \implies HM = BO_2 \cdot \frac{AC}{AB} = b \cdot \frac{2a}{2a+2b} = \frac{ab}{a+b}.$$

Therefore the circle $H(M)$ is Archimedean; similarly for $K(N)$. □

Floor van Lamoen has kindly pointed out that this pair has appeared before in a different construction, as (K_1) and (K_2) in [3] (see also (A_{25a}) and (A_{25b}) in [4]). We show that H and K are intersections of (O_1) and (O_2) with the mid-semicircle with diameter O_1O_2. It is enough to show that $\angle O_1HO_2 = \angle O_1KO_2 = 90°$.

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In Figure 2, O_2 is the midpoint of BC, and BF, CH are parallel. The parallel through O_2 to these lines is the perpendicular bisector of FH. This means that $O_2F = O_2H$, and

$$\begin{aligned}\angle O_1HO_2 &= 180^\circ - \angle O_1HA - \angle O_2HF \\ &= 180^\circ - \angle O_1AH - \angle O_2FH \\ &= \angle AO_2F = 90^\circ.\end{aligned}$$

Similarly, $\angle O_1KO_2 = 90^\circ$.

Theorem 2. *Let P be the intersection of AD with the semicircle with diameter AO_2, and Q that of BD with the semicircle with diameter BO_1. The circles tangent to CD with centers P and Q are Archimedean.*

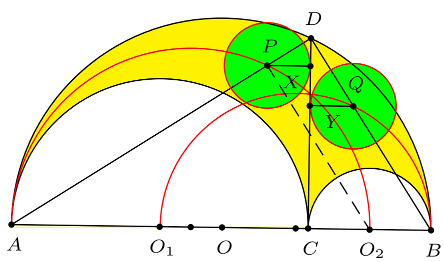

Figure 3

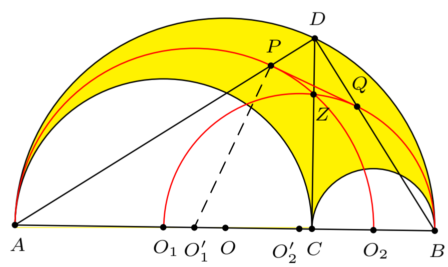

Figure 4

Proof. Let X and Y be the orthogonal projections of P and Q on CD (see Figure 3). Since BD and O_2P are both perpendicular to AD, they are parallel.

$$\frac{PX}{AC} = \frac{DP}{DA} = \frac{BO_2}{BA} \implies PX = AC \cdot \frac{BO_2}{BA} = 2a \cdot \frac{b}{2a+2b} = \frac{ab}{a+b}.$$

Therefore, the circle $P(X)$ is Archimedean; similarly for $Q(Y)$. □

We show that PQ is a common tangent to the semicircles with diameters AO_2 and BO_1 (see [5]). In Figure 4, these two semicircles intersect at a point Z on CD satisfying $CZ^2 = 2a \cdot b = a \cdot 2b$. Now, $DP \cdot DA = DZ(DC + ZC) = DQ \cdot DB$. It follows that $\frac{DP}{DQ} = \frac{DB}{DA}$, so that the right triangles DPQ and DBA are similar. Now, if O_1' is the midpoint of AO_2, then

$$\begin{aligned}\angle O_1'PQ &= 180^\circ - \angle O_1'PA - \angle DPQ \\ &= 180^\circ - \angle BAD - \angle DBA \\ &= \angle ADB = 90^\circ.\end{aligned}$$

Therefore, PQ is tangent to the semicircle on AO_2 at P. Similarly, it is also tangent to the semicircle on BO_1 at Q. It is a common tangent of the two semicircles.

References

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Tran Quang Hung: High school for Gifted students, Hanoi University of Science, Vietnam National University, Hanoi, Vietnam
E-mail address: `analgeomatica@gmail.com`