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Three Constructions of Archimedean Circles in an Arbelos

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Abstract. We give ruler and compass constructions of three Archimedean circles in an arbelos, each with the endpoints of a diameter on the smaller semicircles. In the first case, the diameter contains the intersection of the defining smaller semicircles of the arbelos. In the second case, these endpoints are the intersections of the smaller semicircles with the lines joining the endpoints of the base of the arbelos to a fixed point on the dividing perpendicular line. In the third case, the diameter containing these endpoints is parallel to the base line of the arbelos.

1. Introduction

We consider three constructions of Archimedean circles in an arbelos. Given a segment AB with an interior point C, the semicircles (O), (O_1), (O_2) with diameters AB, AC, CB on the same side of AB bound the arbelos, with dividing line CD perpendicular to AB. Let a and b be the radii of the semicircles (O_1) and (O_2). Circles with radius $t := \frac{ab}{a+b}$ are called Archimedean. They are congruent to the Archimedean twin circles [1, 2, 3]. We shall make use of the Archimedean circles with centers O_1 and O_2 respectively.

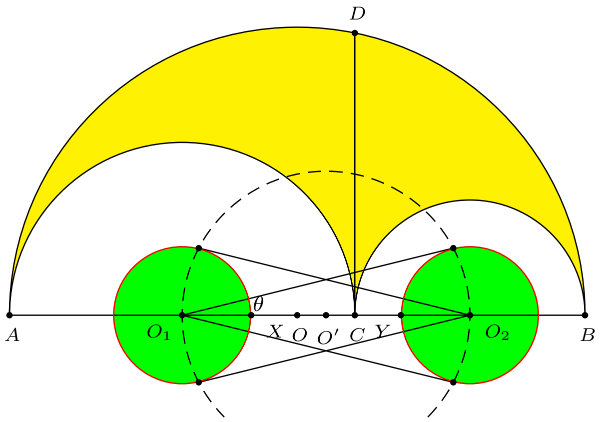

Figure 1.

In particular, we shall encounter below lines making an angle θ with AB defined by

$$\sin\theta = \frac{t}{a+b}. \tag{1}$$

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Such lines are parallel to the tangents from O_1 to the Archimedean circle with center O_2 or vice versa. The points of tangency are the intersections with the circle with diameter O_1O_2 and center O' (see Figure 1). We adopt a Cartesian coordinate system with origin at C, so that the points A and B have coordinates $(-2a, 0)$ and $(2b, 0)$ respectively. The equations of the circles (O_1) and (O_2) are

$$\begin{aligned}(x+a)^2+y^2 &= a^2,\\(x-b)^2+y^2 &= b^2.\end{aligned}$$

2. Archimedean circles with diameter through C and endpoints on (O_1) and (O_2)

Consider the construction of a line $\mathscr{L}$ through C intersecting the circles (O_1) and (O_2) at A' and B' respectively so that the segment $A'B'$ has length $2t$, and the circle with diameter $A'B'$ is Archimedean. If $\mathscr{L}$ has slope m, then these intersections are $A' = \left(\frac{-2a}{1+m^2}, \frac{-2am}{1+m^2}\right)$ and $B' = \left(\frac{2b}{1+m^2}, \frac{2bm}{1+m^2}\right)$. Since the difference between the x-coordinates is $\frac{2(a+b)}{1+m^2}$, $A'B'^2 = \frac{4(a+b)^2}{1+m^2}$. This is equal to $(2t)^2$ if and only if

$$1+m^2 = \frac{(a+b)^4}{a^2b^2} = \frac{(a+b)^2}{t^2} = \csc^2\theta$$

for the angle θ defined by (1). It follows that the slope $m = \pm\cot\theta$, and the line $\mathscr{L}$ is perpendicular to a tangent from O_1 to the Archimedean circle at O_2.

Theorem 1. *A line through C intersecting (O_1) and (O_2) at the endpoints of a diameter of an Archimedean circle if and only if it is perpendicular to a tangent from O_1 to the Archimedean circle with center O_2.*

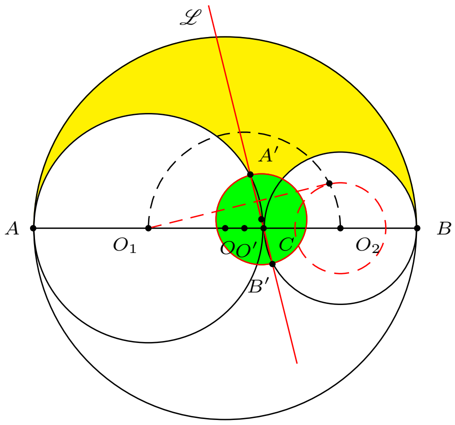

Figure 2(a)

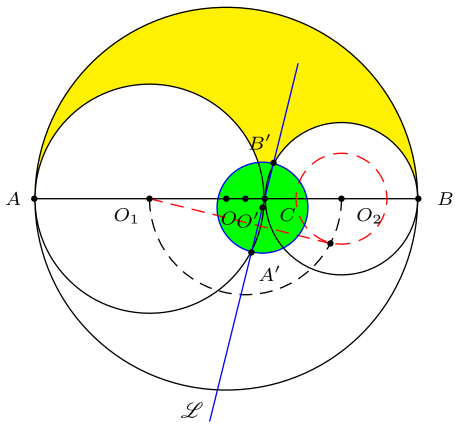

Figure 2(b)

3. Archimedean circle from intersections of QA, QB with Q on CD

We construct a point Q on the line CD such that the intersections of AQ with (O_1) and BQ with (O_2) are the endpoints of a diameter of an Archimedean circle (see Figure 3). Let $Q = (0, q)$. These intersections are $A'' = \left(\frac{-2aq^2}{q^2+4a^2}, \frac{4a^2q}{q^2+4a^2}\right)$ and $B'' = \left(\frac{2bq^2}{q^2+4b^2}, \frac{4b^2q}{q^2+4b^2}\right)$. From these coordinates,

$$A''B''^2 = \frac{4(a+b)^2q^4}{(q^2+4a^2)(q^2+4b^2)}.$$

This is equal to $(2t)^2$ if and only if $(a+b)^4q^4 - a^2b^2(q^2+4a^2)(q^2+4b^2) = 0$. Rewriting this as

$$((a+b)^4 - a^2b^2)q^4 - 4a^2b^2(a^2+b^2)q^2 - 16a^4b^4 = 0, \tag{2}$$

we see that there is a unique positive root.

Theorem 2. *There is a unique point Q on the dividing line CD such that the intersections of QA with (O_1) and QB with (O_2) are the endpoints of a diameter of an Archimedean circle.*

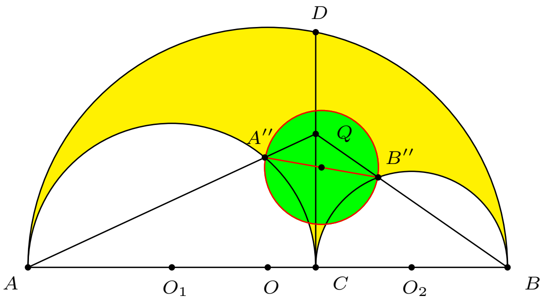

Figure 3.

From (2), we obtain explicitly

$$q^2 = \frac{a^2b^2}{(a+b)^4 - a^2b^2}\left(2(a^2+b^2) + 2(a+b)\sqrt{(a-b)^2 + 4(a+b)^2}\right).$$

Now, $\frac{a^2b^2}{(a+b)^4-a^2b^2} = \frac{t^2}{(a+b)^2-t^2} = \tan^2\theta$ for θ defined by (1). It is enough to construct a segment CX on AB with

$$CX^2 = 2(a^2+b^2) + 2(a+b)\sqrt{(a-b)^2 + 4(a+b)^2}. \tag{3}$$

Let M be the "highest" point of (O), i.e., the intersection of (O) with the perpendicular to AB at O (see Figure 4). For the construction of X, we make use of the following.

(i) $2(a^2+b^2) = (a+b)^2 + (a-b)^2 = OM^2 + OC^2 = CM^2$,

(ii) $(a-b)^2 + 4(a+b)^2 = 4(O'O^2 + OM^2) = 4 \cdot O'M^2$.

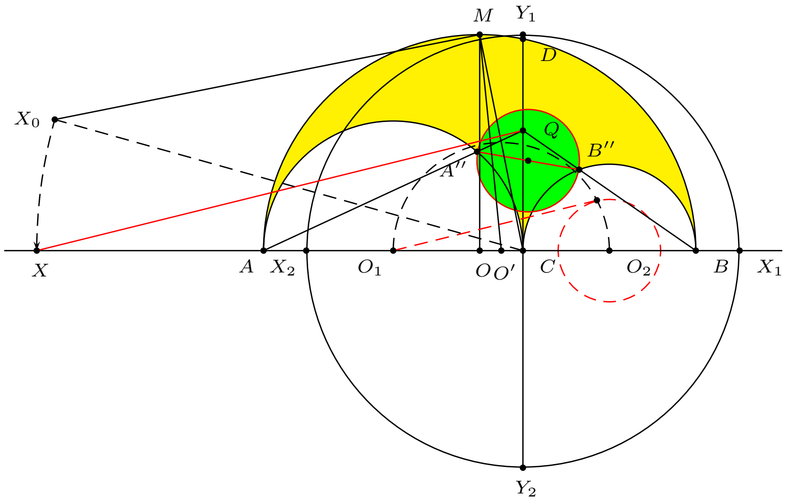

Figure 4.

Construction. (1) On different sides of AB on the perpendicular at C, choose points Y_1 and Y_2 such that $CY_1 = OM$ and $CY_2 = O'M$. Construct the circle with diameter Y_1Y_2, to intersect the line AB in a segment X_1X_2.

(2) On the perpendicular to MC at M, choose a point X_0 such that $MX_0 = X_1X_2$.

(3) Let X be a point on AB such that $CX = CX_0$. The segment CX has length given by (3) above.

(4) Construct a parallel through X to a tangent from O_1 to the Archimedean circle with center O_2, to intersect CD at Q. This is the unique point Q in Theorem 2.

4. Archimedean circle with a diameter parallel to AB and endpoints on (O_1) and (O_2)

We consider the possibility of an Archimedean circle with a diameter parallel to AB having its endpoints one on each of the semicircles (O_1) and (O_2). If the diameter is at a distance d from AB, its endpoints are among the points

$$\begin{array}{ll} X_- = (-a - \sqrt{a^2 - d^2}, d), & X_+ = (-a + \sqrt{a^2 - d^2}, d); \\ Y_- = (b - \sqrt{b^2 - d^2}, d), & Y_+ = (b + \sqrt{b^2 - d^2}, d). \end{array}$$

The differences between the lengths of the various segments and the diameter of an Archimedean circle are

$$\begin{aligned} X_-Y_+ - 2t &= a + b - 2t + \sqrt{a^2 - d^2} + \sqrt{b^2 - d^2}, \\ X_-Y_- - 2t &= a + b - 2t + \sqrt{a^2 - d^2} - \sqrt{b^2 - d^2}, \\ X_+Y_+ - 2t &= a + b - 2t - \sqrt{a^2 - d^2} + \sqrt{b^2 - d^2}, \\ X_+Y_- - 2t &= a + b - 2t - \sqrt{a^2 - d^2} - \sqrt{b^2 - d^2}. \end{aligned}$$

The condition

$$(X_-Y_+ - 2t)(X_-Y_- - 2t)(X_+Y_+ - 2t)(X_+Y_- - 2t) = 0$$

simplifies into

$$4t(a-t)(b-t)(a+b-t) - (a+b-2t)^2 d^2 = 0. \tag{4}$$

This clearly has a unique positive root d.

Theorem 3. *There is a unique Archimedean circle with a diameter parallel to AB, having endpoints one on each of the semicircles (O_1) and (O_2).*

Now, by Heron's formula, $t(a-t)(b-t)(a+b-t)$ is the square of the area of a triangle with sides a, b, and $a+b-2t$. From (4), d is the altitude of the triangle on the side $a+b-2t$. This leads to the following simple construction.

Construction. Let the Archimedean circle with center O_1 intersect O_1C at X and that with center O_2 intersect CO_2 at Y. Construct a point Z (on the same side of the arbelos) such that $XZ = a$ and $YZ = b$. The parallel to AB through Z is the line which intersects (O_1) and (O_2) at two points at a distance $2t$ apart.

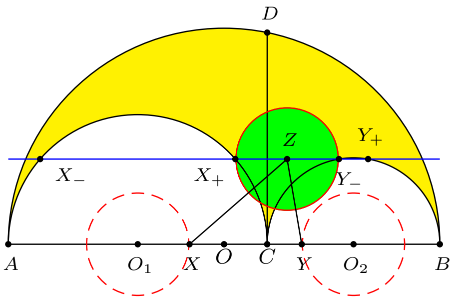

Figure 5a: $a^3 < a^2b + ab^2 + b^3$

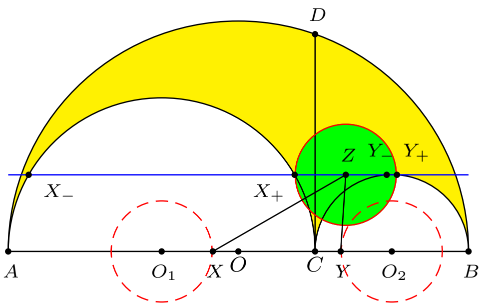

Figure 5b: $a^3 > a^2b + ab^2 + b^3$

The point Z is indeed the center of the Archimedean circle in question.

Assume $a \geq b$ without loss of generality. Note that $O_1X_+ = XZ = a$, and they are parallel since

$$\sin X_+O_1C = \frac{d}{a} = \sin ZXY.$$

This means that O_1XZX_+ is a parallelogram, and $ZX_+ = XO_1 = t$. The circle, center Z, passing through X_+ is Archimedean. The other end of the the diameter is Y_- or Y_+ according as a^3 is less than or greater than $a^2b + ab^2 + b^3$ (see Figures 5a and 5b). This follows from the simple fact

$$\sqrt{b^2 - d^2} = \begin{cases} \frac{-a^3b + a^2b^2 + ab^3 + b^4}{(a+b)(a^2+b^2)} & \text{if } a^3 < a^2b + ab^2 + b^3, \\ \frac{a^3b - a^2b^2 - ab^3 - b^4}{(a+b)(a^2+b^2)}, & \text{if } a^3 > a^2b + ab^2 + b^3. \end{cases}$$

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