A Purely Synthetic Proof of Dao’s Theorem on Six Circumcenters Associated with a Cyclic Hexagon

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Abstract. We present a purely synthetic proof of Dao’s theorem on six circumcenters associated with a cyclic hexagon.

Nikolaos Dergiades [4] has given an elegant proof using complex numbers of the following theorem.

Theorem (Dao [2]). Let six points $A$, $B$, $C$, $D$, $E$, $F$ lie on a circle, and $U = AF \cap BC$, $V = AB \cap CD$, $W = BC \cap DE$, $X = CD \cap EF$, $Y = DE \cap FA$, $Z = EF \cap AB$. Denote by $O_1$, $O_2$, $O_3$, $O_4$, $O_5$, $O_6$ the circumcenters of the six triangles $ABU$, $BCV$, $CDW$, $DEX$, $EFY$, $FAZ$. The three lines $O_1O_4$, $O_2O_5$, $O_3O_6$ are concurrent.

In this note we present a purely synthetic proof.

Lemma 1. Let $A$, $B$, $C$, $A'$, $B'$, $C'$ be six points (in cyclic order) on a circle $(O)$, and $X = AB \cap A'C'$, $X' = A'B' \cap AC'$. Let $O_1$, $O_1'$ be the circumcenters of $(XBC)$, $(X'B'C')$ respectively. The lines $O_1O_1'$, $BB'$, $CC'$ are concurrent (see Figure 2).
Lemma 1. The points \( O \) and \( P \) have \((\text{the triangles } XY, X'Y') \) of \((\text{the circles})\). Clearly, \( XY, X'Y' \) are parallel, and \( XX', YY' \) intersect at a point \( P \) that divides these segments in the ratio of the radii of the circles. Clearly, \( P \) also lies on the segment \( O_1O'_1 \). The lines \( XB, X'B' \) and their perpendiculars \( YB, Y'B' \) meet at the points \( Z', Z \) respectively. If \( AA_0 \) is a diameter of \((O)\), then \( A'A_0 \perp A'A \). Since the points \( Z, B, B', Z' \) are concyclic, we have \( ZZ'||A'A_0 \) because they are both antiparallels to \( BB' \) relative to \( A_0Z, A'Z' \). Hence \( XY'||X'Y'||ZZ' \), and are perpendicular to \( A'A \). By Desargues’ theorem, the triangles \((XYB)\) and \((X'Y'B')\) are perspective. Hence, \( BB' \) passes though \( P \). Similarly we prove that \( CC' \) passes through \( P \). \( \square \)

We reformulate and prove Dao’s theorem in the following form.

**Theorem 2.** Divide a circle in six consecutive arcs \( c_2, a_1, b_2, c_1, a_2, b_1 \) with the arbitrary points \( A, B, C, A', B', C' \). Let the chords of the arcs \( a_2, b_2, c_2 \) bound a triangle \( A_1B_1C_1 \), and those of the arcs \( a_1, b_1, c_1 \) bound a triangle \( A_2B_2C_2 \). If \( O_1, O_1', O_2, O_2', O_3, O_3' \) are the circumcenters of the circles \((A_1BC), (A_2B'C'), (B_1C'A), (B_2CA'), (C_1A'B'), (C_2AB)\) respectively, then the lines \( O_1O_1', O_2O_2', O_3O_3' \) are concurrent (see Figure 3).

**Proof.** Let \( A_3 = BB' \cap CC', B_3 = CC' \cap AA', \) and \( C_3 = AA' \cap BB' \). By Lemma 1 the points \( A_3, B_3, C_3 \) lie on the lines \( O_1O_1', O_2O_2', O_3O_3' \) respectively. Denote

\[
\angle O_1B_3A = A_b, \quad \angle O_2C_3B_3 = B_c, \quad \angle O_3A_3C_3 = C_a,
\]

\[
\angle O_1C_3A_3 = A_c, \quad \angle O_2A_3B_3 = B_a, \quad \angle O_3B_3C_3 = C_b.
\]
We have \( A_b = \angle O_1 BC + \angle CBA_3 = 90^\circ - \angle CA_1 B + \angle CBB' \) or
\[
A_b = 90^\circ - \frac{a_2 + b_1 + c_1 - a_1}{2} + \frac{b_2 + c_1}{2} = 90^\circ + \frac{a_1 + b_2 - a_2 - b_1}{2}.
\]
Similarly,
\[
B_a = 90^\circ - \frac{b_2 + c_1 + a_1 - b_1}{2} + \frac{a_2 + c_1}{2} = 90^\circ - \frac{a_1 + b_2 - a_2 - b_1}{2}.
\]
From these, \( A_b + B_a = 180^\circ \), and \( \sin A_b = \sin B_a \). Similarly, \( \sin B_c = \sin C_b \) and \( \sin C_a = \sin A_c \).

Consider \( O_1 A_3 O'_1 \), \( O_2 B_3 O'_2 \), and \( O_3 C_3 O'_3 \) as lines through the vertices of triangle \( A_3 B_3 C_3 \). Let \( R_1 \) be the radius of the circle \( (O_1) \). Since
\[
\frac{\sin A_b}{\sin C_3 A_3 O'_1} = \frac{\sin A_b}{\sin B A_3 O_1} = \frac{O_1 A_3}{R_1} = \frac{\sin A_c}{\sin O_1 A_3 C} = \frac{\sin A_c}{\sin O'_1 A_3 B_3},
\]
we have \( \frac{\sin C_3 A_3 O'_1}{\sin O'_1 A_3 B_3} = \frac{\sin A_b}{\sin A_c} \). Similarly, \( \frac{\sin A_3 B_3 O'_2}{\sin O'_2 B_3 C_3} = \frac{\sin B_c}{\sin B_a} \), and \( \frac{\sin B_3 C_3 O'_3}{\sin O'_3 C_3 A_3} = \frac{\sin C_a}{\sin C_b} \). Therefore,
\[
\frac{\sin C_3 A_3 O'_1}{\sin O'_1 A_3 B_3} \cdot \frac{\sin A_3 B_3 O'_2}{\sin O'_2 B_3 C_3} \cdot \frac{\sin B_3 C_3 O'_3}{\sin O'_3 C_3 A_3} = \frac{\sin A_b}{\sin A_c} \cdot \frac{\sin B_c}{\sin B_a} \cdot \frac{\sin C_a}{\sin C_b} = 1.
\]
By the converse of Ceva’s theorem, we conclude that the lines \( O_1 O'_1, O_2 O'_2, O_3 O'_3 \) are concurrent. \( \square \)
References


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