Forum Geometricorum
Volume 14 (2014) 261–264.

FORUM GEOM
ISSN 1534-1178

A Purely Synthetic Proof of Dao's Theorem on Six Circumcenters Associated with a Cyclic Hexagon

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Abstract. We present a purely synthetic proof of Dao's theorem on six circumcenters associated with a cyclic hexagon.

Nikolaos Dergiades [4] has given an elegant proof using complex numbers of the following theorem.

Theorem (Dao [2]). *Let six points A, B, C, D, E, F lie on a circle, and $U = AF \cap BC$, $V = AB \cap CD$, $W = BC \cap DE$, $X = CD \cap EF$, $Y = DE \cap FA$, $Z = EF \cap AB$. Denote by O_1, O_2, O_3, O_4, O_5, O_6 the circumceneters of the six triangles ABU, BCV, CDW, DEX , EFY, FAZ. The three lines O_1O_4, O_2O_5, O_3O_6 are concurrent.*

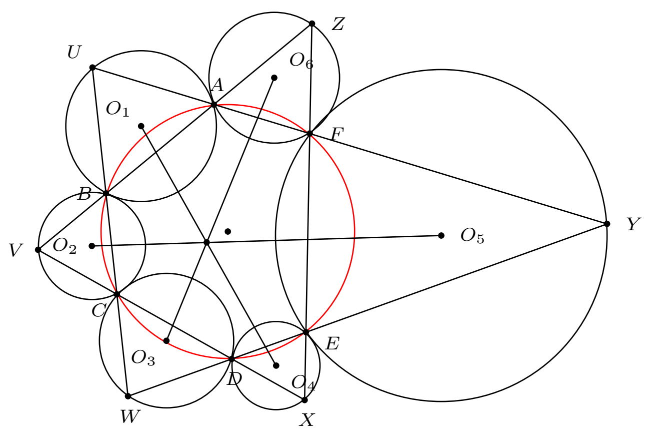

Figure 1

In this note we present a purely synthetic proof.

Lemma 1. *Let A, B, C, A', B', C' be six points* (in cyclic order) *on a circle (O), and $X = AB \cap A'C$, $X' = A'B' \cap AC'$. Let O_1, O_1' be the circumcenters of (XBC), $(X'B'C')$ respectively. The lines O_1O_1', BB', CC' are concurrent* (see Figure 2).

Publication Date: September 30, 2014. Communicating Editor: Nikolaos Dergiades.
The author thanks Messrs. Dao Thanh Oai and Nikolaos Dergiades for their help in the improvement of this paper.

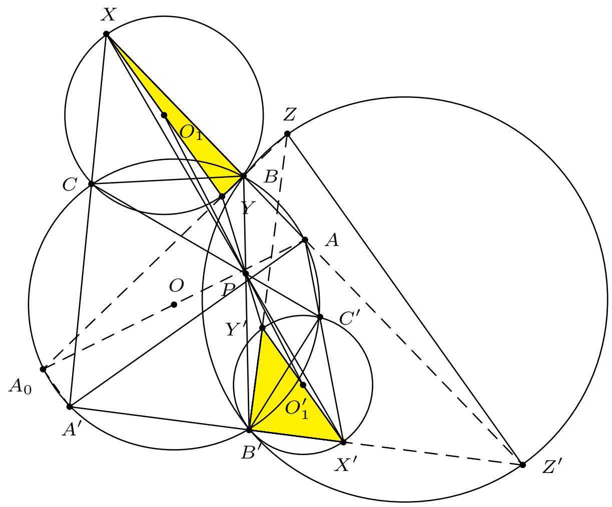

Figure 2

Proof. Since the triangles XBC and $XA'A$ are inversely similar, the diameter XY of (O_1) is an altitude of triangle $XA'A$. Similarly, the diameter $X'Y'$ of (O_1') is an altitude of triangle $XA'A$. Hence, XY and $X'Y'$ are parallel, and XX', YY' intersect at a point P that divides these segments in the ratio of the radii of the circles. Clearly, P also lies on the segment O_1O_1'. The lines $XB, X'B'$ and their perpendiculars $YB, Y'B'$ meet at the points Z', Z respectively. If AA_0 is a diameter of (O), then $A'A_0 \perp A'A$. Since the points Z, B, B', Z' are concyclic, we have $ZZ'||A'A_0$ because they are both antiparallels to BB' relative to A_0Z, $A'Z'$. Hence $XY||X'Y'||ZZ'$, and are perpendicular to $A'A$. By Desargues' theorem, the triangles (XYB) and $(X'Y'B')$ are perspective. Hence, BB' passes though P. Similarly we prove that CC' passes through P. □

We reformulate and prove Dao's theorem in the following form.

Theorem 2. *Divide a circle in six consecutive arcs c_2, a_1, b_2, c_1, a_2, b_1 with the arbitrary points A, B, C, A', B', C'. Let the chords of the arcs a_2, b_2, c_2 bound a triangle $A_1B_1C_1$, and those of the arcs a_1, b_1, c_1 bound a triangle $A_2B_2C_2$. If O_1, O_1', O_2, O_2', O_3, O_3' are the circumcenters of the circles (A_1BC), $(A_2B'C')$, $(B_1C'A)$, (B_2CA'), $(C_1A'B')$, (C_2AB) respectively, then the lines O_1O_1', O_2O_2', O_3O_3' are concurrent* (see Figure 3).

Proof. Let $A_3 = BB' \cap CC'$, $B_3 = CC' \cap AA'$, and $C_3 = AA' \cap BB'$. By Lemma 1 the points A_3, B_3, C_3 lie on the lines O_1O_1', O_2O_2', O_3O_3' respectively. Denote

$$\angle O_1BA_3 = A_b, \qquad \angle O_2C'B_3 = B_c, \qquad \angle O_3A'C_3 = C_a,$$
$$\angle O_1CA_3 = A_c, \qquad \angle O_2AB_3 = B_a, \qquad \angle O_3B'C_3 = C_b.$$

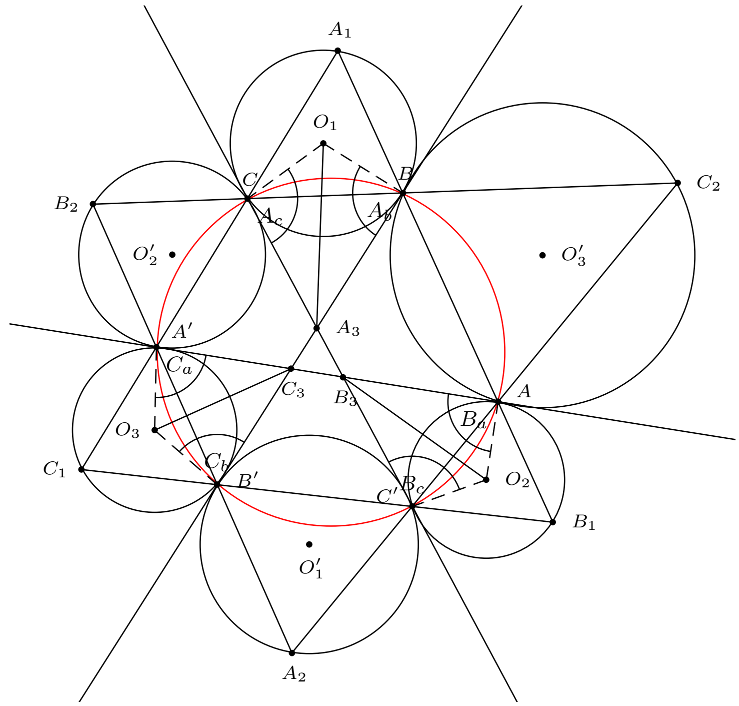

Figure 3

We have $A_b = \angle O_1BC + \angle CBA_3 = 90^o - \angle CA_1B + \angle CBB'$ or

$$A_b = \ 90^\circ - \frac{a_2 + b_1 + c_1 - a_1}{2} + \frac{b_2 + c_1}{2} = 90^\circ + \frac{a_1 + b_2 - a_2 - b_1}{2}.$$

Similarly,

$$B_a = \ 90^\circ - \frac{b_2 + c_1 + a_1 - b_1}{2} + \frac{a_2 + c_1}{2} = 90^\circ - \frac{a_1 + b_2 - a_2 - b_1}{2}.$$

From these, $A_b + B_a = 180^\circ$, and $\sin A_b = \sin B_a$. Similarly, $\sin B_c = \sin C_b$ and $\sin C_a = \sin A_c$.

Consider $O_1A_3O'_1$, $O_2B_3O'_2$, and $O_3C_3O'_3$ as lines through the vertices of triangle $A_3B_3C_3$. Let R_1 be the radius of the circle (O_1). Since

$$\frac{\sin A_b}{\sin C_3A_3O'_1} = \frac{\sin A_b}{\sin BA_3O_1} = \frac{O_1A_3}{R_1} = \frac{\sin A_c}{\sin O_1A_3C} = \frac{\sin A_c}{\sin O'_1A_3B_3},$$

we have $\frac{\sin C_3A_3O'_1}{\sin O'_1A_3B_3} = \frac{\sin A_b}{\sin A_c}$. Similarly, $\frac{\sin A_3B_3O'_2}{\sin O'_2B_3C_3} = \frac{\sin B_c}{\sin B_a}$, and $\frac{\sin B_3C_3O'_3}{\sin O'_3C_3A_3} = \frac{\sin C_a}{\sin C_b}$. Therefore,

$$\frac{\sin C_3A_3O'_1}{\sin O'_1A_3B_3} \cdot \frac{\sin A_3B_3O'_2}{\sin O'_2B_3C_3} \cdot \frac{\sin B_3C_3O'_3}{\sin O'_3C_3A_3} = \frac{\sin A_b}{\sin A_c} \cdot \frac{\sin B_c}{\sin B_a} \cdot \frac{\sin C_a}{\sin C_b} = 1.$$

By the converse of Ceva's theorem, we conclude that the lines $O_1O'_1$, $O_2O'_2$, $O_3O'_3$ are concurrent. □

References

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