Forum Geometricorum
Volume 14 (2014) 265–294.

FORUM GEOM
ISSN 1534-1178

The Triangle of Reflections

Jesus Torres

Abstract. This paper presents some results in triangle geometry discovered with the aids of a dynamic software, namely, the Geometer's Sketchpad, and confirmed with computations using Mathematica 9.0. With the method of barycentric coordinates, we study geometric problems associated with the triangle of reflections $\mathbf{T}^\dagger$ of a given triangle $\mathbf{T}$ (obtained by reflecting the vertices in their opposite sides), resulting in interesting triangle centers and simple loci such as circles and conics. These lead to some new triangle centers with reasonably simple coordinates, and also new properties of some known, classical centers. In particular, we show that the Parry reflection point (reflection of circumcenter in the Euler reflection point) is the common point of two triads of circles, one associated with the tangential triangle, and another with the excentral triangle. More interestingly, we show that a certain rectangular hyperbola through the vertices of $\mathbf{T}^\dagger$ appears as the locus of the perspector of a family of triangles perspective with $\mathbf{T}^\dagger$, and in a different context as the locus of the orthology center of $\mathbf{T}^\dagger$ with another family of triangles.

1. Introduction

This paper is a revision of the author's master thesis [14]. We present some results in triangle geometry discovered with the aids of a dynamic software, namely, the Geometer's Sketchpad©, and confirmed with computations using Mathematica 9.0. With the method of barycentric coordinates, we study geometric problems associated with the triangle of reflections $\mathbf{T}^\dagger$ of a given triangle $\mathbf{T}$ (obtained by reflecting the vertices in their opposite sides). We use the notations and basic formulas in triangle geometry as presented in [15]. In particular, coordinates of triangle centers are expressed in the Conway notation, so as to reduce the degrees of polynomials involved. We obtain a number of interesting triangle centers with reasonably simple coordinates, and also new properties of some known, classical centers.

1.1. *Summary.* Let $\mathbf{T}$ be a given triangle. The triangle of reflections $\mathbf{T}^\dagger$ is the one whose vertices are the reflections of the vertices of $\mathbf{T}$ in their opposite sides. This is introduced in CHAPTER 2. Propositions 2.1 and 2.2 explain the significance of the nine-point center of $\mathbf{T}$ in the geometry of $\mathbf{T}^\dagger$. The homogeneous barycentric coordinates of a few classical centers on the Euler line are computed. While the calculations for the centroid and the circumcenter are easy (Proposition 2.4), the

Publication Date: October 7, 2014. Communicating Editor: Paul Yiu.

coordinates of the orthocenter and nine-point center can only be computed with the aids of Mathematica. These two centers will feature in CHAPTER 7.

CHAPTERS 3 and 4 give a number of simple results related to perspectivity and orthology with $\mathbf{T}^\dagger$. In §4.1, we give a simple computational proof of Sondat's theorem (Theorem 4.1) which states that if two nondegenerate triangles are both perspective and orthologic, then the perpector and the two orthology centers are collinear. This applies to $\mathbf{T}^\dagger$ and the orthic triangle of $\mathbf{T}$ (Theorem 4.3). The line containing these centers has a remarkably simple equation. This line also appears as a locus discussed in §7.2. Also, the orthology center $\mathsf{cev}(H)^\perp(\mathbf{T}^\dagger)$ in Theorem 4.3 is a new center which reappears in a number of places in later chapters.

In CHAPTERS 5 and 6 we construct a number of circles associated with $\mathbf{T}^\dagger$. In §5.3 we construct a triad of circles in relation to the tangential triangle of $\mathbf{T}$, and show that they are concurrent at the Parry reflection point (which is the reflection of the circumcenter of $\mathbf{T}$ in its Euler reflection point). Another triad of circles is constructed in §5.4, this time in connection with the excentral triangle of $\mathbf{T}$. This triad of circles are also concurrent at the same Parry reflection point. A new Tucker circle (through the pedals of the vertices of $\mathbf{T}^\dagger$ on the sidelines of $\mathbf{T}$) is constructed in §6.3. The center of this circle bears a very simple relationship with the Parry reflection point and the Hatzipolakis reflection point in §6.1.

In CHAPTER 7 we present two locus problems related to $\mathbf{T}^\dagger$ and resulting in conic loci, which can be easily identified as rectangular hyperbolas. Specifically, we show that the rectangular circum-hyperbola through the vertices of $\mathbf{T}^\dagger$ and the orthocenter of $\mathbf{T}$ arises as the locus of the perspector of a family of triangles perspective with $\mathbf{T}^\dagger$ (Theorem 7.1), and also as the locus of the orthology center of $\mathbf{T}^\dagger$ with another family of triangles (Theorem 7.4(a)). Some of the triangle centers and lines constructed in earlier chapters also feature in the solutions of the loci problems discussed in this chapter.

Appendix A lists a number of triangle centers catalogued in ETC [7] that feature in this paper with properties related to $\mathbf{T}^\dagger$. Appendix B is a summary of new triangle centers appearing in this thesis, listed in order of their search numbers in ETC.

2. The triangle of reflections and the nine-point center

Given a reference $\mathbf{T} := ABC$, consider the reflections of the vertices in the respective opposite sides. In homogeneous barycentric coordinates, these are the points

$$\begin{aligned}A' &= (-(S_B+S_C) : 2S_C : 2S_B),\\ B' &= (2S_C : -(S_C+S_A) : 2S_A),\\ C' &= (2S_B : 2S_A : -(S_A+S_B)).\end{aligned}$$

The triangle $\mathbf{T}^\dagger := A'B'C'$ is called the *triangle of reflections* of $\mathbf{T}$. It is the main object of study of this paper.

2.1. *Perspectivity with* $\mathbf{T}$. Clearly, $\mathbf{T}$ and $\mathbf{T}^\dagger$ are perspective at the orthocenter H.

Proposition 2.1. *The perspectrix of* $\mathbf{T}$ *and* $\mathbf{T}^\dagger$ *is the trilinear polar of the nine-point center* N.

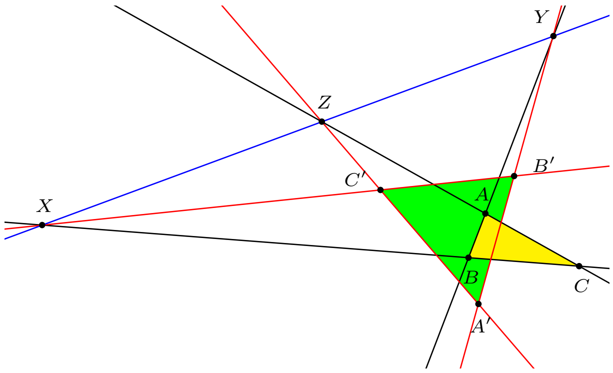

Figure 2.1

Proof. The equation of the line $B'C'$ is

$$(-3S_{AA}+S^2)x+2(S^2+S_{AB})y+2(S^2+S_{CA})z=0.$$

It is clear that $B'C'\cap BC=(0:-(S^2+S_{CA}):S^2+S_{AB})$. The equations of the lines $C'A'$ and $A'B'$, and their intersections with the corresponding sidelines, can be written down easily by cyclic permutations of parameters. These are

$$\begin{aligned} C'A'\cap CA= & \ (S^2+S_{BC}:0:-(S^2+S_{AB})),\\ A'B'\cap AB= & \ (-(S^2+S_{BC}):S^2+S_{CA}:0). \end{aligned}$$

The line containing these points is

$$\frac{x}{S^2+S_{BC}}+\frac{y}{S^2+S_{CA}}+\frac{z}{S^2+S_{AB}}=0,$$

the trilinear polar of the nine-point center N. □

2.2. *Homothety between* $\mathbf{T}^\dagger$ *and the reflection triangle of* N.

Proposition 2.2. *The triangle of reflections* $\mathbf{T}^\dagger$ *is the image of the reflection triangle of the nine-point center* N *under the homothety* $\mathsf{h}(O,2)$.

Proof. If D is the midpoint of BC, it is well known that $2\cdot OD=AH=H_a^\dagger A'$ (see Figure 2.2). If $N_a^\dagger$ is the reflection of N in BC, then

$$NN_a^\dagger = OD+HH_a=\frac{1}{2}(2\cdot OD+2HH_a)=\frac{1}{2}(HH_a+H_aH_a^\dagger+H_a^\dagger A')=\frac{1}{2}HA'.$$

Since N is the midpoint of OH, it follows that $N_a^\dagger$ is the midpoint of OA'.

A similar reasoning shows that $N_b^\dagger$ and $N_c^\dagger$ are the midpoints of OB' and OC'. □

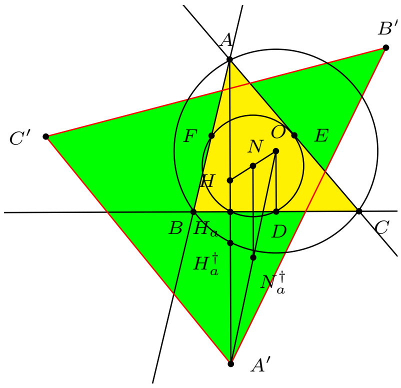

Figure 2.2

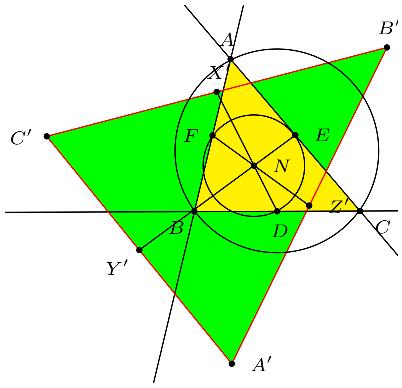

Figure 2.3

Proposition 2.3. *The medial triangles of* $\mathbf{T}$ *and* $\mathbf{T}^{\dagger}$ *are perspective at* N.

Proof. The midpoint of $B'C'$ is the point

$$\begin{aligned}
X' &= \frac{1}{2}(B' + C') \\
&= \frac{1}{2}\left(\frac{(2S_C, -(S_C+S_A), 2S_A)}{S_C+S_A} + \frac{(2S_B, 2S_A, -(S_A+S_B))}{S_A+S_B}\right) \\
&= \frac{(2(S^2+S_{BC}), (S_A-S_B)(S_C+S_A), (S_A-S_C)(S_A+S_B))}{2(S_C+S_A)(S_A+S_B)}.
\end{aligned}$$

In homogeneous barycentric coordinates, this is

$$X' = (2(S^2+S_{BC}) : (S_A-S_B)(S_C+S_A) : (S_A-S_C)(S_A+S_B)).$$

The line joining X' to the midpoint of BC has equation

$$\begin{vmatrix} 2(S^2+S_{BC}) & (S_A-S_B)(S_C+S_A) & (S_A-S_C)(S_A+S_B) \\ 0 & 1 & 1 \\ x & y & z \end{vmatrix} = 0,$$

or

$$S_A(S_B-S_C)x + (S^2+S_{BC})(y-z) = 0.$$

This line clearly contains the nine-point center N, since

$$S_A(S_B-S_C)(S^2+S_{BC}) + (S^2+S_{BC})((S^2+S_{CA}) - (S^2+S_{AB})) = 0.$$

Similarly, the lines joining the midpoints of $C'A'$ and $A'B'$ to those of CA and AB also contain N. We conclude that the two medial triangles are perspective at N (see Figure 2.3). $\square$

2.3. *The Euler line of* $\mathbf{T}^\dagger$.

Proposition 2.4. (a) *The centroid, circumcenter, and orthocenter of* $\mathbf{T}^\dagger$ *are the points*

$$\begin{aligned}
G' =\ & (a^2(S_{AA} - S_A(S_B + S_C) - 3S_{BC}) : \cdots : \cdots),\\
O' =\ & (a^2(-3S_A^3(S_B + S_C) + S_{AA}(5S_{BB} + 6S_{BC} + 5S_{CC})\\
& + 9S_{ABC}(S_B + S_C) + 4S_{BB}S_{CC})) : \cdots : \cdots),\\
H' =\ & (a^2(2(S_A + S_B + S_C)S^6 + S_{BC}(2(-S_A + 4S_B + 4S_C)S^4\\
& + (S_{AA} - 3S^2)((7S_A + 5S_B + 5S_C)S^2 + S_{ABC}))) : \cdots : \cdots),\\
N' =\ & (a^2(3a^4S_A^5 + 2a^2(a^4 + 9S_{BC})S_A^4 - (a^8 + 26a^4S_{BC} - 16S_{BC}^2)S_A^3\\
& - 4a^2S_{BC}(4a^4 + 19S_{BC})S_{AA} - S_AS_{BB}S_{CC}(29a^4 + 48S_{BC}) - 14a^2(S_{BC})^3)\\
& : \cdots : \cdots).
\end{aligned}$$

(b) *The equation of the Euler line of* $\mathbf{T}^\dagger$*:*

$$\sum_{\text{cyclic}} (S_C + S_A)(S_A + S_B)(S_B - S_C)f(S_A, S_B, S_C)x = 0,$$

where

$$f(S_A, S_B, S_C) = 2(8S_A+S_B+S_C)S^4-S_AS_B+S_C)((5S_A+7S_B+7S_C)S^2+S_{ABC}).$$

Remarks. (1) The centroid G' is $X(3060)$ in ETC, defined as the external center of similitude of the circumcircle of $\mathbf{T}$ and the nine-point circle of the orthic triangle.

(2) The circumcenter O' is the reflection of O in N^*.

(3) The orthocenter H' has ETC (6-9-13)-search number $31.1514091170\cdots$.

(4) The nine-point center N' has ETC (6-9-13)-search number $5.99676405896\cdots$.

(5) The Euler line of $\mathbf{T}^\dagger$ also contains the triangle center $X(156)$, which is the nine-point center of the tangential triangle.

2.4. *Euler reflection point of* $\mathbf{T}^\dagger$. A famous theorem of Collings [2] and Longuet-Higgins [8] states that the reflections of a line $\mathscr{L}$ in the sidelines of $\mathbf{T}$ are concurrent if and only if $\mathscr{L}$ contains the orthocenter H. If this condition is satisfied, the point of concurrency is a point on the circumcircle.

Applying this to the Euler line of $\mathbf{T}$, we obtain the Euler reflection point

$$E = \left(\frac{a^2}{b^2 - c^2} : \frac{b^2}{c^2 - a^2} : \frac{c^2}{a^2 - b^2}\right).$$

The Euler reflection point of $\mathbf{T}^\dagger$ is a point E' on the circumcircle of $\mathbf{T}^\dagger$ (see Figure 2.4). Its coordinates involve polynomial factors of degree 12 in S_A, S_B, S_C; it has ETC (6-9-13)-search number $-1.94515015138\cdots$.

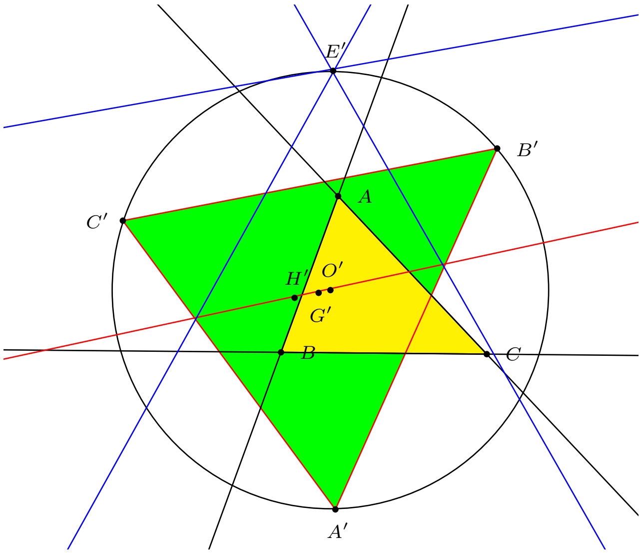

Figure 2.4

3. Perspectivity

3.1. *Triangles perspective with* $\mathbf{T}^\dagger$.

3.1.1. *The excentral triangle.* The excentral triangle of $\mathbf{T}$ has the excenters as vertices. L. Evans [3] has shown that this is perspective with $\mathbf{T}^\dagger$ at the triangle center

$$X(484) = (a(a^3 + a^2(b+c) - a(b^2 + bc + c^2) - (b+c)(b-c)^2) : \cdots : \cdots).$$

This triangle center is often called the Evans perspector. It is the inverse of I in the circumcircle of the excentral triangle, and divides OI in the ratio

$$OX(484) : X(484)I = R + 2r : -4r.$$

3.1.2. *The Fermat triangles.* Hatzipolakis and Yiu [5] have shown that the only Kiepert triangles perspective with $\mathbf{T}^\dagger$ are the Fermat triangles, consisting of vertices of equilateral triangles erected on the sides of $\mathbf{T}$. The perspectors are the isodynamic points, $X(16)$ or $X(15)$ according as the vertices of the equilateral triangles are on the same or opposite sides of the vertices of $\mathbf{T}$.

3.2. *Triangles bounded by reflections of the sidelines of* $\mathbf{T}$ *and* $\mathbf{T}^\dagger$ *in each other.* Let a, b, c be the sidelines BC, CA, AB of triangle $\mathbf{T} := ABC$, and a', b', c' those of $\mathbf{T}^\dagger$. The reflections of these lines in a, b, c (and vice versa) give rise to interesting examples of perspective triangles. Let $\mathscr{L}_a$ be the reflection of a in a', and $\mathscr{L}'_a$ that of a' in a. Similarly, define $\mathscr{L}_b$, $\mathscr{L}'_b$, and $\mathscr{L}_c$, $\mathscr{L}'_c$.

Since a, a' intersect at X, the lines $\mathscr{L}_a$, $\mathscr{L}'_a$ intersect BC at the same point. Similarly, the reflections of b and b' in each other intersect b at Y; so do those of c and c' at Z. By Proposition 2.1, X, Y, Z define the trilinear polar of N.

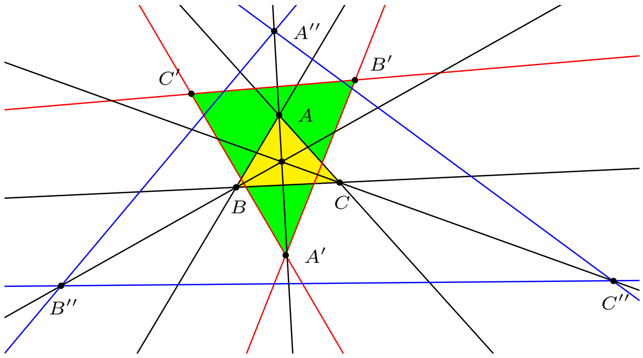

Figure 3.1

Therefore, $\mathbf{T}$, $\mathbf{T}^\dagger$, the triangles $\mathbf{T}^*$ bounded by $\mathscr{L}'_a$, $\mathscr{L}'_b$, $\mathscr{L}'_c$ (see Figure 3.1), and $\mathbf{T}'' = A''B''C''$ bounded by $\mathscr{L}_a$, $\mathscr{L}_b$, $\mathscr{L}_c$ are line-perspective to each other, all sharing the same perspectrix XYZ. They are also vertex-perspective.

The following table gives the $(6-9-13)$-search numbers of the perspectors, with the highest degree of the polynomial factors (in S_A, S_B, S_C) in the coordinates.

	$\mathbf{T}^\dagger$	$\mathbf{T}^*$	$\mathbf{T}''$
$\mathbf{T}$	H	$3.99180618013\cdots(5)$	$8.27975385194\cdots(7)$
$\mathbf{T}^\dagger$		$-9.04876879620\cdots(11)$	$-7.90053389552\cdots(16)$
$\mathbf{T}^*$			$-0.873191727540\cdots(26)$

Here is the perspector of $\mathbf{T}$ and $\mathbf{T}^*$ in homogeneous barycentric coordinates:

$$\left(\frac{a^2}{(3S_A-S_B-S_C)S^4+S_{BC}((6S_A+5S_B+5S_C)S^2+7S_{ABC})}:\cdots:\cdots\right).$$

4. Orthology

4.1. *Orthology and perspectivity.*

Theorem 4.1 (Sondat [12]; see also [13, 9])**.** *If two nondegenerate triangles are both perspective and orthologic, then the perspector and the two orthology centers are collinear.*

Proof. Assume triangles ABC and XYZ are perspective at a point $P=(u:v:w)$ and have perpendiculars from X to BC, Y to CA, and Z to AB concurrent at a point $Q=(u':v':w')$. Now the point X is the intersection of the line AP and the perpendicular from Q to BC. It has coordinates

$$(v((S_B+S_C)w'+S_Bu')-w(S_Cu'+(S_B+S_C)v') : v(S_Bv'-S_Cw') : w(S_Bv'-S_Cw')).$$

Similarly, Y is the intersection of BP and the perpendicular from Q to CA, and Z is that of CP and the perpendicular from Q to AB. Their coordinates can be written down from those of X by cyclic permutations of parameters. Since the triangles are orthologic, we find the second orthology center. The perpendculars from A to YZ, B to ZX, and C to AB are the lines

$$(S_Au - S_Bv)(u'w - w'u)y - (S_Cw - S_Au)(v'u - u'v)z = 0,$$
$$(S_Bv - S_Cw)(v'u - u'v)z - (S_Au - S_Bv)(w'v - v'w)x = 0,$$
$$(S_Cw - S_Au)(w'v - v'w)x - (S_Bv - S_Cw)(u'w - w'u)y = 0.$$

These lines are concurrent at the point

$$Q' = \left(\frac{S_Bv - S_Cw}{w'v - v'w} : \frac{S_Cw - S_Au}{u'w - w'u} : \frac{S_Au - S_Bv}{v'u - u'v}\right).$$

This clearly lies on the line PQ:

$$(w'v - v'w)x + (u'w - w'u)y + (v'u - u'v)z = 0.$$

Therefore the perspector P and the orthology centers Q and Q' are collinear. □

Here is an illustrative example. Let $\mathbf{T}^{\perp}(P)$ be the pedal triangle of a point P. It is clear that the perpendiculars from the vertices of $\mathbf{T}^{\perp}(P)$ to $\mathbf{T}$ are concurrent at P. Therefore, the perpendiculars from A, B, C to the corresponding sides of the pedal triangle are also concurrent. This is the isogonal conjugate of P.

Since the reflection triangle is homothetic to the pedal triangle. The same result holds, namely,

$$(\mathbf{T}^{\dagger}(P))^{\perp}(\mathbf{T}) = P \quad \text{and} \quad \mathbf{T}^{\perp}(\mathbf{T}^{\dagger}(P)) = P^*.$$

4.2. *Orthology with* $\mathbf{T}$. Clearly the perpendiculars from A', B', C' to the sidelines of $\mathbf{T}$ are concurrent at H. We find the other orthology center.

Proposition 4.2. *The orthology center* $\mathbf{T}^{\perp}(\mathbf{T}^{\dagger})$ *is* N^*.

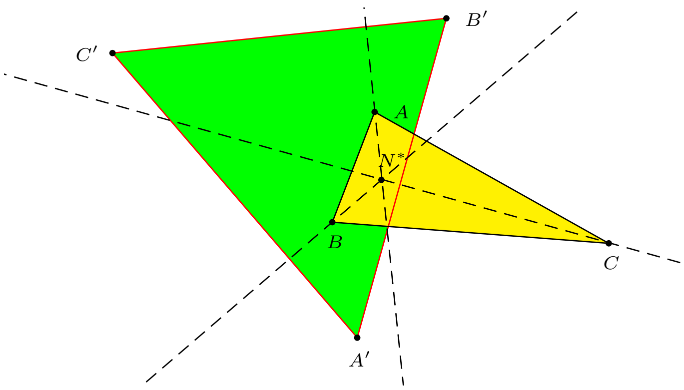

Figure 4.1

Proof. The perpendiculars from A to $B'C'$, B to $C'A'$, and C to $A'B'$ are the lines

$$\begin{array}{rcrcrcl} & & (S_A+S_B)(S^2+S_{CA})y & - & (S_C+S_A)(S^2+S_{AB})z & = & 0,\\ -(S_A+S_B)(S^2+S_{BC})x & & & + & (S_B+S_C)(S^2+S_{AB})z & = & 0,\\ (S_C+S_A)(S^2+S_{BC})x & - & (S_B+S_C)(S^2+S_{CA})y & & & = & 0. \end{array}$$

These are concurrent at

$$\left(\frac{S_B+S_C}{S^2+S_{BC}} : \frac{S_C+S_A}{S^2+S_{CA}} : \frac{S_A+S_B}{S^2+S_{AB}}\right),$$

which is the isogonal conjugate of the nine-point center N. □

4.3. *Orthic triangle.*

Theorem 4.3. *The triangle of reflections* $\mathbf{T}^\dagger$ *is orthologic to the orthic triangle.*

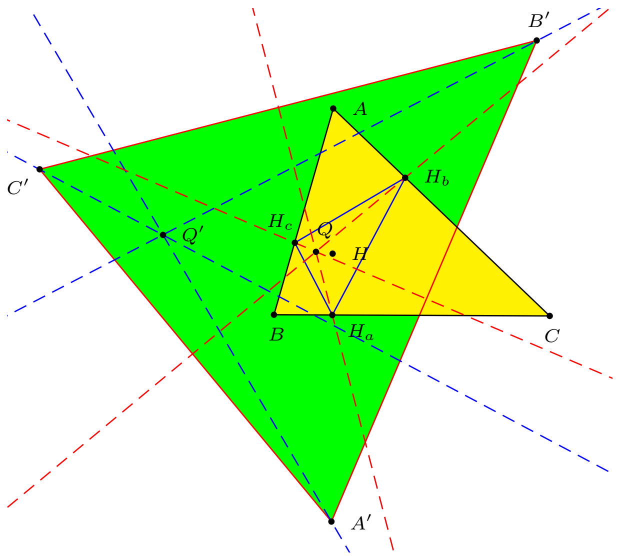

Figure 4.2

Proof. The perpendiculars from H_a to $B'C'$, H_b to $C'A'$, H_c to $A'B'$ are the lines

$$\begin{aligned} 2S^2(S_B-S_C)S_Ax+(S^2(3S_A+S_B+S_C)+S_{ABC})(S_By-S_Cz) &= 0,\\ 2S^2(S_C-S_A)S_By+(S^2(S_A+3S_B+S_C)+S_{ABC})(S_Cz-S_Ax) &= 0,\\ 2S^2(S_A-S_B)S_Cz+(S^2(S_A+S_B+3S_C)+S_{ABC})(S_Ax-S_By) &= 0. \end{aligned}$$

These are concurrent at

$Q :=\ (S_{BC}(S_B+S_C)(S_{AA}-3S^2)(S^2(3S_A+S_B+S_C)+S_{ABC}) : \cdots : \cdots).$

The perpendiculars from A' to H_bH_c, B' to H_cH_a, C' to H_aH_b are the lines

$$2S^2(S_B - S_C)(x+y+z) = (S_B+S_C)(-(S_A+S_B)S_C y + (S_C+S_A)S_B z),$$
$$2S^2(S_C - S_A)(x+y+z) = (S_C+S_A)(-(S_B+S_C)S_A z + (S_A+S_B)S_C x),$$
$$2S^2(S_A - S_B)(x+y+z) = (S_A+S_B)(-(S_C+S_A)S_B x + (S_B+S_C)S_A y).$$

These lines are concurrent at

$$Q' = ((S_B+S_C)(3S_A^3(S_B+S_C) - S_{AA}(S_B-S_C)^2 - 5S_{ABC}(S_B+S_C) - 4S_{BC}^2) : \cdots : \cdots).$$

$\square$

Remarks. (1) The orthology center $Q := \mathsf{cev}(H)^{\perp}(\mathbf{T}^{\dagger})$ has ETC (6-9-13)-search number $12.4818250323\cdots$. This also appears in Proposition 6.3 and §7.1.1 below.

(2) The orthology center $Q' := \mathbf{T}^{\dagger\perp}(\mathsf{cev}(H))$ has ETC (6-9-13)-search number $-8.27009636449\cdots$.

(3) Since the two triangles are perspective at H, the line joining these two orthology centers contains H. This is the line

$$\sum_{\text{cyclic}} a^2 S_A(S_B - S_C)(3S^2 - S_{AA})x = 0.$$

See Theorem 7.4(b) below.

4.4. *Tangential triangle.* Since the tangential triangle is homothetic to the orthic triangle, the results of §4.2 also shows that the triangle of reflections is orthologic to the tangential triangle. Clearly, $(\mathbf{T}^{\dagger})^{\perp}(\mathsf{cev}^{-1}(K)) = (\mathbf{T}^{\dagger})^{\perp}(\mathsf{cev}(H)) = Q'$.

Proposition 4.4. *The orthology center* $\mathsf{cev}^{-1}(K)^{\perp}(\mathbf{T}^{\dagger})$ *is the circumcenter of* $\mathbf{T}^{\dagger}$.

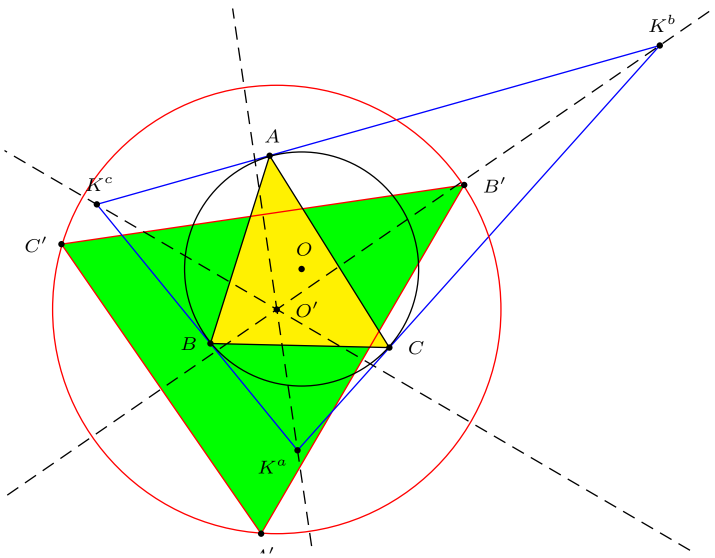

Figure 4.3

Proof. The vertex $K^a = (-(S_B + S_C) : S_C + S_A : S_A + S_B)$ is equidistant from B' and C'. In fact,

$$K^a B'^2 = K^a C'^2 = \frac{S_A(9S^2 + S_{BB} + S_{BC} + S_{CC}) + (S_B + S_C)S_{BC}}{4S_{AA}}.$$

Therefore, K^a lies on the perpendicular bisector of $B'C'$. Similarly, K^b and K^c lie on the perpendicular bisectors of $C'A'$ and $A'B'$ respectively. From this the result follows. □

5. Triads of circles

In this chapter we consider triads of circles related to $\mathbf{T}^\dagger$. The circumcircle of the reflection flanks are considered in §5.1. In §5.2, we construct a triad of coaxial circles associated with pedals and with the line HK as axis. In §5.3,4, we show that a common triangle center, the Parry reflection point $X(399)$, the reflection of the circumcenter in the Euler reflection point of $\mathbf{T}$, occurs as the point of concurrence of two triads of circles, one associated with the tangential triangle (Proposition 5.7), and another with the excentral triangle (§5.4).

We shall make frequent use of the following fundamental theorem.

Theorem 5.1 ([5, Proposition 18])**.** *If the circles XBC, AYC, ABZ have a common point, so do the circles AYZ, XBZ, XYC.*

5.1. *The reflection flanks and their circumcircles.* We shall refer to the triangles $\mathbf{T}^{\dagger a} := AB'C'$, $\mathbf{T}^{\dagger b} := A'BC'$, and $\mathbf{T}^{\dagger c} := A'B'C$ as the reflection flanks.

Proposition 5.2. *The reflection flank $\mathbf{T}^{\dagger a}$ is degenerate if and only if $A = \frac{\pi}{3}$ or $\frac{2\pi}{3}$.*

Proof. The line $B'C'$ has equation

$$(-3S_{AA} + S^2)x + 2(S^2 + S_{AB})y + 2(S^2 + S_{CA})z = 0.$$

(See §2.1). This contains the vertex A if and only if $3S_{AA} = S^2$, $\cot^2 A = \frac{1}{3}$, i.e., $A = \frac{\pi}{3}$ or $\frac{2\pi}{3}$. □

The circumcircle of $\mathbf{T}^{\dagger a}$:

$$\begin{aligned}&(S^2 - 3S_{AA})(a^2yz + b^2zx + c^2xy)\\ &- 2(x + y + z)(c^2(S^2 + S_{CA})y + b^2(S^2 + S_{AB})z) = 0.\end{aligned}$$

Its center is the point

$$O_a = 2S^2(3S^2 - S_{AA})(1, 0, 0) + b^2c^2(S^2 + S_{BC}, S^2 + S_{CA}, S^2 + S_{AB}).$$

Proposition 5.3. (a) *The circumcenters of the reflection flanks form a triangle perspective with ABC at the nine-point center N.*

(b) *The orthocenters of the reflection flanks form a triangle perspective with ABC at N^*.*

Proof. From the coordinates of O_a, we note that the line AO_a contains the nine-point center $N = (S^2 + S_{BC} : S^2 + S_{CA} : S^2 + S_{AB})$. Similarly, for the circumcenters O_b and O_c of $\mathbf{T}^{\dagger b}$ and $\mathbf{T}^{\dagger c}$ the lines BO_b and CO_c also contain the nine-point center.

(b) is equivalent to the orthology of $\mathbf{T}$ and $\mathbf{T}^\dagger$. It follows from Proposition 4.2. □

Since the circles $(A'BC)$, $(AB'C)$, (ABC') are concurrent at H, the circumcircles of the reflection flanks are also concurrent.

Proposition 5.4. *The circumcircles of the reflection flanks are concurrent at* $X(1157)$, *the inverse of* N^* *in the circumcircle of* $\mathbf{T}$.

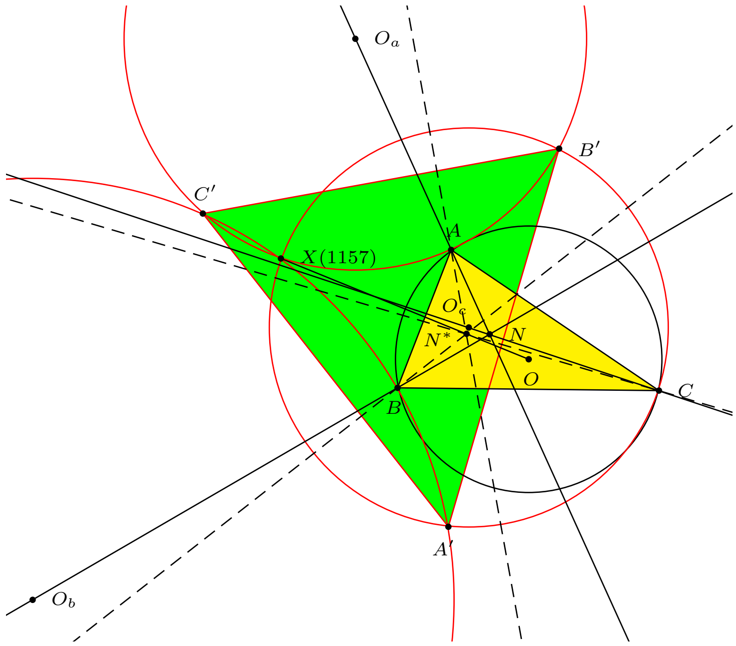

Figure 5.1

Proof. The point of concurrency is necessarily the radical center of the circles. From the equations of the circumcircles of the reflection flanks:

$$(S^2 - 3S_{AA})(a^2yz + b^2zx + c^2xy) - 2(x + y + z)(c^2(S^2 + S_{CA})y + b^2(S^2 + S_{AB})z) = 0,$$
$$(S^2 - 3S_{BB})(a^2yz + b^2zx + c^2xy) - 2(x + y + z)(a^2(S^2 + S_{AB})z + c^2(S^2 + S_{BC})x) = 0,$$
$$(S^2 - 3S_{CC})(a^2yz + b^2zx + c^2xy) - 2(x + y + z)(b^2(S^2 + S_{BC})x + a^2(S^2 + S_{CA})y) = 0,$$

we obtain the radical center as the point $(x : y : z)$ satisfying

$$\frac{c^2(S^2+S_{CA})y+b^2(S^2+S_{AB})z}{S^2-3S_{AA}} = \frac{a^2(S^2+S_{AB})z+c^2(S^2+S_{BC})x}{S^2-3S_{BB}} = \frac{b^2(S^2+S_{BC})x+a^2(S^2+S_{CA})y}{S^2-3S_{CC}}.$$

Rewriting this as

$$\frac{\frac{(S^2+S_{CA})y}{b^2}+\frac{(S^2+S_{AB})z}{c^2}}{a^2(S^2-3S_{AA})} = \frac{\frac{(S^2+S_{AB})z}{c^2}+\frac{(S^2+S_{BC})x}{a^2}}{b^2(S^2-3S_{BB})} = \frac{\frac{(S^2+S_{BC})x}{a^2}+\frac{(S^2+S_{CA})y}{b^2}}{c^2(S^2-3S_{CC})},$$

we have

$$\frac{\frac{(S^2+S_{CA})y}{b^2}+\frac{(S^2+S_{AB})z}{c^2}}{a^2(S^2-3S_{AA})} = \frac{\frac{(S^2+S_{AB})z}{c^2}+\frac{(S^2+S_{BC})x}{a^2}}{b^2(S^2-3S_{BB})} = \frac{\frac{(S^2+S_{BC})x}{a^2}+\frac{(S^2+S_{CA})y}{b^2}}{c^2(S^2-3S_{CC})}.$$

From these,

$$\begin{aligned}
&\frac{\frac{(S^2+S_{BC})x}{a^2}}{-a^2(S^2-3S_{AA})+b^2(S^2-3S_{BB})+c^2(S^2-3S_{CC})}\\
=\ &\frac{\frac{(S^2+S_{CA})y}{b^2}}{a^2(S^2-3S_{AA})-b^2(S^2-3S_{BB})+c^2(S^2-3S_{CC})}\\
=\ &\frac{\frac{(S^2+S_{AB})z}{c^2}}{a^2(S^2-3S_{AA})+b^2(S^2-3S_{BB})-c^2(S^2-3S_{CC})},
\end{aligned}$$

and

$$x : y : z = \frac{a^2(-a^2(S^2-3S_{AA})+b^2(S^2-3S_{BB})+c^2(S^2-3S_{CC}))}{S^2+S_{BC}} : \cdots : \cdots .$$

This gives the triangle center $X(1157)$ in ETC, the inverse of N^* in the circumcircle of **T**. □

5.2. *Three coaxial circles.* Let $H_aH_bH_c$ be the orthic triangle, and H'_a, H'_b, H'_c the pedals of A on $B'C'$, B on $C'A'$, and C on $A'B'$ respectively.

Proposition 5.5. *The lines $H_aH'_a$, $H_bH'_b$, $H_cH'_c$ are concurrent at*

$$(a^2S_{BC}((5S_A+S_B+S_C)S^4+S_{ABC}(S^2-2S_{AA})) : \cdots : \cdots).$$

Remark. This has ETC (6-9-13)-search number $3.00505308538\cdots$.

Theorem 5.6. *The three circles $AH_aH'_a$, $BH_bH'_b$, $CH_cH'_c$ are coaxial with radical axis HK.*

Proof. The centers of the circles AXX' are the point

$$\begin{aligned}
O'_a &= (S_A(S_B-S_C) : -(S^2+S_{CA}) : S^2+S_{AB}),\\
O'_b &= (S^2+S_{BC} : S_B(S_C-S_A) : -(S^2+S_{AB}),\\
O'_c &= (-(S^2+S_{BC}) : S^2+S_{CA} : S_C(S_A-S_B)).
\end{aligned}$$

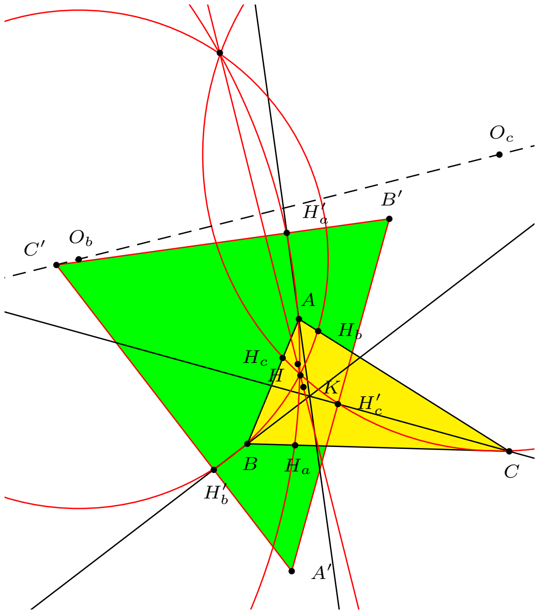

Figure 5.2

These centers lie on the line

$$a^2 S_A x + b^2 S_B y + c^2 S_C z = 0.$$

The circles have equations

$$\begin{aligned}
S_A(S_B - S_C)(a^2yz + b^2zx + c^2xy) - (x+y+z)(S_B(S^2+S_{AB})y - S_C(S^2+S_{CA})z) &= 0,\\
S_B(S_C - S_A)(a^2yz + b^2zx + c^2xy) - (x+y+z)(S_C(S^2+S_{BC})z - S_A(S^2+S_{AB})x) &= 0,\\
S_C(S_A - S_B)(a^2yz + b^2zx + c^2xy) - (x+y+z)(S_A(S^2+S_{CA})x - S_B(S^2+S_{BC})y) &= 0.
\end{aligned}$$

The radical axis is

$$(S_B - S_C)S_{AA}x + (S_C - S_A)S_{BB}y + (S_A - S_B)S_{CC}z = 0,$$

which clearly contains H and K (see Figure 5.2). □

Remark. The radical axes of the circumcircle with these three circles are concurrent at

$$X(53) = \left(\frac{S^2 + S_{BC}}{S_A} : \frac{S^2 + S_{CA}}{S_B} : \frac{S^2 + S_{AB}}{S_C}\right).$$

5.3. $\mathbf{T}^\dagger$ *and the tangential triangle.* Let $\mathsf{cev}^{-1}(K) := K^aK^bK^c$ be the tangential triangle. The centers of the circles $K^aB'C'$, $K^bC'A'$, $K^cA'B'$ are the points

$$\begin{aligned}
O_a'' &= (S_{ABC} + (3S_A - S_B - S_C)S^2 : b^2c^2S_C : b^2c^2S_B),\\
O_b'' &= (c^2a^2S_C : S_{ABC} + (3S_B - S_C - S_A)S^2 : c^2a^2S_A),\\
O_c'' &= (a^2b^2S_B : a^2b^2S_A : S_{ABC} + (3S_C - S_A - S_B)S^2).
\end{aligned}$$

The triangle $O_a''O_b''O_c''$ is perspective with $\mathbf{T}$ at H. Therefore, the two triangles are orthologic. The perpendiculars from O_a'', O_b'', O_c'' to BC, CA, AB respectively are concurrent at

$$X(265) = \left(\frac{S_A}{3S_{AA} - S^2} : \frac{S_B}{3S_{BB} - S^2} : \frac{S_C}{3S_{CC} - S^2}\right).$$

Proposition 5.7 ([5, §5.1.2])**.** *the circles $K^aB'C'$, $K^bC'A'$, $K^cA'B'$ are concurrent at the Parry reflection point $X(399)$.*

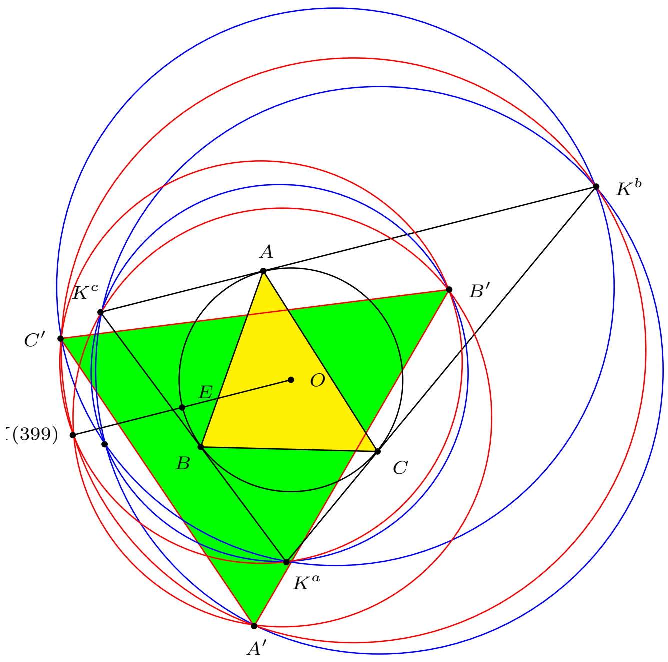

Figure 5.3

Proof. The equations of the circles are

$$\begin{aligned}
2S_A(a^2yz + b^2zx + c^2xy) + (x+y+z)(b^2c^2x + 2c^2S_Cy + 2b^2S_Bz) &= 0,\\
2S_B(a^2yz + b^2zx + c^2xy) + (x+y+z)(2c^2S_Cx + c^2a^2y + 2a^2S_Az) &= 0,\\
2S_C(a^2yz + b^2zx + c^2xy) + (x+y+z)(2b^2S_Bx + 2a^2S_Ay + a^2b^2z) &= 0.
\end{aligned}$$

The radical center of the three circles is the point $(x:y:z)$ satisfying

$$\frac{b^2c^2x+2c^2S_Cy+2b^2S_Bz}{S_A}=\frac{2c^2S_Cx+c^2a^2y+2a^2S_Az}{S_B}=\frac{2b^2S_Bx+2a^2S_Ay+a^2b^2z}{S_C}.$$

This is

$$(x:y:z)=(a^2(-8S^4+3b^2c^2(S^2+3S_{BC})):\ \cdots:\ \cdots),$$

the triangle center $X(399)$, the Parry reflection point, which is the reflection of O in the Euler reflection point E (see Figure 5.3). □

The circles $A'K^bK^c$, $B'K^cK^a$, $C'K^aK^b$ are also concurrent (see [11]). The point of concurrency is a triangle center with coordinates

$$(a^2f(S_A,S_B,S_C):b^2f(S_B,S_C,S_A):c^2f(S_C,S_A,S_B)),$$

with ETC (6-9-13)-search number $1.86365616601\cdots$. The polynomial $f(S_A,S_B,S_C)$ has degree 10.

5.4. $\mathbf{T}^\dagger$ *and the excentral triangle.*

Proposition 5.8 ([5, §5.1.3]). *The circles $A'I^bI^c$, $I^aB'I^c$, I^aI^bC' have the Parry reflection point as a common point.*

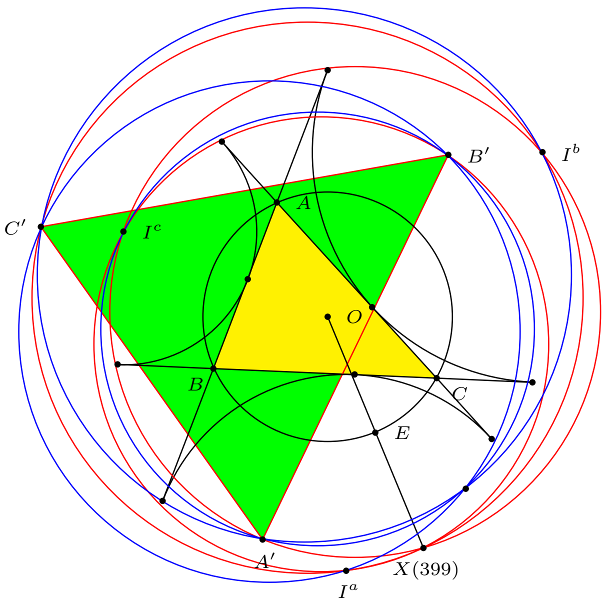

Figure 5.4

The centers of these circles are perspective with the excentral triangle at a point with ETC (6-9-13)-search number $-27.4208873972\cdots$.

On the other hand, the circles $I^aB'C'$, $A'I^bC'$ and $A'B'I^c$ have a common point with ETC (6-9-13)-search number $7.08747856659\cdots$. Their centers are perspective with ABC at the point $X(3336)$ which divides OI in the ratio $OX(3336) : X(3336)I = 2R + 3r : -4r$.

6. Pedals of vertices of $\mathbf{T}^\dagger$ on the sidelines of $\mathbf{T}$

6.1. *The triad of triangles AB_aC_a, BC_bA_b, CA_cB_c.* Consider the pedals of A', B', C' on the sidelines of $\mathbf{T}$. These are the points

	BC	CA	AB
A'		$B_a = (S_{CC} - S^2 : 0 : 2S^2)$,	$C_a = (S_{BB} - S^2 : 2S^2 : 0)$;
B'	$A_b = (0 : S_{CC} - S^2 : 2S^2)$		$C_b = (2S^2 : S_{AA} - S^2 : 0)$;
C'	$A_c = (0 : 2S^2 : S_{BB} - S^2)$,	$B_c = (2S^2 : 0 : S_{AA} - S^2)$.	

Proposition 6.1. *The Euler lines of the triangles AB_aC_a, BC_bA_b, CA_cB_c are concurrent at the Hatzipolakis reflection point $X(1986)$.*

Proof. The circumcenter of AB_aC_a is $H_a = (0 : S_C : S_B)$. The centroid is the point

$$2S^2(S_B + S_C, S_C + S_A, S_A + S_B) - (3S_{ABC} + S^2(S_A + S_B + S_C))(1, 0, 0).$$

From these we find the equation of its Euler line; similarly for the other two triangles. The Euler lines of the triangles are the lines

$$\begin{aligned}
2S^2 \cdot S_A(S_B - S_C)x + (3S_{ABC} - S^2(-S_A + S_B + S_C))(S_By - S_Cz) &= 0,\\
2S^2 \cdot S_B(S_C - S_A)y + (3S_{ABC} - S^2(S_A - S_B + S_C))(S_Cz - S_Ax) &= 0,\\
2S^2 \cdot S_C(S_A - S_B)z + (3S_{ABC} - S^2(S_A + S_B - S_C))(S_Ax - S_By) &= 0.
\end{aligned}$$

These three lines are concurrent at a point with coordinates given above. It is the triangle center $X(1986)$ (see Figure 6.1). □

Remark. Here is a definition of the Hatzipolakis reflection point $X(1986)$ equivalent to the one given in ETC. Let $H_aH_bH_c$ be the orthic triangle. $X(1986)$ is the common point of the reflections of the circles AB_bH_c in H_bH_c, BH_cH_a in H_cH_a, and CH_aH_b in H_aH_b.

Proposition 6.2. *The circumcircles of the triangles AB_aC_a, BC_bA_b, CA_cB_c have radical center $X(68)$.*

Proof. These are the circles with centers H_a, H_b, H_c, passing through A, B, C respectively.

$$\begin{aligned}
a^2(a^2yz + b^2zx + c^2xy) - (x + y + z)((S_{BB} - S^2)y + (S_{CC} - S^2)z) &= 0,\\
b^2(a^2yz + b^2zx + c^2xy) - (x + y + z)((S_{CC} - S^2)z + (S_{AA} - S^2)x) &= 0,\\
c^2(a^2yz + b^2zx + c^2xy) - (x + y + z)((S_{AA} - S^2)x + (S_{BB} - S^2)y) &= 0.
\end{aligned}$$

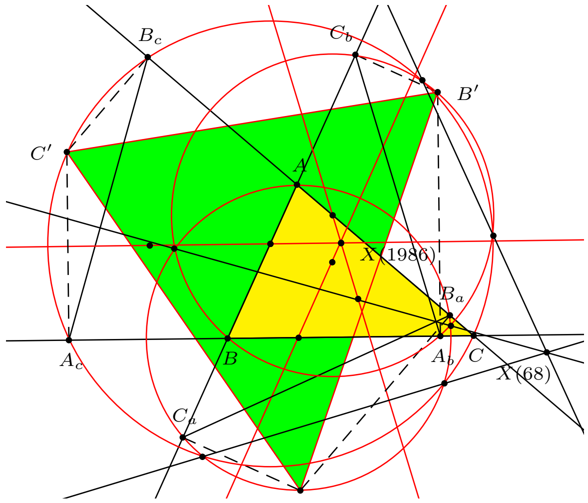

Figure 6.1

The radical center is the point defined by

$$\frac{(S_{BB}-S^2)y+(S_{CC}-S^2)z}{a^2} = \frac{(S_{CC}-S^2)z+(S_{AA}-S^2)x}{b^2} = \frac{(S_{AA}-S^2)x+(S_{BB}-S^2)y}{c^2}.$$

From these,

$$\frac{(S_{AA}-S^2)x}{b^2+c^2-a^2} = \frac{(S_{BB}-S^2)y}{c^2+a^2-b^2} = \frac{(S_{CC}-S^2)z}{a^2+b^2-c^2}$$

and

$$x:y:z = \frac{S_A}{S_{AA}-S^2} : \frac{S_B}{S_{BB}-S^2} : \frac{S_C}{S_{CC}-S^2}.$$

This is the triangle center $X(68)$. □

6.2. *The triad of triangles $A'B_aC_a$, $B'C_bA_b$, $C'A_cB_c$.*

Theorem 6.3. *The Euler lines of the triangles $A'B_aC_a$, $B'C_bA_b$, $C'A_cB_c$ are concurrent at*

$$Q = (S_{BC}(S_B+S_C)(S_{AA}-3S^2)(S_{ABC}+S^2(3S_A+S_B+S_C)) : \cdots : \cdots).$$

Proof. The circumcenter of $A'B_aC_a$ is $H_a = (0 : S_C : S_B)$, the same as AB_aC_a. The centroid is the point

$$(-(S_{ABC}+S^2(3S_A+S_B+S_C)) : 2(S_C+S_A)(2S^2-S_{AB}) : 2(S_A+S_B)(2S^2-S_{CA})).$$

From these we find the equation of its Euler line; similarly for the other two triangles. The Euler lines of the triangles are the lines

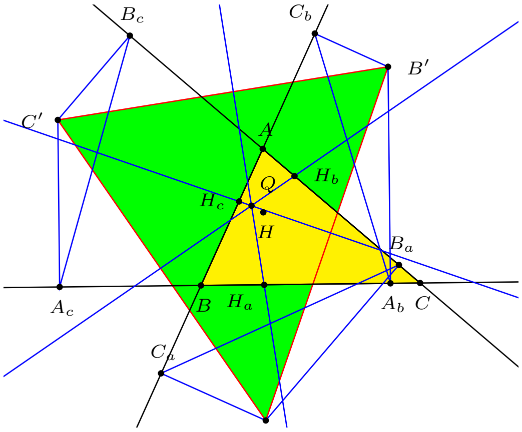

Figure 6.2

$$\begin{aligned}
2S^2 \cdot S_A(S_B - S_C)x + (S_{ABC} + S^2(3S_A + S_B + S_C))(S_By - S_Cz) &= 0,\\
2S^2 \cdot S_B(S_C - S_A)y + (S_{ABC} + S^2(S_A + 3S_B + S_C))(S_Cz - S_Ax) &= 0,\\
2S^2 \cdot S_C(S_A - S_B)z + (S_{ABC} + S^2(S_A + S_B + 3S_C))(S_Ax - S_By) &= 0.
\end{aligned}$$

These three lines are concurrent at a point with coordinates given above. □

Remark. This is the same as the orthology center $\text{cev}(H)^{\perp}(\mathbf{T}^{\dagger})$ in Theorem 4.3.

6.3. *A Taylor-like circle.* It is well known that the six pedals of H_a, H_b, H_c on the sidelines of $\mathbf{T}$ are concyclic. The circle containing them is the Taylor circle (see, for example, [6, §9.6]). The center of the circle, called the Taylor center, is the triangle center

$$X(389) = (S^4 - S_{AA}S_{BC} : S^4 - S_{BB}S_{CA} : S^4 - S_{CC}S_{AB}).$$

Analogous to the Taylor circle, the pedals of A', B', C' on the sidelines of $\mathbf{T}$ are also concyclic (see [1]). In fact, the circle containing them is a Tucker circle (Figure 6.3), since
(i) the segments B_cC_b, C_aA_c, A_bB_a are parallel to BC, CA, AB respectively, and
(ii) the segments B_aC_a, C_bA_b, A_cB_c are antiparallel to BC, CA, AB respectively, *i.e.*, they are parallel to the corresponding sides of the orthic triangle $H_aH_bH_c$.

The equation of the circle containing these six pedals is

$$\begin{aligned}
&a^2b^2c^2(a^2yz + b^2zx + c^2xy)\\
&- 2S^2(x + y + z)\left((S_{AA} - S^2)x + (S_{BB} - S^2)y + (S_{CC} - S^2)z\right) = 0.
\end{aligned}$$

The center of the circle is the point

$$(a^2(S_{AA}-S^2)(S^2+S_{BC}) : b^2(S_{BB}-S^2)(S^2+S_{CA}) : c^2(S_{CC}-S^2)(S^2+S_{AB})).$$

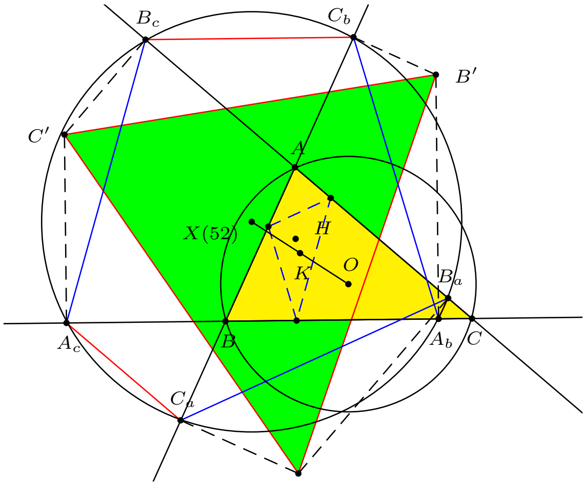

Figure 6.3

This is the triangle center $X(52)$. It is the reflection of O in the center of the Taylor circle. It is also the orthocenter of the orthic triangle (see Figure 6.3).

7. Some locus problems leading to conics

The website [4] CATALOGUE OF TRIANGLE CUBICS of B. Gibert contains a vast number of cubic and higher degree curves arising from locus problems in triangle geometry. In this chapter we consider a few loci related to perspectivity and orthology with $\mathbf{T}^\dagger$ which are conics. To avoid presenting excessively complicated algebraic manipulations, we present two problems in which the conic loci can be easily identified as rectangular hyperbolas. For this we recall a basic fact in triangle geometry: A (circum-)conic passing through the vertices of a triangle is a rectangular hyperbola if and only if it also passes through the orthocenter of the triangle.

7.1. *Reflection of* $\mathbf{T}$ *in a point.* Let P be a point with homogeneous barycentric coordinates $(x : y : z)$. The reflections of $\mathbf{T}$ in P is the triangle $\mathbf{T}_P^\dagger$ with vertices

$$\begin{aligned} A_P^\dagger &= (x-y-z : 2y : 2z), \\ B_P^\dagger &= (2x : y-z-x : 2z), \\ C_P^\dagger &= (2x : 2y : z-x-y). \end{aligned}$$

7.1.1. *Perspectivity of* $\mathbf{T}^\dagger$ *with reflection of* $\mathbf{T}$ *in a point.*

Theorem 7.1. *The locus of P for which $\mathbf{T}_P^\dagger$ is perspective with $\mathbf{T}^\dagger$ is the rectangular circum-hyperbola of the orthic triangle containing the orthocenter H.*

Proof. The equation of the line $A'A_P^\dagger$ is

$$\begin{vmatrix} -(S_B+S_C) & 2S_C & 2S_B \\ x-y-z & 2y & 2z \\ \mathbb{X} & \mathbb{Y} & \mathbb{Z} \end{vmatrix} = 0$$

or

$$2(S_By - S_Cz)\mathbb{X} - (S_Bx - S_By + S_Cz)\mathbb{Y} + (S_Cx - S_By + S_Cz)\mathbb{Z} = 0.$$

Similarly, we have the equations of the lines $B'B_P^\dagger$ and $C'C_P^\dagger$. These three lines are concurrent if and only if

$$\begin{vmatrix} 2(S_By - S_Cz) & -(S_Bx - S_By + S_Cz) & S_Cx - S_By + S_Cz \\ S_Ax + S_Ay - S_Cz & 2(S_Cz - S_Ax) & -(S_Ax + S_Cy - S_Cz) \\ -(-S_Ax + S_By + S_Az) & -S_Ax + S_By + S_Bz & 2(S_Ax - S_By) \end{vmatrix} = 0.$$

Expanding this determinant, we obtain

$$(x+y+z)\left(\sum_{\text{cyclic}}(S_B - S_C)(S_{AA}x^2 + S_{BC}yz)\right) = 0.$$

Since P is a finite point, $x+y+z \neq 0$. Therefore $(x:y:z)$ must satisfy

$$\sum_{\text{cyclic}}(S_B - S_C)(S_{AA}x^2 + S_{BC}yz) = 0.$$

The clearly defines a conic. By setting $x = 0$, we obtain

$$(S_By - S_Cz)(S_B(S_C - S_A)y - S_C(S_A - S_B)z) = 0.$$

It is clear that the conic contains $H_a = (0 : S_C : S_B)$. Similarly, it also contains H_b and H_c. Therefore, it is a circumconic of the orthic triangle $\mathsf{cev}(H) = H_aH_bH_c$. We also verify that the conic contains the following two points:
(i) the orthocenter $H = \left(\frac{1}{S_A} : \frac{1}{B} : \frac{1}{S_C}\right)$ (easy),
(ii) the triangle center

$$X(52) = ((S_B + S_C)(S^2 - S_{AA})(S^2 - S_{AA})(S^2 + S_{BC}) : \cdots : \cdots)$$

(with the help of Mathematica). This latter, according to ETC, is the orthocenter $H^\perp$ of the orthic triangle (see Figure 7.1).

It follows that the locus of P is the rectangular circum-hyperbola of the orthic triangle containing H. □

Remarks. (1) For $P = H^\perp = X(52)$, the perspector of $\mathbf{T}^\dagger$ and $\mathbf{T}_P^\dagger$ is the orthology center $Q = (\mathbf{T}^\dagger)^\perp(\mathsf{cev}(H))$ in Proposition 4.3.

(2) Since the conic intersects the sidelines of $\mathbf{T}$ at the traces of H, the second intersections with the sidelines are the traces of another point. This is $X(847)$.

(3) The center of the conic is the point $X(1112)$.

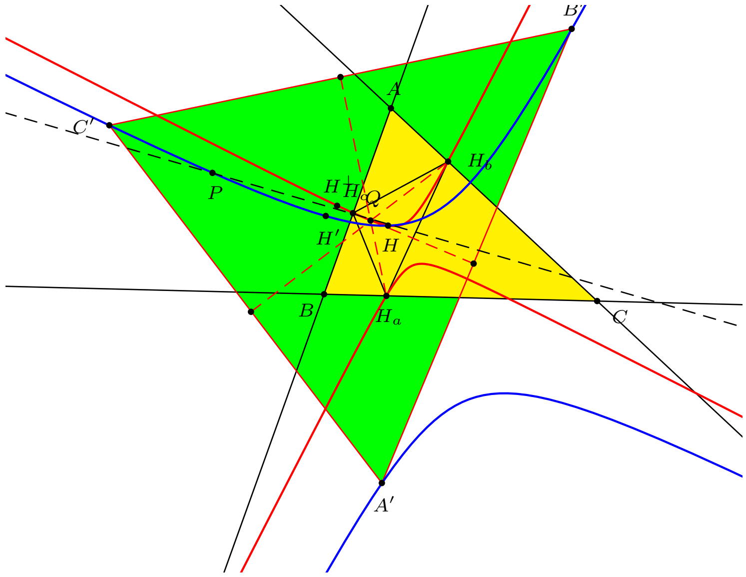

Figure 7.1

7.1.2. *Locus of perspector.*

Theorem 7.2. *For P on the rectangular circum-hyperbola of the orthic triangle containing H, the locus of the perspector of $\mathbf{T}^\dagger$ and $\mathbf{T}^\dagger_P$ is the rectangular circum-hyperbola of $\mathbf{T}^\dagger$ containing the orthocenter H.*

Proof. Rearranging the equation of $A'A^\dagger_P$ in the form

$$(S_B\mathbb{Y} - S_C\mathbb{Z})x - (2\mathbb{X} + \mathbb{Y} + \mathbb{Z})(S_By - S_Cz) = 0,$$

and likewise those of $B'B^\dagger_P$ and $C'C^\dagger_P$, we obtain the condition for concurrency:

$$\begin{vmatrix} S_B\mathbb{Y} - S_C\mathbb{Z} & -S_B(2\mathbb{X} + \mathbb{Y} + \mathbb{Z}) & S_C(2\mathbb{X} + \mathbb{Y} + \mathbb{Z}) \\ S_A(\mathbb{X} + 2\mathbb{Y} + \mathbb{Z}) & S_C\mathbb{Z} - S_A\mathbb{X} & -S_C(\mathbb{X} + 2\mathbb{Y} + \mathbb{Z}) \\ -S_A(\mathbb{X} + \mathbb{Y} + 2\mathbb{Z}) & S_B(\mathbb{X} + \mathbb{Y} + 2\mathbb{Z}) & S_A\mathbb{X} - S_B\mathbb{Y} \end{vmatrix} = 0.$$

Expanding the determinant, we obtain

$$(\mathbb{X} + \mathbb{Y} + \mathbb{Z})\left(\sum_{\text{cyclic}}(S_B - S_C)(2S_{AA}\mathbb{X}^2 - (S_A(S_B + S_C) - 2S_{BC})\mathbb{Y}\mathbb{Z})\right) = 0.$$

Since this perspector cannot be an infinite point, it must lie on the conic

$$\sum_{\text{cyclic}}(S_B - S_C)(2S_{AA}\mathbb{X}^2 - (S_A(S_B + S_C) - 2S_{BC})\mathbb{Y}\mathbb{Z}) = 0. \tag{7.1}$$

Construct the parallels to $A'C'$ and $A'B'$ through H_c and H_b respectively to intersect at a point X. The reflections of B and C in X lie respectively on $A'C'$

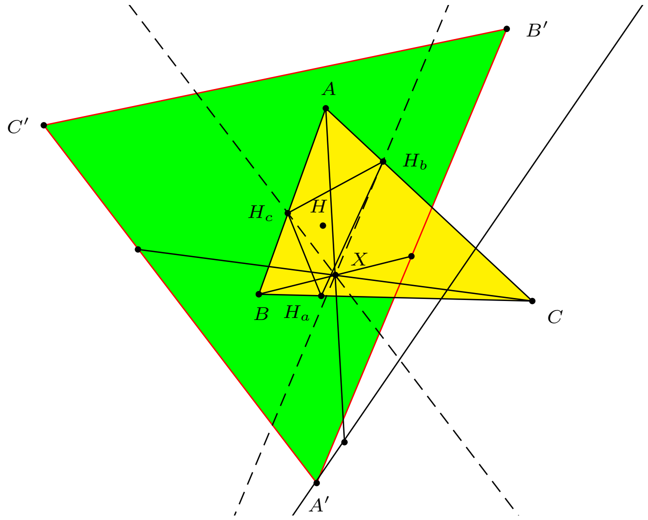

Figure 7.2

and $A'B'$ (Figure 7.2). Therefore, $\mathbf{T}^{\dagger}_X$ is perspective to $\mathbf{T}^{\dagger}$ at A'. This shows that the conic (7.1) contains A'. The same reasoning shows that it also contains B' and C'. It is a circumconic of $\mathbf{T}^{\dagger}$.

Now we claim that this conic contains the following two points:
(i) the orthocenter H (easy, take $P = H$),
(ii) the orthocenter H' of $\mathbf{T}^{\dagger}$.

In Proposition 4.3, we have constructed the orthology center $Q = (\mathsf{cev}(H))^{\dagger}(\mathbf{T}^{\dagger})$. We claim that the reflection $\mathbf{T}^{\dagger}_Q$ is perspective with $\mathbf{T}^{\dagger}$ at the orthocenter of $\mathbf{T}^{\dagger}$. Note that in triangle $AA'A^{\dagger}_Q$, Q and H_a are the midpoints of $AA^{\dagger}_Q$ and AA'. Therefore, $A'A^{\dagger}_Q$ is parallel to H_aQ. Since H_aQ is perpendicular to $B'C'$, $A'A^{\dagger}_Q$ is the altitude of $\mathbf{T}^{\dagger}$ (through A'). Similarly, $B'B^{\dagger}_Q$ and $C'C^{\dagger}_Q$ are also altitudes of the same triangle (see Figure 7.3). The three lines are concurrent at H', the orthocenter of $\mathbf{T}^{\dagger}$, which therefore lies on the conic defined by (7.1).

It follows that (7.1) defines the rectangular circum-hyperbola of $\mathbf{T}^{\dagger}$ containing H. □

7.2. *Orthology of* $\mathbf{T}^{\dagger}$ *with reflection triangle of* P. For a given point P, the reflection triangle $\mathbf{T}^{\dagger}(P)$ has vertices the reflections of P in the sidelines:

$$\begin{aligned}
P_a^{\dagger} &= (-(S_B+S_C)x : 2S_Cx+(S_B+S_C)y : 2S_Bx+(S_B+S_C)z),\\
P_b^{\dagger} &= (2S_Cy+(S_C+S_A)x : -(S_C+S_A)y : 2S_Ay+(S_C+S_A)z),\\
P_c^{\dagger} &= (2S_Bz+(S_A+S_B)x : 2S_Az+(S_A+S_B)y : -(S_A+S_B)z).
\end{aligned}$$

It is well known that the reflection triangle $\mathbf{T}^{\dagger}(P)$ is degenerate if and only if P lies on the circumcircle. It is clear that the perpendiculars from A, B, C to the

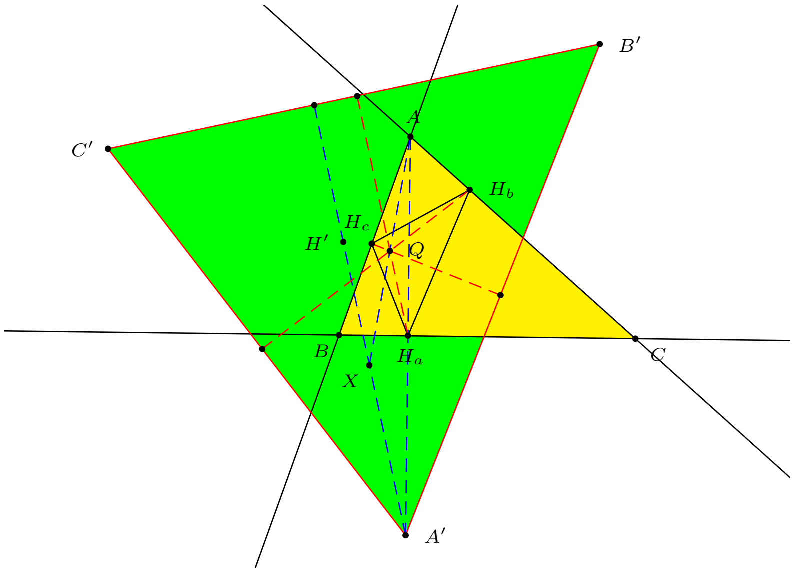

Figure 7.3

line containing $P_a^\dagger$, $P_b^\dagger$, $P_c^\dagger$ are parallel. However, the perpendiculars from these points to the sidelines of $\mathbf{T}$ are concurrent if and only if P is an intersection of the Euler line with the circumcircle. Henceforth, we shall consider P not on the circumcircle, so that its reflection triangle is nondegenerate. We study the locus of P for which $\mathbf{T}^\dagger(P)$ is orthologic to $\mathbf{T}^\dagger$.

7.2.1. *Locus of P whose reflection triangle is orthologic to $\mathbf{T}^\dagger$.*

Theorem 7.3. *The reflection triangle of P is orthologic to $\mathbf{T}^\dagger$ if and only if P lies on the Euler line.*

Proof. Let P be a point outside the circumcircle, and with homogeneous barycentric coordinates $(x : y : z)$. The perpendiculars from the vertices of $\mathbf{T}^\dagger$ to the sidelines of $\mathbf{T}^\dagger(P)$ are the lines

$$\begin{aligned}
2(-c^2S_Cy+b^2S_Bz)\mathbb{X}+(c^2(S_B-S_C)y+2b^2S_Bz)\mathbb{Y}+(-2c^2S_Cy+b^2(S_B-S_C)z)\mathbb{Z}=&\ 0,\\
(-2a^2S_Az+c^2(S_C-S_A)x)\mathbb{X}+2(-a^2S_Az+c^2S_Cx)\mathbb{Y}+(a^2(S_C-S_A)z+2c^2S_Cx)\mathbb{Z}=&\ 0,\\
(b^2(S_A-S_B)x+2a^2S_Ayx)\mathbb{X}+(-2b^2S_Bx+a^2(S_A-S_B)y)\mathbb{Y}+2(-b^2S_Bx+a^2S_Ay)\mathbb{Z}=&\ 0.
\end{aligned}$$

These are concurrent if and only if

$$\begin{vmatrix}
2(-c^2S_Cy+b^2S_Bz) & (c^2(S_B-S_C)y+2b^2S_Bz) & (-2c^2S_Cy+b^2(S_B-S_C)z)\\
(-2a^2S_Az+c^2(S_C-S_A)x) & 2(-a^2S_Az+c^2S_Cx) & (a^2(S_C-S_A)z+2c^2S_Cx)\\
(b^2(S_A-S_B)x+2a^2S_Ayx) & (-2b^2S_Bx+a^2(S_A-S_B)y) & 2(-b^2S_Bx+a^2S_Ay)
\end{vmatrix}=0.$$

Expanding this determinant, we obtain

$$-a^2b^2c^2(a^2yz+b^2zx+c^2xy)(S_A(S_B-S_C)x+S_B(S_C-S_A)y+S_C(S_A-S_B)z)=0.$$

Apart from the circumcircle, this defines the Euler line. □

7.2.2. *Loci of orthology centers.*

Theorem 7.4. *Let P be a point on the Euler line.*

(a) *The locus of the orthology center $\mathbf{T}^{\dagger\perp}(\mathbf{T}^\dagger(P))$ is the rectangular circum-hyperbola of $\mathbf{T}^\dagger$ containing the orthocenter H of $\mathbf{T}$.*

(b) *The locus of the orthology center $\mathbf{T}^\dagger(P)^\perp(\mathbf{T}^\dagger)$ is the line joining the orthocenter H of $\mathbf{T}$ to the nine-point center of $\mathbf{T}^\dagger$.*

Proof. Let $P=(S_{BC}+t, S_{CA}+t, S_{AB}+t)$ be a point on the Euler line.

(a) The perpendiculars from A' to the line $P_b^\dagger P_c^\dagger$ is

$$\begin{aligned}
&2S_A(S_B-S_C)(S^2+t)x\\
&+(S_A(S_A(2S_{BB}+S_{BC}-S_{CC})+S_{BC}(3S_B-S_C))+(S_A(3S_B-S_C)+S_B(S_B+S_C)t)y\\
&+(S_A(S_A(S_{BB}-S_{BC}-2S_{CC})+S_{BC}(S_B-3S_C))+(S_A(S_B-3S_C)-S_C(S_B+S_C))t)z\\
&=0.
\end{aligned}$$

The coefficients are linear in t; similarly for the equations of the perpendiculars from B', C' to $P_c^\dagger P_a^\dagger$ and $P_a^\dagger P_b^\dagger$. The point of concurrency of the three perpendiculars therefore has coordinates given by quadratic functions in t. Therefore, the locus of the orthology center $\mathbf{T}^{\dagger\perp}(\mathbf{T}^\dagger(P))$ is a conic. Note that for $P=O$, this orthology center is H. Also, for $P=N$, this is the orthocenter of $\mathbf{T}^\dagger$. (Proof: Since the reflection triangle of N is homothetic to $\mathbf{T}^\dagger$ (Proposition 2.2), they are orthologic, and $(\mathbf{T}^\dagger)^\perp(\mathbf{T}^\dagger(N))=H'$, the orthocenter of $\mathbf{T}^\dagger$).

We claim that A', B', C' are also three such orthology centers. Consider the Jerabek hyperbola (the rectangular circum-hyperbola of $\mathbf{T}$ through O and H. This is the isogonal conjugate of the Euler line. It also contains the symmedian point K. If the lines $A'K, B'K, C'K$ intersect the hyperbola again at X^*, Y^*, Z^* respectively, then their isogonal conjugates X, Y, Z are on the Euler line. If $X_a^\dagger X_b^\dagger X_c^\dagger$ is the reflection triangle of X, then the lines $X_a^\dagger X_b^\dagger$ and $X_a^\dagger X_c^\dagger$ are perpendicular to $A'C'$ and $A'B'$ respectively. This shows that A' is the orthology center $(\mathbf{T}^\dagger)^\perp(\mathbf{T}^\dagger(X))$, and it lies on the conic locus. The same reasoning shows that $B'=(\mathbf{T}^\dagger)^\perp(\mathbf{T}^\dagger(Y))$ and $C'=(\mathbf{T}^\dagger)^\perp(\mathbf{T}^\dagger(Z))$ are also on the same conic. This shows that the conic is a circumconic of $\mathbf{T}^\dagger$, containing its orthocenter H' and H.

(b) On the other hand, we compute the equations of the lines from $P_a^\dagger$ to $B'C'$, $P_b^\dagger$ to $C'A'$, and $P_c^\dagger$ to $A'B'$. These three lines are concurrent at a point whose coordinates are linear functions of t. Therefore, the locus of the orthology center is a line. This line has equation

$$\sum_{\text{cyclic}} a^2S_A(S_B-S_C)(3S^2-S_{AA})\mathbb{X}=0.$$

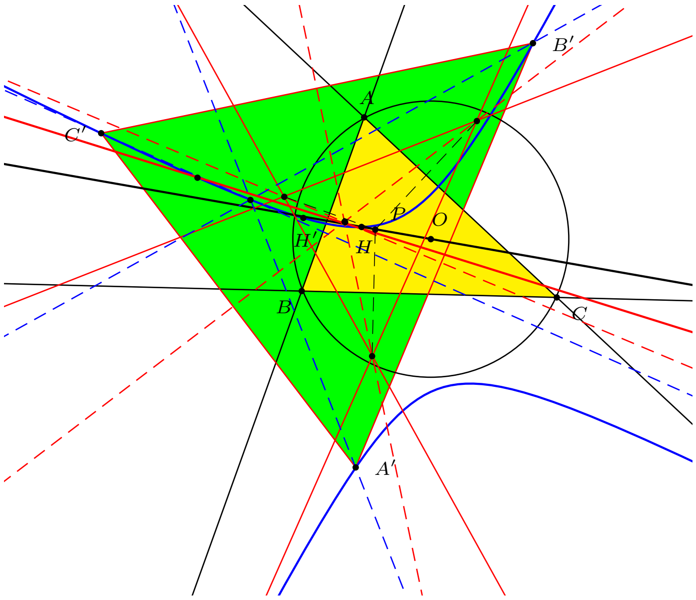

Figure 7.4

It is routine to check that this line contains the orthocenter H. It also contains the nine-point center N' of $\mathbf{T}^\dagger$ (with coordinates given in Proposition 2.4). $\square$

Remarks. (1) This is the same line joining the orthology centers of $\mathbf{T}^\dagger$ and the orthic triangle in Theorem 4.3.

(2) The orthology center $\mathbf{T}^\dagger(P)^\perp(\mathbf{T}^\dagger)$ is
(i) the orthocenter H for $P = X(3520) = (a^2S_{BC}(5S_{AA}+S^2) : \cdots : \cdots)$, which can be constructed as the reflection of H in $X(1594)$, which is the intersection of the Euler line with the line joining the Jerabek and Taylor centers,
(ii) the nine-point center of $\mathbf{T}^\dagger$ for

$$P = (-S^4(S_A+S_B+S_C)+S_{BC}((16S_A+7S_B+7S_C)S^2+S_{ABC}) : \cdots : \cdots)$$

with ETC (6-9-13)-search number $7.47436627857\cdots$.

8. Epilogue

In two appendices we list the triangle centers encountered in this paper. Appendix A lists those catalogued in ETC that feature in this paper with properties related to $\mathbf{T}^\dagger$. Appendix B lists new triangle centers in order of their search numbers in ETC. Here we present two atlases showing the positions of some of the centers in Appendix A in two groups.

Figure 8.1 shows a number of centers related to N and its isogonal conjugate N^*. On the line ON^* there are $X(195)$ (the circumcenter of $\mathbf{T}^\dagger$) and $X(1157)$ (the common point of the circumcircles of the reflection flanks (Proposition 5.4)). The line NN^* intersects the circumcircle at $X(1141)$, which also lies on the rectangular

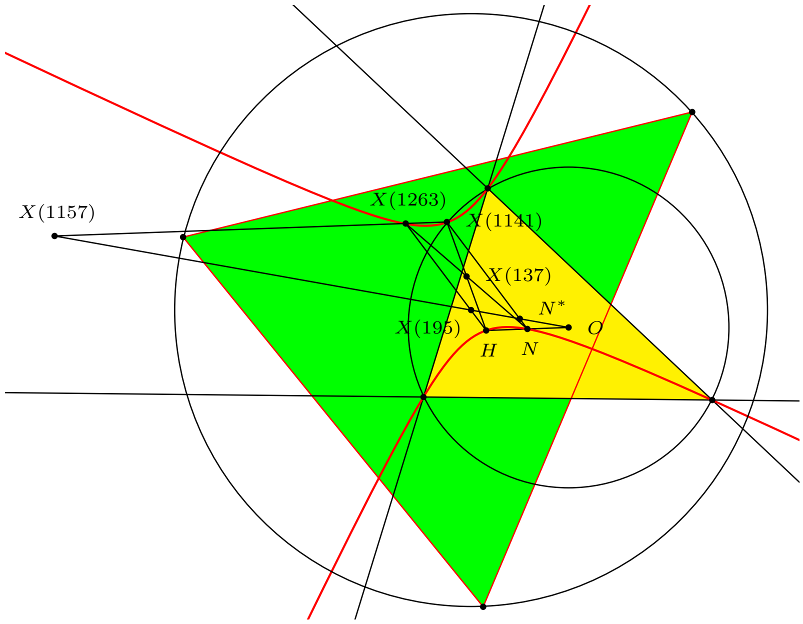

Figure 8.1

circum-hyperbola of $\mathbf{T}$ containing N. The center of the hyperbola is $X(137)$ on the nine-point circle.

The line joining $X(1141)$ to $X(1157)$ is parallel to the Euler line of $\mathbf{T}$. This line intersects the hyperbola at the antipode of N, which is the triangle center $X(1263)$.

Figure 8.2 shows the Euler line and its isogonal conjugate, the Jerabek hyperbola with center $X(125)$. The four triangle centers O, $X(125)$, $X(52)$ (center of the Taylor-like circle in §6.3), and $X(1986)$ (the Hatzipolakis reflection point) form a parallelogram, with center $X(389)$, the center of the Taylor circle. The line joining O to $X(125)$ intersects the hyperbola at $X265)$ (the reflection conjugate of O), which appears several times in this paper.

It is a well known fact that the line the hyperbola intersects the circumcircle at $X(74)$, the antipode of H. The antipode of $X(74)$ on the circumcircle is the Euler reflection point E. The Parry reflection point $X(399)$ is the reflection of O in E.

Figure 8.2 also shows the construction given in §7.2.2 of $X(3520)$ on the Euler line.

Appendix A: Triangle centers in ETC *associated with* $\mathbf{T}^\dagger$.

(1) $X(5) = N$: nine-point center.
- Reflection triangle homothetic to $\mathbf{T}^\dagger$ (Proposition 2.2).
- Perspector of the medial triangles of $\mathbf{T}^\dagger$ and $\mathbf{T}$ (Proposition 2.3).
- Perspector of the circumcenters of $AB'C'$, $A'BC'$, $A'B'C$ (Proposition 5.3(a)).

(2) $X(15)$: isodynamic point.

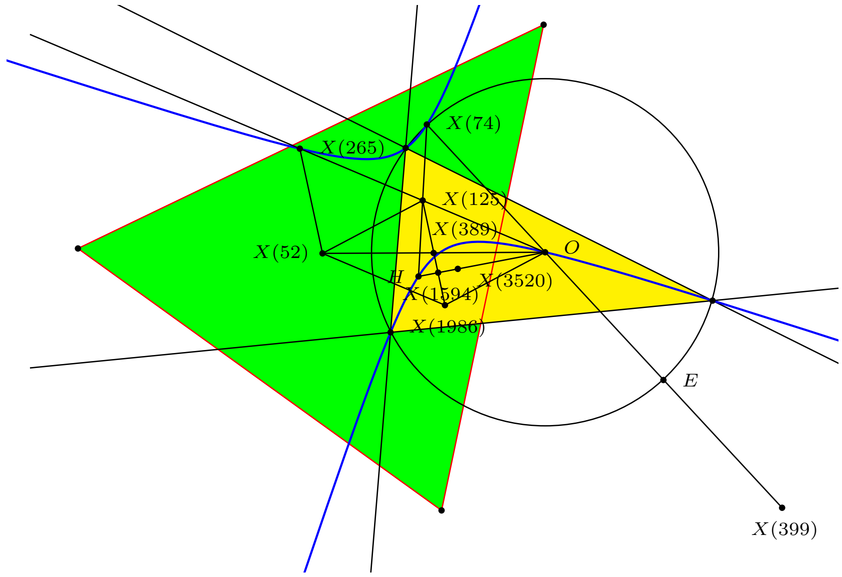

Figure 8.2

- Perspector of negative Fermat triangle with $\mathbf{T}^\dagger$ (§3.1.2).

(3) $X(16)$: isodynamic point.
- Perspector of positive Fermat triangle with $\mathbf{T}^\dagger$ (§3.1.2).

(4) $X(52)$: orthocenter of orthic triangle.
- Center of the circle through the 6 pedals of A' on CA, AB etc. (§6.3).

(5) $X(53)$: symmedian point of orthic triangle.
- Point of concurrency of the radical axes of the circumcircle with the circles AH_aH_a', BH_bH_b', CH_cH_c', where $H_aH_bH_c$ is the orthic triangle, and H_a', H_b', H_c' are the pedals of A on $B'C'$ etc (Theorem 5.6).

(6) $X(54) = N^*$.
- orthology center from ABC to $\mathbf{T}^\dagger$ (Proposition 4.2).
- Perspector of the orthocenters of $AB'C'$, $A'BC'$, $A'B'C$ (Proposition 5.3(b)).
- perspector of the orthocenters of the reflection flanks (Proposition 5.3(b)).
- radical center of the nine-point circles of $A'BC$, $AB'C$ and ABC'.
- Perspector of the triangle bounded by the radical axes of the circumcircle with the circles each through one vertex and the reflections of the other two in their opposite sides. (The triangle in question is indeed the anticevian triangle of N^*.)

(7) $X(68)$:
- radical center of the circles AB_aC_aA', BC_bA_bB', CA_cB_cC' (Proposition 6.2).

(8) $X(195)$: reflection of O in N^*.
- The circumcenter of $\mathbf{T}^\dagger$ (§2.3).
- Orthology center from tangential triangle to $\mathbf{T}^\dagger$ (Proposition 4.4).

(9) $X(265)$: reflection conjugate of O.

- Orthology center from circumcenters of $K^aB'C'$, $K^bC'A'$, $K^cA'B'$ to $\mathbf{T}$ (§5.3).
- Point of concurrency of parallels through A to Euler line of $A'BC$ etc.

(10) $X(399)$: Parry reflection point (reflection of O in Euler reflection point):
- Common point of circles $K^aB'C'$, $A'K^bC'$, $A'B'K^c$ (Proposition 5.7).
- Common point of circles $A'I^bI^c$, $B'I^cI^a$, $C'I^aI^b$ (Proposition 5.8).

(11) $X(484)$ Evans perspector
- Perspector of $\mathbf{T}^\dagger$ and the excentral triangle (§3.1.1).

(12) $X(1141)$: intersection of circumcircle with the rectangular circum-hyperbola through N.
- Perspector of the reflection triangle of $X(1157)$.
- Cevian triangle perspective with $\mathbf{T}^\dagger$ at $X(1157)$.

(13) $X(1157)$: inverse of N^* in circumcircle.
- Common point of the circles $AB'C'$, $A'BC'$ and $A'B'C$ (Proposition 5.4).
- Perspector of $\mathbf{T}^\dagger$ with the anticevian triangle of N^*.
- Perspector of $\mathbf{T}^\dagger$ with the cevian triangle of $X(1141)$.

(14) $X(1986)$: Hatzipolakis reflection point.
- Reflection of Jerabek point in Taylor center.
- Common point of the Euler lines of AB_aC_a etc., where B_a and C_a are the pedals of A' on AC and AB respectively (Theorem 6.1).

(15) $X(3060)$
- Centroid $\mathbf{T}^\dagger$ (§2.3).

(16) $X(3336)$
- Perspector of centers of circles $I^aB'C'$, $I^bC'A'$, $I^cA'B'$ (§5.4).

(17) $X(3520)$
- Point on the Euler line with orthology center $(\mathbf{T}^\dagger)^\perp(\mathbf{T}^\dagger(X(3520)) = H$ (§7.2.2, Remark 2(i)).

Appendix B: Triangle centers not in ETC.

(1) $-27.4208873972\cdots$: perspector of excentral triangle with centers of circles $A'I^bI^c$, $B'I^cI^a$, $C'I^aI^b$ (§5.4).
(2) $-9.04876879620\cdots$: perspector of $\mathbf{T}^\dagger$ and triangle bounded by reflections of a$'$ in a etc (§3.2).
(3) $-8.27009636449\cdots$: orthology center from $\mathbf{T}^\dagger$ to orthic triangle (§4.3).
(4) $-7.90053389552\cdots$: perspector of $\mathbf{T}^\dagger$ and triangle bounded by reflections of a in a$'$ etc ((§3.2)).
(5) $-5.94879118842\cdots$: symmedian point of $\mathbf{T}^\dagger$.
(6) $-1.94515015138\cdots$: Euler reflection point of $\mathbf{T}^\dagger$ (§2.4).
(7) $-0.873191727540\cdots$: perspector of the triangles bounded by the reflections of a' in a etc and those of a in a' etc (§3.2).
(8) $1.86365616601\cdots$: common point of circles $A'K^bK^c$, $B'K^cK^a$, $C'K^aK^b$ (§5.4).
(9) $2.99369649092\cdots$: radical center of pedal circles of A', B', C'.
(10) $3.00505308538\cdots$: point of concurrency of $H_aH'_a$, $H_bH'_b$, $H_cH'_c$, where H'_a, H'_b, H'_c are the pedals of A on $B'C'$, B on $C'A'$, and C on $A'B'$ respectively (Proposition 5.5).
(11) $3.99180618013\cdots$: perspector of $\mathbf{T}$ and triangle bounded by reflections of a$'$ in a etc (§3.2).
(12) $4.22924780831\cdots$: perspector of $\mathbf{T}$ and the reflection triangle of $X(1263)$ (§4.1).
(13) $5.99676405896\cdots$: nine-point center of $\mathbf{T}^\dagger$ (§2.3).

(14) $7.08747856659\cdots$: common point of circles $I^aB'C'$, $A'I^bC'$ and $A'B'I^c$ (§5.4).
(15) $7.47436627857\cdots$: P for which the orthology center $\mathbf{T}^\dagger(P)^\perp(\mathbf{T}^\dagger) = N^\dagger$ (§7.2.2, Remark 2(ii)).
(16) $8.27975385194\cdots$: perspector of $\mathbf{T}$ and triangle bounded by reflections of a in a' etc (§3.2).
(17) $12.48182503\cdots$: $Q :=$ orthology center from orthic triangle to $\mathbf{T}^\dagger$ (§4.3).
- orthology center from reflection triangle of $X(1594)$ to $\mathbf{T}^\dagger$,
- point of concurrency of the Euler lines of triangles $A'B_aC_a$, $B'C_bA_b$, $C'A_cB_c$ (Proposition 6.3).
- reflection of $\mathbf{T}$ in this point is perspective with $\mathbf{T}^\dagger$ (at the orthocenter of $\mathbf{T}^\dagger$) (Proof of Theorem 7.3).

(18) $31.1514091170\cdots$:
- orthocenter of $\mathbf{T}^\dagger$ (§2.3).
- perspector of $\mathbf{T}^\dagger$ and reflection of $\mathbf{T}$ in Q (§7.1.2).

References

[1] M. E. Akengin, Z. K. Köroğlu and Y. Yargiç, Three natural homotheties of the nine-point circle, *Forum Geom.*, 13 (2013) 209–218.
[2] S. N. Collings, Reflections on a triangle, part 1, *Math. Gazette*, 57 (1973) 291–293.
[3] L. S. Evans, A tetrahedral arrangement of triangle centers, *Forum Geom.*, 3 (2003) 181–186.
[4] B. Gibert, *Catalogue of Triangle Cubics*,
[5] A. P. Hatzipolakis and P. Yiu, Reflections in triangle geometry, *Forum Geom.*, 9 (2009) 301–348.
[6] R. Honsberger, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, MAA, 1995.
[7] C. Kimberling, *Encyclopedia of Triangle Centers*,
[8] M. S. Longuet-Higgins, Reflections on a triangle, part 2, *Math. Gazette*, 57 (1973) 293–296.
[9] D. Mitrea and M. Mitrea, A generalization of a theorem of Euler, *Amer. Math. Monthly*, 101 (1994) 55–58.
[10] D. Pedoe, *Geometry, A Comprehensive Course*, 1970; Dover reprint, 1988.
[11] C. Pohoata, On the Parry reflection point, *Forum Geom.*, 8 (2008) 43–48.
[12] P. Sondat and B. Sollerstinsky, Question 38, *L'intermédiaire des mathématiciens*, 1 (1894) 10; solution, *ibid*, 44–45.
[13] V. Thébault, Perspective and orthologic triangles and tetrahedrons, *Amer. Math. Monthly*, 59 (1952) 24–28.
[14] J. Torres, *The Triangle of Reflections*, M. Sc Thesis, Florida Atlantic University, vi+78 pages, 2014.
[15] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University Lecture Notes, 2001; with corrections, 2013, available at
`http://math.fau.edu/Yiu/Geometry.html`
[16] P. Yiu, The circles of Lester, Evans, Parry, and their generalizations, *Forum Geom.*, 10 (2010) 175–209.

Jesus Torres: Calvary Christian Academy, 2401 W. Cypress Creek Road, Fort Lauderdale, Florida 33309, USA
E-mail address: `jtorres2255@gmail.com`