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The Diagonal Point Triangle Revisited

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Abstract. We derive a formula for the area of the diagonal point triangle belonging to a tangential quadrilateral in terms of the four tangent lengths, and prove a characterization for a tangential trapezoid.

1. Introduction

Consider a convex quadrilateral with no pair of opposite parallel sides. Let the two diagonals intersect at E and the extensions of opposite sides intersect at F and G. Then the triangle EFG is called the *diagonal point triangle* or sometimes just the diagonal triangle (see Figure 1).

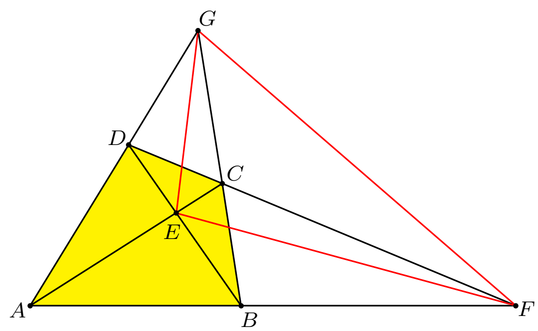

Figure 1. The diagonal point triangle EFG

The significance of the diagonal point triangle is most evident in projective geometry, where it is studied in connection with the complete quadrilateral. It is for instance a well known property that the diagonal point triangle associated with a cyclic quadrilateral is self-conjugate.

In [5] we derived a formula for the area of the diagonal point triangle belonging to a cyclic quadrilateral in terms of the four sides. In this note we shall derive a formula for this triangle area in connection with a tangential quadrilateral (a quadrilateral with an incircle), but here it will be in terms of the tangent lengths instead. The tangent lengths e, f, g, h in a tangential quadrilateral are defined to be the distances from the vertices to the points where the incircle is tangent to the sides (see Figure 2).

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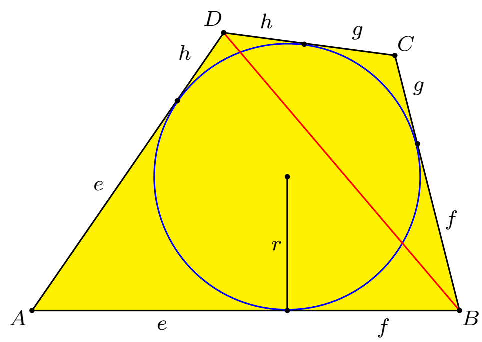

Figure 2. A tangential quadrilateral with its tangent lengths and a diagonal

2. More on the area of the diagonal point triangle

We will use Richard Guy's version of Hugh ApSimon's formula to derive a formula for the area of the diagonal point triangle belonging to a tangential quadrilateral. According to it (see [2]), the diagonal point triangle belonging to a convex quadrilateral $ABCD$ has the area

$$T = \frac{2T_1T_2T_3T_4}{K(T_1T_2 - T_3T_4)} \tag{1}$$

where T_1, T_2, T_3, T_4 are the areas of triangles ABC, ACD, ABD, BCD respectively, and K is the area of the quadrilateral.

Theorem 1. *If e, f, g, h are the tangent lengths in a tangential quadrilateral with no pair of opposite parallel sides, then the associated diagonal point triangle has the area*

$$T = \frac{2efghK}{|ef - gh||eh - fg|}$$

where

$$K = \sqrt{(e+f+g+h)(efg+fgh+ghe+hef)}$$

is the area of the quadrilateral.

Proof. In a tangential quadrilateral, triangle ABD has the area (see Figure 2)

$$T_3 = \frac{1}{2}(e+f)(e+h)\sin A = (e+f)(e+h)\sin\frac{A}{2}\cos\frac{A}{2}.$$

According to Theorem 8 in [4], we have that

$$\sin^2\frac{A}{2} = \frac{efg+fgh+ghe+hef}{(e+f)(e+g)(e+h)}.$$

Using the trigonometric Pythagorean theorem yields

$$\cos^2\frac{A}{2} = 1-\sin^2\frac{A}{2} = \frac{(e+f)(e+g)(e+h)-(efg+fgh+ghe+hef)}{(e+f)(e+g)(e+h)}$$
$$= \frac{e^2(e+f+g+h)}{(e+f)(e+g)(e+h)}.$$

Thus we get the subtriangle area

$$T_3 = \frac{(e+f)(e+h)e\sqrt{(efg+fgh+ghe+hef)(e+f+g+h)}}{(e+f)(e+g)(e+h)} = \frac{eK}{e+g}.$$

The last equality is due to formula (2) in [4] which gives the area of a tangential quadrilateral in terms of the tangent lengths. By symmetry we also have

$$T_1 = \frac{fK}{f+h}, \qquad T_2 = \frac{hK}{f+h}, \qquad T_4 = \frac{gK}{e+g}.$$

Combining the last four formulas gives

$$T_1T_2 - T_3T_4 = \frac{fhK^2}{(f+h)^2} - \frac{egK^2}{(e+g)^2} = K^2\left(\frac{(e+g)^2fh - eg(f+h)^2}{(e+g)^2(f+h)^2}\right).$$

Expanding the numerator, canceling the two double products and factoring it yields

$$(e+g)^2fh - eg(f+h)^2 = e^2fh + fg^2h - ef^2g - egh^2 = (ef-gh)(eh-fg).$$

Now by inserting the expressions for the triangle areas T_1, T_2, T_3, T_4 into (1), we get the area of the diagonal point triangle belonging to a tangential quadrilateral. Hence this is

$$T = \frac{2efghK^4}{K(e+g)^2(f+h)^2}\cdot\frac{(e+g)^2(f+h)^2}{K^2(ef-gh)(eh-fg)}$$

and the formula in the theorem follows by simplification and adding an absolute value to the denominator to cover all cases. □

Except in projective geometry, where the notion of area is irrelevant, we have only found one source where the diagonal point triangle associated with a tangential quadrilateral is treated. This is in the old extensive paper [1] on quadrilateral geometry by Dostor. He derives a formula for this triangle area,[1] but that formula is wrong. It states incorrectly (using other notations) that the area is given by

$$T = \frac{4efghK}{(e^2-g^2)(f^2-h^2)}$$

where e, f, g, h are the tangent lengths and K is the area of the quadrilateral. In [5] we concluded that Dostor's formula for the area of the diagonal point triangle belonging to a cyclic quadrilateral is also wrong, and then derived the correct formula.

An interesting observation is that the correct formula in [5] for a cyclic quadrilateral has the exact same *form* as Dostor's incorrect formula for a tangential

[1]Formula CCXVII on page 308 in [1]. We used e, f, g, h in the citation of his formula to easily be able to compare it to Theorem 1 in this note.

quadrilateral (except for a factor of 2). But there is one big difference. The letters used in Dostor's formula stands for the tangent lengths in a tangential quadrilateral, whereas in Theorem 1 in [5], we used a, b, c, d which stands for the side lengths in a cyclic quadrilateral.

3. A characterization of tangential trapezoids

If two opposite sides in the quadrilateral are parallel, then one of the points F or G becomes a point at infinity. Then the area of the diagonal point triangle is infinite. This is equivalent to having a denominator in Theorem 1 that is zero, so we get a necessary and sufficient condition for parallel opposite sides this way. Hence opposite sides are parallel if and only if $ef = gh$ or $eh = fg$.

Now we give a second proof of these characterizations of a *tangential trapezoid* (a trapezoid with an incircle; see Figure 3) where it is easier to determine which pair of opposite sides that are parallel in each case.

Theorem 2. *The opposite sides AB and CD in a tangential quadrilateral $ABCD$ with tangent lengths e, f, g, h are parallel if and only if*

$$eh = fg.$$

The opposite sides AD and BC are parallel if and only if

$$ef = gh.$$

Proof. According to Theorem 3 in [6], the opposite sides AB and CD in a convex quadrilateral are parallel if and only if

$$\tan\frac{A}{2}\tan\frac{D}{2} = \tan\frac{B}{2}\tan\frac{C}{2}.$$

Since $\tan\frac{A}{2} = \frac{r}{e}$, $\tan\frac{B}{2} = \frac{r}{f}$, $\tan\frac{C}{2} = \frac{r}{g}$ and $\tan\frac{D}{2} = \frac{r}{h}$ in a tangential quadrilateral with inradius r (see Figure 2), we have that AB and CD are parallel if and only if

$$\frac{r}{e}\cdot\frac{r}{h} = \frac{r}{f}\cdot\frac{r}{g} \qquad \Leftrightarrow \qquad eh = fg.$$

The second condition is proved in the same way using the angle characterization

$$\tan\frac{A}{2}\tan\frac{B}{2} = \tan\frac{C}{2}\tan\frac{D}{2}$$

for when AD and BC are parallel in a convex quadrilateral $ABCD$. □

In [4, p.129] we concluded that the inradius in a tangential trapezoid with tangent lengths e, f, g, h is given by

$$r = \sqrt[4]{efgh}.$$

Combining this with Theorem 2 yields that the formula for the inradius in a tangential trapezoid $ABCD$ with bases AB and CD can be simplified to

$$r = \sqrt{eh} = \sqrt{fg}.$$

These formulas can also be derived without the use of trigonometry. We give two other short proofs. Let the incircle be tangent to AB and CD at W and Y respectively, and I be the incenter (see Figure 3). Then triangles AWI and IYD are similar (AAA), so $\frac{r}{h} = \frac{e}{r}$. Whence $r^2 = eh$ and the second formula follows in a similar way. Another derivation starts by noting that the angle AID is a right angle when $AB \parallel CD$. Using the Pythagorean theorem in the three triangles AWI, DYI and AID yields $AI^2 = e^2 + r^2$, $DI^2 = h^2 + r^2$ and $AI^2 + DI^2 = (e+h)^2$. Combining these, we have $r^2 + e^2 + r^2 + h^2 = (e+h)^2$, and thus $r^2 = eh$.

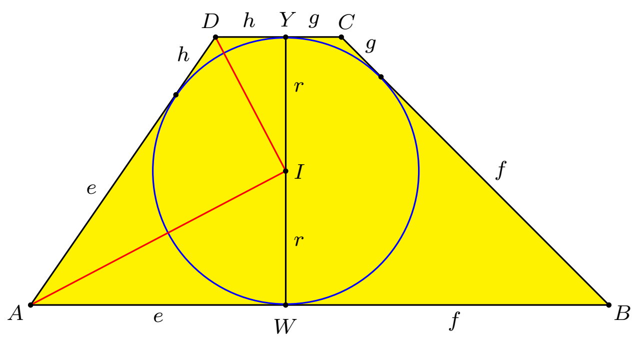

Figure 3. A tangential trapezoid

When the bases of the tangential trapezoid instead are AD and BC, the corresponding formulas are

$$r = \sqrt{ef} = \sqrt{gh}.$$

They can be derived in the same way by any of the three methods used above.

As a final remark, we note that the related equality $eg = fh$ gives another necessary and sufficient condition in tangential quadrilaterals. Two different proofs (both using other notations) were given in [7] and [3, p.104] that this is a characterization for when a tangential quadrilateral is also cyclic.

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