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Congruent Contiguous Excircles

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Abstract. In this paper we present some interesting lines in a triangle and we give some of their properties.

1. The e-property

For a given triangle ABC, and a point X on the side BC, consider the contiguous subtriangles ABX and AXC, with their excircles on the sides BX and XC. We shall say that X has the e-property if these two excircles are congruent (see Figure 1).

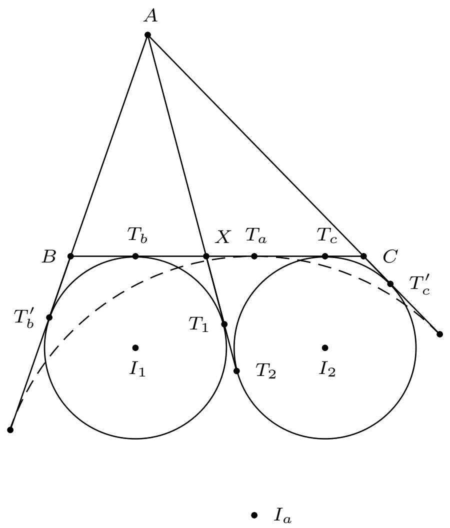

Figure 1

Denote by a, b, c the lengths of the sides BC, CA, AB. The radii of the incircle and excircle on BC are $r = \frac{\Delta}{s}$ and $r_a = \frac{\Delta}{s-a}$, where s and Δ are the semiperimeter and area of triangle ABC. It is known that $BT_a = s - c$ and $T_aC = s - b$.

In the following, we shall denote by s_T and $\Delta(T)$ the semiperimeter and area of a triangle T.

Let I_1, I_2, and I_a be the centers of the excircles of triangles ABX, AXC, and ABC on the sides BX, XC, and BC respectively, with points of tangency indicated in Figure 1. We denote by ρ_a the common exradius of the congruent contiguous excircles of ABX and ACX.

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Theorem 1. *Let X be the point on BC with the e-property.*
(a) $AX = \sqrt{s(s-a)}$.
(b) $\rho_a = \frac{\Delta}{(\sqrt{s}+\sqrt{s-a})\sqrt{s-a}}$.
(c) $BX = \frac{a\left(c+\sqrt{s(s-a)}\right)}{\left(\sqrt{s}+\sqrt{s-a}\right)^2}$, *and* $CX = \frac{a\left(b+\sqrt{s(s-a)}\right)}{\left(\sqrt{s}+\sqrt{s-a}\right)^2}$.

Proof. (a) In triangles ABX and AXC we have

$$\begin{aligned}\rho_a = \frac{\Delta(ABX)}{s_{ABX}-BX} &= \frac{\Delta(AXC)}{s_{AXC}-XC}\\ &= \frac{\Delta(ABX)+\Delta(AXC)}{s_{ABX}-BX+s_{AXC}-XC}\\ &= \frac{\Delta}{AX+s-a}.\end{aligned} \tag{1}$$

Since I_1I_2 is parallel to BC,

$$\begin{aligned}\frac{r_a-\rho_a}{r_a} = \frac{I_1I_2}{a} &= \frac{T_bX+XT_c}{a}\\ &= \frac{(s_{ABX}-AX)+(s_{AXC}-AX)}{a} = \frac{s-AX}{a}.\end{aligned}$$

From this, $\frac{r_a-\rho_a}{r_a} = \frac{s-AX}{a}$, and

$$AX-(s-a) = \frac{a\rho_a}{r_a}. \tag{2}$$

From (1) and (2), we have $AX^2-(s-a)^2 = \frac{a\Delta}{r_a} = a(s-a)$. Therefore, $AX^2 = (s-a)^2+a(s-a) = s(s-a)$. This proves (a).

(b) follows from (1) and (a).

(c) Let $BX = x$. We have

$$\rho_a = \frac{\Delta(ABX)}{s_{ABX}-BX} = \frac{\Delta(AXC)}{s_{AXC}-XC} \implies \frac{x}{c+AX-x} = \frac{a-x}{b+AX-(a-x)}.$$

This reduces to $x(b+AX-a+x) = (a-x)(c+AX-x)$, and $x = \frac{a(c+AX)}{b+c+2\cdot AX}$. Making use of (a), we obtain the expression for BX given in (b); similarly for CX. $\square$

Analogous to the congruent contiguous excircles, the problem of congruent contiguous incircles has been studied by P. Yiu ([3] and [4]). It was shown [3, §9.1.5] that for a point X on BC, the incircles of triangles ABX and AXC are congruent if and only if their excircles on the sides BX and XC are congruent, i.e., X has the e-property.

Theorem 2. *For the points of tangency of the excircles indicated in* Figure 1*, we have*
(a) $\frac{BT_b}{CT_c} = \frac{s-c}{s-b}$,
(b) $\frac{AT_b'}{AT_c'} = \frac{BX}{CX} = \frac{c+\sqrt{s(s-a)}}{b+\sqrt{s(s-a)}}$.

Proof. (a) From the excircle of triangle ABX on the side BX, we have, making use of Theorem 1 (a) and (c),

$$\begin{aligned} BT_b =&\ s_{ABX} - c = \frac{AX + BX - c}{2} \\ =&\ \frac{\sqrt{s(s-a)} + \frac{a\left(c+\sqrt{s(s-a)}\right)}{\left(\sqrt{s}+\sqrt{s-a}\right)^2} - c}{2} \\ =&\ \frac{(s-c)\sqrt{s-a}}{\sqrt{s}+\sqrt{s-a}} \end{aligned} \tag{3}$$

after simplification. Similarly,

$$CT_c = \frac{(s-b)\sqrt{s-a}}{\sqrt{s}+\sqrt{s-a}}. \tag{4}$$

From (3) and (4), $\frac{BT_b}{CT_c} = \frac{s-c}{s-b}$. This proves (a).

(b) Note that

$$AT_b' = AB + BT_b' = AB + BT_b = c + \frac{(s-c)\sqrt{s-a}}{\sqrt{s}+\sqrt{s-a}} = \frac{\sqrt{s}\left(c+\sqrt{s(s-a)}\right)}{\sqrt{s}+\sqrt{s-a}}.$$

Similarly, $AT_c' = \frac{\sqrt{s}\left(b+\sqrt{s(s-a)}\right)}{\sqrt{s}+\sqrt{s(s-a)}}$. From these, (b) follows. □

Theorem 3. *For the points of tangency of the excircles indicated in* Figure 1*, we have*

(a) $T_bT_c = I_1I_2 = \frac{a\sqrt{s}}{\sqrt{s}+\sqrt{s-a}}$,

(b) $T_1T_2 = \frac{|b-c|\sqrt{s}}{\sqrt{s}+\sqrt{s-a}}$.

Proof. (a) Since the excircles are congruent, $I_1I_2T_cT_b$ is a rectangle. Using (3) and (4), we obtain

$$\begin{aligned} T_bT_c =&\ BC - BT_b - CT_c \\ =&\ a - \frac{(s-c)\sqrt{s-a}}{\sqrt{s}+\sqrt{s-a}} - \frac{(s-b)\sqrt{s-a}}{\sqrt{s}+\sqrt{s-a}} \\ =&\ \frac{a\left(\sqrt{s}+\sqrt{s-a}\right) - a\sqrt{s-a}}{\sqrt{s}+\sqrt{s-a}} \\ =&\ \frac{a\sqrt{s}}{\sqrt{s}+\sqrt{s-a}}. \end{aligned}$$

(b) If the excircles of triangles ABX and AXC touch the line AX at T_1 and T_2 respectively, then, making use of (a) above and Theorem 1(b), we have

$$\begin{aligned} T_1T_2^2 =&\ I_1I_2^2 - (\rho_a + \rho_a)^2 \\ =&\ \left(\frac{a\sqrt{s}}{\sqrt{s}+\sqrt{s-a}}\right)^2 - \frac{4\Delta^2}{\left(\sqrt{s}+\sqrt{s-a}\right)^2(s-a)} \\ =&\ \frac{a^2s - 4s(s-b)(s-c)}{\left(\sqrt{s}+\sqrt{s-a}\right)^2} \\ =&\ \frac{s(b-c)^2}{\left(\sqrt{s}+\sqrt{s-a}\right)^2}. \end{aligned}$$

□

Theorem 4. *If X has the e-property, the line AX bisects I_1I_2.*

Proof. Let the bisectors AI_1 and AI_2 of angles BAX and XAC intersect BC at D and E respectively. In triangle ABX with bisector AD, we have, by Theorem 1(a) and (c),

$$DX = \frac{BX \cdot AX}{AB + AX} = \frac{a\sqrt{s(s-a)}}{\left(s+\sqrt{s-a}\right)^2}.$$

Similarly, $EX = \frac{a\sqrt{s(s-a)}}{\left(\sqrt{s}+\sqrt{s-a}\right)^2} = DX$. This shows that X is the midpoint of DE. Since I_1I_2 is parallel to DE, the line AX bisects I_1I_2. □

2. Some identities and inequalities involving points with the e-property

Analogous to the point X on BC with the e-property, there are also points Y on CA, and Z on AB with the e-property, i.e, the contiguous triangles BCY, BYA have congruent excircles on CY, YA, and the pair CAZ, CZB also with congruent excircles on AZ, ZB (see Figure 2). In general, the cevians AX, BY, and CZ are not concurrent. If triangle ABC is isosceles, they do, because of obvious symmetry.

Proposition 5. *The area of triangle XYZ is*

$$\frac{\begin{array}{r}(a+\sqrt{s(s-c)})(b+\sqrt{s(s-a)})(c+\sqrt{s(s-b)}) \\ +(a+\sqrt{s(s-b)})(b+\sqrt{s(s-c)})(c+\sqrt{s(s-a)})\end{array}}{(\sqrt{s}+\sqrt{s-a})^2(\sqrt{s}+\sqrt{s-b})^2(\sqrt{s}+\sqrt{s-c})^2} \cdot \Delta.$$

We establish some identities and inequalities involving the cevian lines through through the points with the e-property. Denote by R the circumradius of triangle ABC.

Theorem 6. (a) $AX \cdot BY \cdot CZ = s\Delta$.
(b) $AX^2 + BY^2 + CZ^2 = s^2$.
(c) $\frac{1}{AX^2} + \frac{1}{BY^2} + \frac{1}{CZ^2} = \frac{4R+r}{rs^2}$.
(d) $\frac{a}{AX^2} + \frac{b}{BY^2} + \frac{c}{CZ^2} = \frac{2(2R-r)}{\Delta}$.

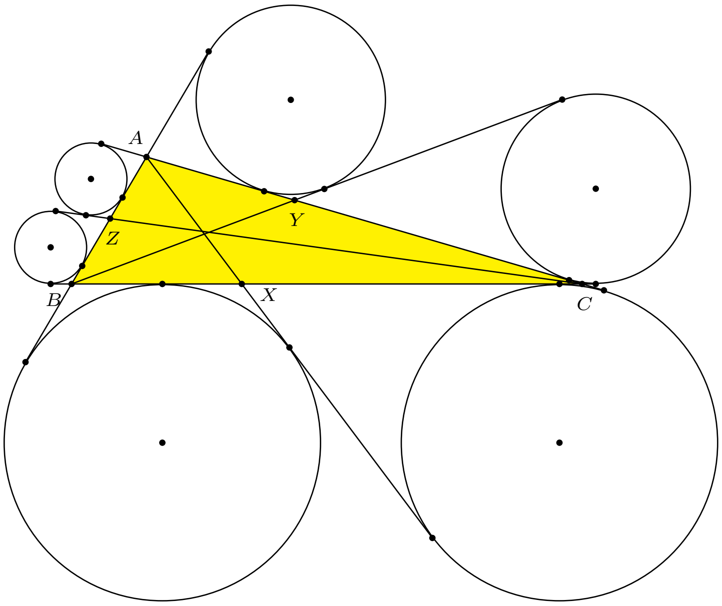

Figure 2

(e) $AX^4 + BY^4 + CZ^4 = s^2(s^2 - 2r^2 - 8Rr)$.
(f) $AX^6 + BY^6 + CZ^6 = s^4(s^2 - 12Rr)$.

Proof. These follow from Theorem 1(a), (c), and the basic relations

$$\begin{aligned} ab + bc + ca &= s^2 + r^2 + 4Rr, \\ a^2 + b^2 + c^2 &= 2(s^2 - r^2 - 4Rr), \\ a^3 + b^3 + c^3 &= 2s(s^2 - 3r^2 - 6Rr); \end{aligned}$$

see [1]. □

Theorem 7. *Let m_a and w_a be the lengths of the median on BC and angle bisector of angle A. If X is the point on BC with the e-property, then $w_a \le AX \le m_a$.*

Proof. For the bisector of angle A,

$$w_a = \frac{2bc}{b+c}\cos\frac{A}{2} = \frac{2\sqrt{bc}}{b+c}\cdot\sqrt{s(s-a)} = \frac{2\sqrt{bc}}{b+c}\cdot AX \le AX,$$

since $2\sqrt{bc} \le b + c$.

On the other hand, for the median on BC, the inequality $AX \le m_a$ is equivalent to $s(s-a) \le \frac{2(b^2+c^2)-a^2}{4}$. After some rearrangement, this reduces to $(b+c)^2 \le 2(b^2+c^2)$, which is clearly valid. □

Theorem 8. (a) $AX \cdot BY + BY \cdot CZ + CZ \cdot AX \le s^2$.
(b) $3\sqrt[3]{s\Delta} \le AX + BY + CZ \le \sqrt{3}s$.

Proof. These follow from applying the well known inequalities $(x+y+z)^2 \leq 3(x^2+y^2+z^2)$ and $xy+yz+zx \leq x^2+y^2+z^2$, applied to $x = AX$, $y = BY$, and $z = CZ$, and making use of Theorem 6 (a), (b). ☐

Remarks. (1) By using Schwarz's Inequality, we have

$$(AX^2 + BY^2 + CZ^2)\left(\frac{1}{AX^2} + \frac{1}{BY^2} + \frac{1}{CZ^2}\right) \geq 9.$$

By using the formulae in Theorem 6(b), (c), we have $s^2 \cdot \frac{4R+r}{rs^2} \geq 9$. From this the well known Euler's inequality $R \geq 2r$ follows.

(2) Again, it is easy to establish

$$AX^2 \cdot BY^2 + BY^2 \cdot CZ^2 + CZ^2 \cdot AX^2 = (4R+r)rs^2.$$

From the inequality

$$AX^2 \cdot BY^2 + BY^2 \cdot CZ^2 + CZ^2 \cdot AX^2 \leq AX^4 + BY^4 + CZ^4,$$

and the formula in Theorem 6(e), we have $(4R+r)rs^2 \leq s^2(s^2 - 2r^2 - 8Rr)$. This leads to another known inequality

$$3(4R+r)r \leq s^2;$$

see [1, 2].

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