Forum Geometricorum
Volume 14 (2014) 403–408.

FORUM GEOM
ISSN 1534-1178

Some Circles Associated with the Feuerbach Points

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Abstract. Consider a triangle with its nine-point circle tangent to the incircle and excircles at the Feuerbach point. We show that the four circles each through the circumcenter, nine-point, and Feuerbach point contain the nine-point center of the intouch triangle or the corresponding extouch triangle. Furthermore, the lines joining these Feuerbach points to the corresponding nine-point centers are concurrent on the nine-point circle of the given triangle.

1. Four coaxial circles through the Feuerbach points

The starting point of this note is the famous Feuerbach theorem, that for a given triangle, the nine-point circle is tangent internally to the incircle and externally to each of the excircles. Given triangle ABC, with incenter I, excenters I_a, I_b, I_c, and nine-point N, the points of tangency of the nine-point circle with these circles are the Feuerbach points $F_{\rm e}$, F_a, F_b, F_c on the lines NI, NI_a, NI_b, NI_c with ratios of division

$$NF_{\rm e} : F_{\rm e}I = \frac{R}{2} : -r,$$

$$NF_a : F_aI_a = \frac{R}{2} : r_a, \quad NF_b : F_bI_b = \frac{R}{2} : r_b, \quad NF_c : F_cI_c = \frac{R}{2} : r_c,$$

where R, r, r_a, r_b, r_c are the circumradius, inradius, and exradii.

Proposition 1. *Let O be the circumcenter of triangle ABC.*
(a) $OI^2 = R(R - 2r)$.
(b) $OI_a^2 = R(R + 2r_a)$.
(c) $II_a^2 = 4R(r_a - r)$.
(d) *The excentral triangle $I_aI_bI_c$ has circumcenter I' at the reflection of I in O, and circumradius $2R$.*

For (a-c), see [1, Theorems 152-154]. For (d), see [4, §4.6.1].

Consider the circle through O, N, and $F_{\rm e}$. Since I is an interior point of the segment $NF_{\rm e}$, it is an interior point of the circle. Our first result relates the endpoint of the chord through O and I with the intouch triangle, whose vertices are the points of tangency of the incircle with the sidelines.

Theorem 2. *The line OI intersects the circle $ONF_{\rm e}$ again at the nine-point center of the intouch triangle.*

Publication Date: December 11, 2014. Communicating Editor: Paul Yiu.

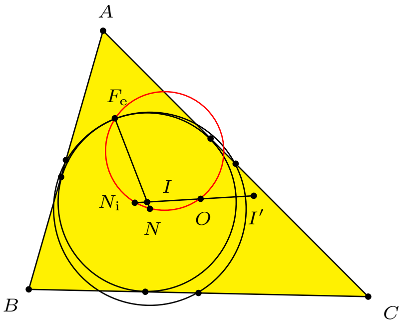

Figure 1

Proof. Let $N_{\rm i}$ be the second intersection of the line OI with the circle $ONF_{\rm e}$. By the intersecting chords theorem, $OI \cdot IN_{\rm i} = NI \cdot IF_{\rm e}$. Therefore,

$$\frac{IN_{\rm i}}{OI} = \frac{NI \cdot IF_{\rm e}}{OI^2} = \frac{\left(\frac{R}{2} - r\right) r}{R(R-2r)} = \frac{r}{2R}.$$

Note that the intouch triangle is homothetic to the excentral triangle. Since the excentral triangle has circumcenter I' and nine-point center O, its Euler line is the line OI. Since the intouch triangle has circumcenter I, its Euler line is also the line OI. From

$$\frac{IN_{\rm i}}{I'O} = \frac{IN_{\rm i}}{OI} = \frac{r}{2R},$$

the homothetic ratio of the intouch and excentral triangles, we conclude that $N_{\rm i}$ is nine-point center of the intouch triangle. □

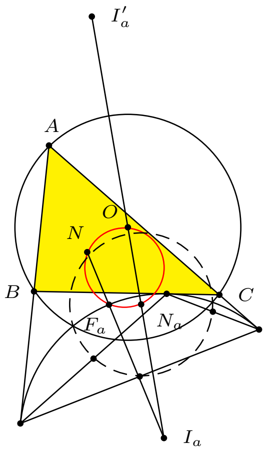

Figure 2.

Analogous results hold if we replace the Feuerbach point $F_{\rm e}$ by the other Feuerbach points, say, F_a. if the circle through O, N, F_a intersects the line OI_a at N_a, then $\frac{I_aN_a}{I_aO} = \frac{r_a}{2R}$. Now, triangle II_bI_c is homothetic to the A-extouch triangle formed by the points of tangency of the A-excircle with the sidelines of triangle ABC, with homothetic ratio $-\frac{r}{2R}$, since the circumradius of II_bI_c is also $2R$. In fact, the circumcenter of II_bI_c is the reflection of I_a in O. It follows that $\frac{I_aN_a}{I_a'O} = -\frac{I_aN_a}{OI_a} = -\frac{r}{2R}$, and N_a is the nine-point center of the A-extouch triangle (see Figure 2).

Theorem 3. *The points O, N, F_a and N_a are concyclic; so are O, N, F_b, N_b, and O, N, F_c, N_c.*

2. Concurrency of four lines on the nine-point circle

In preparation for the proof of our next main result (Theorem 7 below), we establish an interesting relation between the centers O, N, I, I_a given in Proposition 5. The reformulation as Proposition 6 in terms of directed angles ([2, §§16-19]) makes the proof of Theorem 7 independent of the relative position of O and N with respect to the bisector of angle A.

Lemma 4. *Let N be the nine-point center of triangle ABC.*
(a) *If $A > 60°$, then N and O lie on the same side of the bisector of angle A.*
(b) *If $A = 60°$, then N lies on the bisector of angle A.*
(c) *If $A < 60°$, then N and O lie on opposite sides of the bisector of angle A.*

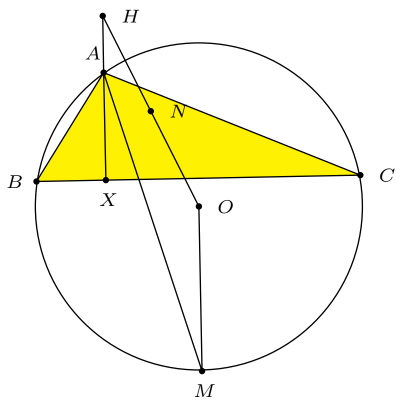

Figure 3A

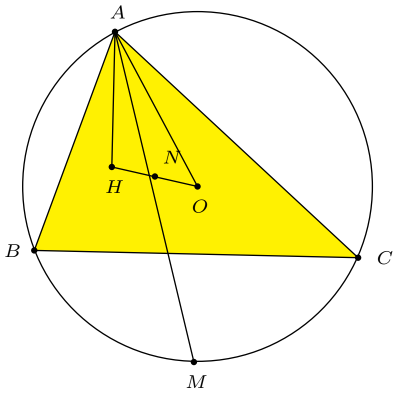

Figure 3B

Proof. First consider the case when A is an obtuse angle (see Figure 3A). Construct the perpendicular from O to BC, to intersect the circumcircle at M on the opposite side of A. The line AM is clearly the bisector of angle A. If X is the orthogonal projection of A on BC, then the orthocenter H and X are on opposite sides of A. It follows that O and H, and their midpoint N, all are on the same side of the bisector AM.

The same conclusion holds if $A = 90°$, since the orthocenter H coincides with A.

Now we assume A an acute angle (see Figure 3B). It is known that $AH = 2R\cos A$, and that the bisector of angle A also bisects the angle OAH. It divides

OH in the ratio $R : 2R\cos A = 1 : 2\cos A$. Therefore, O and N are on the same side of the bisector if and only if $2\cos A < 1$, i.e., $A > 60°$.

If $A = 60°$, then $AH = AO$, and N lies on the bisector. This completes the proof of the theorem. □

Proposition 5. (a) *The angle IOI_a is acute, right, or obtuse according as A is less than, equal to, or greater than* $60°$.

(b) $\angle INI_a = \begin{cases} 2\angle IOI_a, & \text{if } A \leq 60°, \\ 360° - 2\angle IOI_a, & \text{if } A > 60°. \end{cases}$.

Proof. (a) Applying the law of cosines to triangle IOI_a, and using the expressions for the lengths given in Proposition 2, we have

$$\begin{aligned} \cos IOI_a = &\ \frac{OI^2 + OI_a^2 - II_a^2}{2 \cdot OI \cdot OI_a} \\ = &\ \frac{R(R-2r) + R(R+2r_a) - 4R(r_a - r)}{2 \cdot OI \cdot OI_a} \\ = &\ \frac{R(R - r_a + r)}{OI \cdot OI_a}. \end{aligned} \tag{1}$$

The angle IOI_a is acute, right, or obtuse according as $r_a - r$ is less than, equal to or greater than R. Since $\frac{r_a - r}{R} = 4\sin^2\frac{A}{2}$, these are the cases according as A is less than, equal to, greater than $60°$.

(b) From (1), we have

$$\begin{aligned} \cos 2 \cdot IOI_a = &\ 2\cos^2 IOI_a - 1 \\ = &\ \frac{(OI^2 + OI_a^2 - II_a^2)^2 - 2 \cdot OI^2 \cdot OI_a^2}{2 \cdot OI^2 \cdot OI_a^2} \\ = &\ \frac{(2R(R - r_a + r))^2 - 2R(R-2r) \cdot R(R+2r_a)}{2R(R-2r) \cdot R(R+2r_a)} \\ = &\ \frac{R^2 - 6R(r_a - r) + 2(r_a^2 + r^2)}{(R-2r)(R+2r_a)}. \end{aligned} \tag{2}$$

On the other hand,

$$\begin{aligned} \cos INI_a = &\ \frac{NI^2 + NI_a^2 - II_a^2}{2NI \cdot NI_a} \\ = &\ \frac{\left(\frac{R}{2} - r\right)^2 + \left(\frac{R}{2} + r_a\right)^2 - 4R(r_a - a)}{2\left(\frac{R}{2} - r\right)\left(\frac{R}{2} + r_a\right)} \\ = &\ \frac{R^2 - 6R(r_a - r) + 2(r_a^2 + r^2)}{(R-2r)(R+2r_a)}. \end{aligned} \tag{3}$$

Comparison of (2) and (3) shows that $\cos INI_a = \cos 2 \cdot IOI_a$. Therefore, $\angle INI_a = 2\angle IOI_a$ or $360° - 2\angle IOI_a$. Taking (a) into account, we have (b). □

Remark. When $A = 60°$, $\angle IOI_a = 90°$, and $\angle INI_a = 180°$ (see Lemma 4(b)).

In terms of directed angles, we reformulate Proposition 5 below.

Proposition 6. $(NI,\ NI_a) = -2(OI,\ OI_a)$.

Theorem 7. *The four lines $F_{\mathrm{e}}N_{\mathrm{i}}$, F_aN_a, F_bN_b, F_cN_c are concurrent at a point on the nine-point circle.*

Proof. It is enough to prove this for the lines $F_{\mathrm{e}}N_{\mathrm{i}}$ and F_aN_a (see Figure 4). Let P be the intersection of the line $F_{\mathrm{e}}N_{\mathrm{i}}$ with the nine-point circle. We show that it also lies on the line F_aN_a. For this, it is enough to verify $(F_aN,\ F_aP) = (F_aN,\ F_aN_a)$.

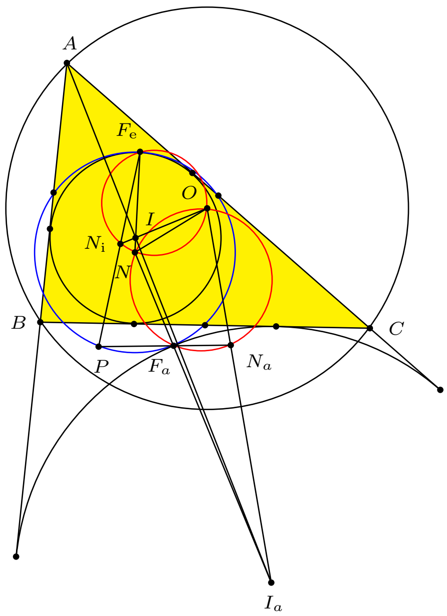

Figure 4.

$$\begin{aligned}
(F_aN,\ F_aP) &= (PF_a,\ PN) && \text{triangle } NF_aP \text{ isosceles}\\
&= (PF_a,\ PF_{\mathrm{e}}) + (PF_{\mathrm{e}},\ PN) &&\\
&= \frac{1}{2}(NF_a,\ NF_{\mathrm{e}}) + (PF_{\mathrm{e}},\ PN) && N = \text{center of circle } PF_{\mathrm{e}}F_a\\
&= \frac{1}{2}(NI_a,\ NI) + (PF_{\mathrm{e}},\ PN) &&\\
&= (OI,\ OI_a) + (PF_{\mathrm{e}},\ PN) && \text{Proposition 6}\\
&= (OI,\ OI_a) + (NF_{\mathrm{e}},\ N_{\mathrm{i}}F_{\mathrm{e}}) && \text{triangle } NPF_{\mathrm{e}} \text{ isosceles}\\
&= (OI,\ OI_a) + (NO,\ N_{\mathrm{i}}O) && O,\ N,\ F_{\mathrm{e}},\ N_{\mathrm{i}} \text{ concyclic}\\
&= (OI,\ ON_a) + (ON,\ OI) &&\\
&= (ON,\ ON_a) &&\\
&= (F_aN,\ F_aN_a) && O,\ N,\ F_a,\ N_a \text{ concyclic.}
\end{aligned}$$

The same reasoning shows that P also lies on the lines F_bN_b and F_cN_c. □

We summarize the main results in this note in Figure 5 below, and conclude with an identification of the triangle centers $N_{\rm i}$ and P. According to the ENCYCLOPEDIA OF TRIANGLE CENTERS [3], the intouch triangle has orthocenter $X(65)$. Its nine-point center $N_{\rm i}$, being the midpoint of $IX(65)$, is $X(942)$. This point lies on the line through $X(11) = F_{\rm e}$ and $X(113)$, which is on the nine-point circle, Therefore P is $X(113)$, which is also the midpoint of $HX(110)$.

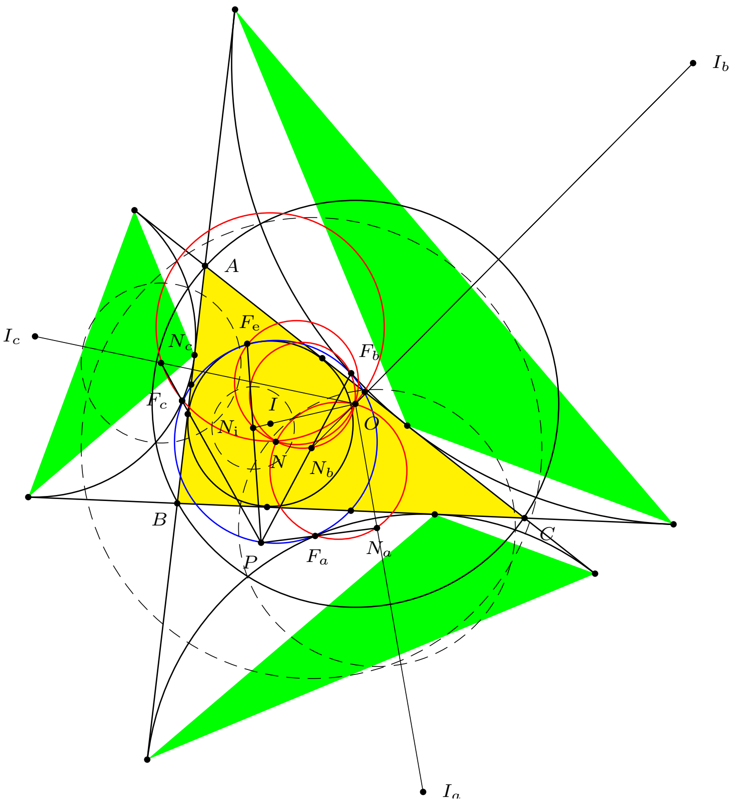

Figure 5.

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