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Angle and Circle Characterizations of Tangential Quadrilaterals

Martin Josefsson

Abstract. We prove five necessary and sufficient conditions for a convex quadrilateral to have an incircle that concerns angles or circles.

1. Introduction

A tangential quadrilateral is a convex quadrilateral with an incircle, i.e., a circle inside the quadrilateral that is tangent to all four sides. In [4] and [5] we reviewed and proved a total of 20 different necessary and sufficient conditions for a convex quadrilateral to be tangential. Of these there were 14 dealing with different distances (sides, line segments, radii, altitudes), four were about circles (excluding their radii), and only two were about angles. In this paper we will prove five more such characterizations concerning angles and circles. First we review two that can be found elsewhere.

A characterization involving the four angles and all four sides of a quadrilateral appeared as part of a proof of an inverse altitude characterization of tangential quadrilaterals in [6, p.115]. According to it, a convex quadrilateral $ABCD$ with sides $a = AB$, $b = BC$, $c = CD$ and $d = DA$ is tangential if and only if
\[ a \sin A \sin B + c \sin C \sin D = b \sin B \sin C + d \sin D \sin A. \]

In the extensive monograph [9, p.133] on quadrilateral geometry, the following characterization is attributed to Simionescu. A convex quadrilateral is tangential if and only if its consecutive sides $a$, $b$, $c$, $d$ and diagonals $p$, $q$ satisfies
\[ |ac - bd| = pq \cos \theta \]
where $\theta$ is the acute angle between the diagonals. The proof is a simple application of the quite well known identity $2pq \cos \theta = |b^2 + d^2 - a^2 - c^2|$ that holds in all convex quadrilaterals. Rewriting it as
\[ 2pq \cos \theta = |(b + d)^2 - (a + c)^2 + 2(ac - bd)|, \]
we see that Simionescu’s theorem is equivalent to Pitot’s theorem $a + c = b + d$ for tangential quadrilaterals. In Theorem 2 we will prove another characterization for the angle between the diagonals, but it only involves four different distances instead of six.

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2. Angle characterizations of tangential quadrilaterals

It is well known that a convex quadrilateral has an incircle if and only if the four angle bisectors of the internal vertex angles are concurrent. If this point exist, it is the incenter. Here we shall prove a necessary and sufficient condition for an incircle regarding the intersection of two opposite angle bisectors which characterize the incenter in terms of two angles in two different ways. To prove that one of these equalities holds in a tangential quadrilateral (the direct theorem) was a problem in [1, p.67].

**Theorem 1.** A convex quadrilateral $ABCD$ is tangential if and only if

$$\angle AIB + \angle CID = \pi = \angle AID + \angle BIC$$

where $I$ is the intersection of the angle bisectors at $A$ and $C$.

**Proof.** ($\Rightarrow$) In a tangential quadrilateral the four angle bisectors intersect at the incenter. Using the sum of angles in a triangle and a quadrilateral, we have

$$\angle AIB + \angle CID = \pi - \left(\frac{A}{2} + \frac{B}{2}\right) + \pi - \left(\frac{C}{2} + \frac{D}{2}\right) = 2\pi - \frac{2\pi}{2} = \pi.$$

The second equality can be proved in the same way, or we can use that the four angles in the theorem make one full circle, so $\angle AID + \angle BIC = 2\pi - \pi = \pi$.

($\Leftarrow$) In a convex quadrilateral where $I$ is the intersection of the angle bisectors at $A$ and $C$, and the equality

$$\angle AIB + \angle CID = \angle AID + \angle BIC$$

holds, assume without loss of generality that $AB > AD$ and $BC > CD$. Construct points $D'$ and $D''$ on $AB$ and $BC$ respectively such that $AD' = AD$ and $CD'' = CD$ (see Figure 1). Then triangles $AID'$ and $AID$ are congruent, and so are triangles $CID''$ and $CID$. Thus $ID' = ID = ID''$. These two pairs

1If instead there is equality in one of these inequalities, then it’s easy to see that the quadrilateral is a kite. It’s well known that kites have an incircle.
of congruent triangles and (1) yields that \( \angle BID' = \angle BID'' \), so triangles \( BID' \) and \( BID'' \) are congruent. Thus \( BD' = BD'' \). Together with \( AD' = AD \) and \( CD'' = CD \), we get

\[
AD' + D'B + CD = AD + BD'' + D''C \quad \Rightarrow \quad AB + CD = AD + BC.
\]

Then \( ABCD \) is a tangential quadrilateral according to Pitot’s theorem.

The idea for the proof of the converse comes from [8], where Goutham used this method to prove the converse of a related characterization of tangential quadrilaterals concerning areas. That characterization states that if \( I \) is the intersection of the angle bisectors at \( A \) and \( C \) in a convex quadrilateral \( ABCD \), then it has an incircle if and only if

\[
S_{AIB} + S_{CID} = S_{AID} + S_{BIC},
\]

where \( S_{XYZ} \) stands for the area of triangle \( XYZ \). According to [9, p.134], this theorem is due to V. Pop and I. Gavrea. In [6, pp.117–118] a similar characterization concerning the same four areas was proved, but it also includes the four sides. It states that \( ABCD \) is a tangential quadrilateral if and only if

\[
c \cdot S_{AIB} + a \cdot S_{CID} = b \cdot S_{AID} + d \cdot S_{BIC},
\]

where \( a = AB \), \( b = BC \), \( c = CD \) and \( d = DA \).

The next characterization is about the angle between the diagonals. We will assume we know the lengths of the four parts that the intersection of the diagonals divide them into. Then the question is, what size the angle between the diagonals shall have for the quadrilateral to have an incircle? This means that the sides of the quadrilateral are not fixed and the lengths of them changes as we vary the angle between the diagonals. See Figure 2. If \( \theta \to 0 \), then clearly \( a + c < b + d \); and if \( \theta \to \pi \), then \( a + c > b + d \). Hence for some \( 0 < \theta < \pi \) we have \( a + c = b + d \), and the quadrilateral has an incircle.

![Figure 2. The diagonal parts](image-url)
Theorem 2. If the diagonals of a convex quadrilateral are divided into parts \( w, x \) and \( y, z \) by their point of intersection, then it is a tangential quadrilateral if and only if the angle \( \theta \) between the diagonals satisfies

\[
\cos \theta = \frac{(w - x)(y - z)(2wx + yz) - \sqrt{(w + x)^2(y + z)^2 + 4(wx - yz)^2}}{(w + x)(y + z)^2 - 16wxz}.
\]

Proof. A convex quadrilateral is tangential if and only if its consecutive sides \( a, b, c, d \) satisfies Pitot’s theorem

\[
a + c = b + d. \tag{2}
\]

The sides of the quadrilateral can be expressed in terms of the diagonal parts and the angle between the diagonals using the law of cosines, according to which (see Figure 2)

\[
a^2 = w^2 + y^2 - 2wy \cos \theta,
\]
\[
b^2 = x^2 + y^2 + 2xy \cos \theta,
\]
\[
c^2 = x^2 + z^2 - 2xz \cos \theta,
\]
\[
d^2 = w^2 + z^2 + 2wz \cos \theta.
\]

Here we used \( \cos(\pi - \theta) = -\cos \theta \) in the second and fourth equation. Inserting these into (2) yields

\[
\sqrt{w^2 + y^2 - 2wy \cos \theta} + \sqrt{x^2 + z^2 - 2xz \cos \theta}
\]

\[
= \sqrt{x^2 + y^2 + 2xy \cos \theta} + \sqrt{w^2 + z^2 + 2wz \cos \theta}.
\]

The algebra involved in solving this equation including four square roots is not simple. For this reason we will use a computer calculation to solve it. Squaring both sides results in a new equation with only two square roots. Collecting them alone on one side of the equality sign and squaring again gives another equation, this time with only one square root. The last step in the elimination of the square roots is to separate that last one from the other terms, on one side, and squaring a third time. This results in a polynomial equation in \( \cos \theta \) that has 115 terms!

Factoring that with the computer, we obtain

\[
(w + x)^2(y + z)^2(-1 + T)(1 + T)
\]
\[
\cdot (-w^2y^2 + 2wy^2x - y^2x^2 + 2w^2yz - 4wyxz + 2x^2z - w^2z^2 + 2wz^2)
\]
\[
- x^2z^2 - 4w^2yzT + 4wyx^2T - 4wy^2zT + 4u^2xT + 4y^2xzT - 4wx^2zT
\]
\[
+ 4wy^2zT - 4y^2T - 4yx^2T + w^2y^2T^2 + 2wy^2xT^2 + y^2x^2T^2 + 2w^2yzT^2
\]
\[
- 12wyxzT^2 + 2y^2x^2T^2 + w^2z^2T^2 + 2wxz^2T^2 + x^2z^2T^2) = 0
\]

where we put \( T = \cos \theta \). None of the factors but the last parenthesis gives any valid solutions. Solving the quadratic equation in the last parenthesis with the computer yields

\[
T = \frac{4(w - x)(y - z)(wx + yz) \pm \sqrt{4(w - x)^2(y - z)^2P_1}}{2P_2}, \tag{3}
\]
where
\[ P_1 = w^2y^2 + 2wyz + 2w^2x^2 + y^2x^2 + 2w^2yz + 4w^2x^2z + 2w^2y^2x \]
\[ + w^2z^2 + 4y^2z^2 + 2w^2z^2 + x^2z^2 \]
and
\[ P_2 = w^2y^2 + 2wyz + 2w^2x^2 + y^2x^2 + 2w^2yz - 12w^2x^2z + 2w^2z^2 + 2w^2x^2z + x^2z^2 \]
\[ = (wz + xz)^2 + (wz + yx)^2 + 2(wz + xz)(wz + yx) - 4wxz - 12wxyz \]
\[ = (w + x + y + y)^2 - 16wxyz = (w + x)^2(y + z)^2 - 16wxyz. \]

Thus
\[ P_1 = (w + x)^2(y + z)^2 - 8wxz + 4w^2x^2 + 4y^2z^2 \]
\[ = (w + x)^2(y + z)^2 + 4(wx - yz)^2. \]

Inserting the simplified expressions for \( P_1 \) and \( P_2 \) into the solutions (3) and factoring them, we get
\[ \cos \theta = \frac{(w - x)(y - z)(2wx + yz)}{(w + x)^2(y + z)^2 - 16wxyz}. \]

To determine the correct sign, we study a special case. In an isosceles tangential trapezoid where \( w = y = 2u \) and \( x = z = u \) (here \( u \) is an arbitrary positive number), we have
\[ \cos \theta = \frac{u^2(8u^2 - 9u^2 + 9u^4)}{9u^2 - 16 - 4u^4} = \frac{8 \pm 9}{17}. \]

For the solution with the plus sign, we get \( \cos \theta = 1 \). Thus \( \theta = 0 \) which is not a valid solution. Hence the correct solution is the one with the minus sign. \( \square \)

**Corollary 3.** A convex quadrilateral where one diagonal bisect the other has an incircle if and only if it is a kite.

**Proof.** (\( \Rightarrow \)) If in a tangential quadrilateral \( w = x \) or \( y = z \), then the formula in the theorem indicates that \( \cos \theta = 0 \). Thus \( \theta = \frac{\pi}{2} \), so one diagonal is the perpendicular bisector of the other. Then the quadrilateral must be a kite, since one diagonal is a line of symmetry.

(\( \Leftarrow \)) If the quadrilateral is a kite (they always have the property that one diagonal bisect the other), then it has an incircle according to Pitot’s theorem. \( \square \)

### 3. Circle characterizations of tangential quadrilaterals

To prove the first circle characterization we need the following theorem concerning the extended sides, which we reviewed in [4] and [5]. Since it is quite rare to find a proof of it in modern literature (particularly the converse), we start by proving it here. It has been known at least since 1846 according to [10].

---

2Here we used that \( \sqrt{(w - x)^2(y - z)^2} = (w - x)(y - z) \). We don’t have to put absolute values since there is \( \pm \) in front of the square root and we don’t yet know which sign is correct.
**Theorem 4.** In a convex quadrilateral \(ABCD\) that is not a trapezoid, let the extensions of opposite sides intersect at \(E\) and \(F\). If exactly one of the triangles \(AEF\) and \(CEF\) is outside of the quadrilateral \(ABCD\), then it is a tangential quadrilateral if and only if
\[
AE + CF = AF + CE.
\]

![Figure 3. Tangential quadrilateral with extended sides](image)

**Proof.** \((\Rightarrow)\) In a tangential quadrilateral, let the incircle be tangent to the sides \(AB, BC, CD, DA\) at \(W, X, Y, Z\) respectively. We apply the two tangent theorem (that two tangents to a circle through an external point are congruent) several times to get (see Figure 3)
\[
AE + CF = AW + EW + FX - CX = AZ + EY + FZ - CY = AF + CE.
\]

\((\Leftarrow)\) We do an indirect proof of the converse. In a convex quadrilateral where \(AE + CF = AF + CE\), we draw a circle tangent to the sides \(AB, BC, CD\). If this circle is not tangent to \(DA\), draw a tangent to the circle parallel to \(DA\). This tangent intersect \(AB, CD\) and \(BF\) at \(A', D'\) and \(F'\) respectively (see Figure 4). We assume \(DA\) does not cut the circle; the other case can be proved in the same way. Also, let \(G\) be a point on \(DA\) such that \(A'G\) is parallel to (and thus equal to) \(F'F\). From the direct part of the theorem we now have
\[
A'F' + CE = A'E + CF'.
\]
Subtracting this from \(AE + CF = AF + CE\), we get
\[
AG = AA' + A'G.
\]
This equality is a contradiction since it violates the triangle inequality in triangle \(AGA'\). Hence the assumption that \(DA\) was not tangent to the circle must be incorrect. Together with a similar argument in the case when \(DA\) cuts the circle this completes the proof. \(\Box\)

\(^3\)And thus not a parallelogram, rhombus, rectangle or a square either.
Remark. If both triangles $AEF$ and $CEF$ are outside of the quadrilateral $ABCD$, then the characterization for a tangential quadrilateral is $BE + DF = BF + DE$. It is obtained by relabeling the vertices according to $A \rightarrow B \rightarrow C \rightarrow D \rightarrow A$ in comparison to Theorem 4.

The direct part of the first circle characterization was a problem proposed and solved at [7]. We will use Theorem 4 to give a very short proof including the converse as well.

**Theorem 5.** In a convex quadrilateral $ABCD$ that is not a trapezoid, let the extensions of opposite sides intersect at $E$ and $F$. If exactly one of the triangles $AEF$ and $CEF$ is outside of the quadrilateral $ABCD$, then it is a tangential quadrilateral if and only if the incircles in triangles $AEF$ and $CEF$ are tangent to $EF$ at the same point.

**Proof.** It is well known that in a triangle, the distance from a vertex to the point where the incircle is tangent to a side is equal to the semiperimeter of the triangle subtracted by the side opposite to that vertex [2, p.184]. Now assume the incircles in triangles $AEF$ and $CEF$ are tangent to $EF$ at $G$ and $H$ respectively. Then we have (see Figure 5)

$$2(FG - FH) = (EF + AF - AE) - (EF + CF - CE) = AF + CE - AE - CF.$$

Hence

$$G \equiv H \iff FG = FH \iff AE + CF = AF + CE$$

which proves that the two incircles are tangent at the same point on $EF$ if and only if the quadrilateral is tangential according to Theorem 4. □

Remark. If both triangles $AEF$ and $CEF$ are outside of the quadrilateral $ABCD$ (this happens if $F$ is below $AB$ or $E$ is to the left of $AD$ in Figure 5), then the
Theorem 5. In a convex quadrilateral $ABCD$ that is not a trapezoid, let the extensions of opposite sides $AB$ and $DC$ intersect at $E$, and the extensions of opposite sides $BC$ and $AD$ intersect at $F$. Let the incircle in triangle $AEF$ be tangent to $AE$ and $AF$ at $K$ and $L$ respectively, and the incircle in triangle $CEF$ be tangent to $BF$ and $DE$ at $M$ and $N$ respectively. If exactly one of the triangles $AEF$ and $CEF$ is outside of the quadrilateral $ABCD$, then it is a tangential quadrilateral if and only if $KLMN$ is a cyclic quadrilateral.

Figure 5. Two tangent points at $EF$

Figure 6. Here $ABCD$ is a tangential quadrilateral
Proof. (\(\Rightarrow\)) In a tangential quadrilateral \(ABCD\), the incircles in triangles \(AEF\) and \(CEF\) are tangent to \(EF\) at the same point \(G\) according to Theorem 5. This together with the two tangent theorem yields that \(EK = EG = EN\) and \(FL = FG = FM\), so the triangles \(EKN\) and \(FLM\) are isosceles (see Figure 6). Thus \(\angle ENK = \frac{A+D}{2}\) and \(\angle FLM = \frac{A+B}{2}\). Triangles \(ALK\) and \(CNM\) are also isosceles, so \(\angle ALK = \frac{\pi-A}{2}\) and \(\angle CNM = \frac{\pi-C}{2}\). Hence for two opposite angles in quadrilateral \(KLMN\), we get

\[
\angle KLM + \angle KNM = \left(\pi - \frac{A + B}{2} - \frac{\pi - A}{2}\right) + \left(\frac{\pi - C}{2} + \frac{\pi - A + D}{2}\right)
= 2\pi - \frac{A + B + C + D}{2} = \pi
\]

where we used the sum of angles in a quadrilateral. This means that \(KLMN\) is a cyclic quadrilateral according to a well known characterization.

\[\text{Figure 7. Here } ABCD \text{ is not a tangential quadrilateral}\]

\((\Leftarrow)\) If \(ABCD\) is not a tangential quadrilateral, we shall prove that \(KLMN\) is not a cyclic quadrilateral. When \(ABCD\) is not tangential, the incircles in triangles \(AEF\) and \(CEF\) are tangent to \(EF\) at different points \(G\) and \(H\) respectively. We assume without loss of generality that \(G\) is closer to \(F\) than \(H\) is. Thus \(EK = EG > EH = EN\) and \(FL = FG < FH = FM\) (see Figure 7). Applying that in a triangle, a longer side is opposite a larger angle, we get \(\angle ENK > \frac{A+D}{2}\) and \(\angle FLM > \frac{A+B}{2}\). Triangles \(ALK\) and \(CNM\) are still isosceles. This yields that

\[
\angle KLM < \pi - \frac{A + B}{2} - \frac{\pi - A}{2} = \frac{\pi - B}{2}
\]

\[\text{The other case can be dealt with in the same way. What happens is that all inequalities below will be reversed.}\]
and \( \angle KND < \pi - \frac{A+D}{2} \). Hence for two opposite angles in \( KLMN \),

\[
\angle KLM + \angle KNM < \frac{\pi - B}{2} + \left( \frac{\pi - C}{2} + \pi - \frac{A+D}{2} \right) = \pi,
\]

again using the sum of angles in a quadrilateral. This proves that if \( ABCD \) is not a tangential quadrilateral, then \( KLMN \) is not a cyclic quadrilateral. \( \square \)

**Corollary 7.** The incircle in \( ABCD \) and the circumcircle to \( KLMN \) in Theorem 6 are concentric.

![Figure 8. The two concentric circles](image)

**Proof.** The incircle in \( ABCD \) is also an incircle in triangles \( AED \) and \( AFB \) (see Figure 8). The perpendicular bisectors of the sides \( KN \) and \( LM \) are also angle bisectors to the angles \( AED \) and \( AFB \), hence they intersect at the incenter of \( ABCD \). This proves that the two circles are concentric. \( \square \)

Next we will study a related configuration to the one in Theorem 6, with two other incircles. In [4, pp.66–67] we proved that in a convex quadrilateral \( ABCD \), the two incircles in triangles \( ABD \) and \( CBD \) are tangent to \( BD \) at the same point if and only if \( ABCD \) is a tangential quadrilateral. These two incircles are also tangent to all four sides of the quadrilateral (two tangency point per circle). In [11, pp.197–198] it was proved that if \( ABCD \) is a tangential quadrilateral, then these four tangency points are the vertices of a cyclic quadrilateral that is concentric with the incircle in \( ABCD \). Another proof of the concyclic property of the four tangency points was given in [9, pp.272–273]. Now we shall prove that the converse is true as well and thus get another characterization of tangential quadrilaterals.

**Theorem 8.** In a convex quadrilateral \( ABCD \), let the incircles in triangles \( ABD \) and \( CBD \) be tangent to the sides of \( ABCD \) at \( K, L, M, N \). Then \( ABCD \) is a tangential quadrilateral if and only if \( KLMN \) is a cyclic quadrilateral.
Angle and circle characterizations of tangential quadrilaterals

Figure 9. Here $ABCD$ is not a tangential quadrilateral

\textbf{Proof.} Only the proof of the converse is given, but a proof of the direct theorem is obtained by simply changing all the inequalities below to equalities. Thus we prove that if $ABCD$ is not a tangential quadrilateral, then $KLMN$ is not a cyclic quadrilateral.

Let the incircles in triangles $ABD$ and $CBD$ be tangent to $BD$ at $G$ and $H$ respectively, and assume without loss of generality that $G$ is closer to $D$ than $H$ is. If $K, L, M, N$ are the tangency points at $AB, AD, CD$ and $CB$ respectively, then according to the two tangent theorem $BK = BG > BH = BN$ and $DL = DG < DH = DM$ (see Figure 9). Since a larger angle in a triangle is opposite a longer side, we have that $\angle BKN < \frac{\pi - B}{2}$. Also, $\angle AKL = \frac{\pi - A}{2}$ since triangle $AKL$ is isosceles. Thus

$$\angle LKN > \pi - \frac{\pi - B}{2} + \frac{\pi - A}{2} = \frac{A + B}{2}.$$ 

In the same way we have $\angle DML < \frac{\pi - D}{2}$ and $\angle CMN = \frac{\pi - C}{2}$, so

$$\angle LMN > \pi - \frac{\pi - D}{2} + \frac{\pi - C}{2} = \frac{C + D}{2}.$$ 

Hence for two opposite angles in $KLMN$,

$$\angle LKN + \angle LMN > \frac{A + B + C + D}{2} = \pi.$$ 

This proves that if $ABCD$ is not a tangential quadrilateral, then $KLMN$ is not a cyclic quadrilateral. \hfill \Box

\section{4. A related characterization of a bicentric quadrilateral}

A bicentric quadrilateral is a convex quadrilateral that is both tangential and cyclic, i.e., it has both an incircle and a circumcircle. In a tangential quadrilateral $ABCD$, let the incircle be tangent to the sides $AB, BC, CD, DA$ at $W, X, Y, Z$ respectively. It is well known that the quadrilateral $ABCD$ is also cyclic (and hence bicentric) if and only if the tangency chords $WY$ and $XZ$ are perpendicular.
[3, p.124]. Now we will prove a similar characterization concerning the configuration of Theorem 6.

**Theorem 9.** In a tangential quadrilateral $ABCD$ that is not a trapezoid, let the extensions of opposite sides $AB$ and $DC$ intersect at $E$, and the extensions of opposite sides $BC$ and $AD$ intersect at $F$. Let the incircle in triangle $AEF$ be tangent to $AE$ and $AF$ at $K$ and $L$ respectively, and the incircle in triangle $CEF$ be tangent to $BF$ and $DE$ at $M$ and $N$ respectively. If exactly one of the triangles $AEF$ and $CEF$ is outside of the quadrilateral $ABCD$, then it is a bicentric quadrilateral if and only if the extensions of $KN$ and $LM$ are perpendicular.

![Diagram of quadrilateral ABCD with incircles and tangents](image)

**Figure 10. Angle between extensions of opposite sides of $KLMN$**

**Proof.** Let $J$ be the intersection of the extensions of $KN$ and $LM$, and $v$ the angle between them. Then $\angle JNC = \angle ENK = \frac{A+D}{2}$ and $\angle JMC = \angle FML = \frac{A+B}{2}$ (see Figure 10). Thus, using the sum of angles in quadrilateral $CMJN$, we have

$$v = 2\pi - C - \frac{A + B}{2} - \frac{A + D}{2} = 2\pi - \frac{A + B + C + D}{2} - \frac{A + C}{2} = \pi - \frac{A + C}{2}.$$ 

Hence

$$v = \frac{\pi}{2} \iff A + C = \pi$$

so the extensions of $KN$ and $LM$ are perpendicular if and only if the tangential quadrilateral $ABCD$ is also cyclic. \qed

**References**


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The Associated Harmonic Quadrilateral

Paris Pamfilos

Abstract. In this article we study a natural association of a harmonic quadrilateral to every non-parallelogrammic quadrilateral. In addition we investigate the corresponding association in the case of cyclic quadrilaterals and the reconstruction of the quadrilateral from its harmonic associated one. Finally, we associate to a generic quadrilateral a cyclic one.

1. Harmonic quadrilaterals

Harmonic quadrilaterals, introduced by Tucker and studied by Neuberg ([1, p.206], [6]) can be defined in various equivalent ways. A simple one is to draw the tangents $FA, FC$ to a circle $\kappa$ from a point $F$ (can be at infinity) and draw also an additional secant $FBD$ to the circle (see Figure 1(I)). Another definition starts with an arbitrary triangle $ABD$ and its circumcircle $\kappa$ and defines $C$ as the intersection of $\kappa$ with the symmedian from $A$. These convex quadrilaterals have several interesting properties exposed in textbooks and articles ([5, p.100, p.306], [8]). One of them, used in the sequel, is their characterization as convex cyclic quadrilaterals, for which the products of opposite side-lengths are equal $|AB||CD| = |BC||DA|$ or, equivalently, the ratios of adjacent side-lengths are equal $\frac{|AB|}{|AD|} = \frac{|CB|}{|CD|}$. Another property, also used below, deals with a dissection of the quadrilateral in similar triangles (see Figure 1(II)), which I formulate as a lemma without a proof.

Lemma 1. Let $ABCD$ be a harmonic quadrilateral and $P$ be the projection of its circumcenter $K$ onto the diagonal $BD$. Then triangles $ADC, APB$ and $BPC$ are similar. Furthermore, the tangents of its circumcircle at points $A$ and $C$ intersect.
at a point $F$ of the diagonal $BD$ and the circumcircle $\delta$ of $ACF$ passing through $K$ and $P$.

Figure 2. Determination by $\omega$ and $r = \frac{|AB|}{|AD|}$

Kite

Note that, up to similarity, a harmonic quadrilateral is uniquely determined by its angle $\omega = \angle BCD$ and the ratio $r = \frac{|AB|}{|AD|} = \frac{|CB|}{|CD|}$ (see Figure 2(I)). In fact, fixing the circle $\kappa$ and taking an inscribed angle of measure $\omega$, the angle-sides determine a chord $BD$ of length depending only on $\kappa$ and $\omega$. Then, points $A, C$ on both sides of $BD$ are determined by intersecting $\kappa$ with the Apollonius circle ([2, p.15]), dividing $BD$ in the given ratio $r$.

A special class of harmonic quadrilaterals, comprising the squares, is the one of kites, which are symmetric with respect to one of their diagonals (see Figure 2(II)). Excluding this special case, for all other harmonic quadrilaterals there is a kind of symmetry with respect to the two diagonals, having the consequence, that in all properties, including one of the diagonals, it is irrelevant which one of the two is actually chosen.

Figure 3. Harmonic trapezia

Another class of special harmonic quadrilaterals is the one of harmonic trapezia, comprising all equilateral trapezia with side lengths satisfying $ac = b^2$ (see Figure
The associated harmonic quadrilateral 17

3. This, up to similarity, is also a one-parameter family of harmonic quadrilaterals. Given the circle \( \kappa(r) \), each harmonic trapezium, inscribed in \( \kappa \), is determined by the ratio \( k = \frac{a}{b} < 1 \) of the small parallel to the non-parallel side-length. A short calculation shows that to each such trapezium corresponds a special triangle \( ABD \) with data

\[
a = k' r, \quad b = \frac{k'}{k} r, \quad \cos B = \frac{1 - k^2}{2k},
\]

where \( k' = \sqrt{\frac{4k^2 - (1 - k^2)^2}{2}} \).

2. The associated harmonic quadrilateral

In the sequel we restrict ourselves to non-parallellogrammic convex quadrilaterals. For every such quadrilateral \( p = ABCD \) there is a harmonic quadrilateral \( q \), naturally associated to \( p \). The next theorem shows how to construct it.

**Theorem 2.** The two centers \( Z_1, Z_2 \) of the similarities \( f_1, f_2 \), mapping respectively \( f_1(A) = C, f_1(B) = D, f_2(B) = D, f_2(C) = A \), of a non-parallelogrammic quadrilateral \( p = ABCD \), together with the midpoints \( M, N \) of its diagonals \( AC, BD \), are the vertices of a harmonic quadrilateral \( q = N Z_1 M Z_2 \), whose circumcircle \( \kappa \) passes through the intersection point \( E \) of the diagonals.

In fact, let \( \kappa \) be the circle passing through the midpoints \( M, N \) of the diagonals and also passing through their intersection point \( E \). Some special cases in which point \( E \) is on line \( MN \) are handled below. Point \( Z_1 \) is the center of similarity \( f_1 \) ([3, p.72], [11, II, p.43]) mapping the triangle \( ABZ_1 \) correspondingly onto \( CDZ_1 \) (see Figure 4). Analogously, point \( Z_2 \) is the center of the similarity mapping the triangle \( BCZ_2 \) onto \( DAZ_2 \). It follows easily, that the triangles based on the diagonals, \( ACZ_1 \) and \( BDZ_1 \), are also similar, their similarity ratio being equal to those of their medians from \( Z_1 \), as well as their corresponding bases coinciding with the diagonals \( \lambda = \frac{|Z_1 N|}{|Z_1 M|} = \frac{|AC|}{|BD|} \). This implies also that the angles formed by corresponding medians of the two similar triangles are equal, i.e., \( ANZ_1 \) and...
$EMZ_1$ are equal angles. This implies that $Z_1$ is on $\kappa$. Analogously is seen that $Z_2$ is also on $\kappa$ and that the ratio $\frac{|Z_2N|}{|Z_2M|} = \lambda$. Thus,

$$\frac{|Z_1N|}{|Z_1M|} = \frac{|Z_2N|}{|Z_2M|}$$

which means that the cyclic quadrilateral $Z_1MZ_2N$ is harmonic.

In the case one of the midpoints of the diagonals coincides with their intersection point ($N = E$) the circle $\kappa$ passes through the midpoints $M, N$ of the diagonals and is tangent to the diagonal $(AC)$, whose midpoint coincides with $E$ (see Figure 5(I)). Another particular class is the one of trapezia, characterized by the fact that one of the similarity centers ($Z_1$) coincides with the intersection $E$ of the diagonals (see Figure 5(II)).

3. The inverse construction

Fixing a harmonic quadrilateral $q$ and selecting two opposite vertices $Z_1, Z_2$ of it, we can easily construct all convex quadrilaterals $p$ having the given one as their associated. This reconstruction is based on the following lemma.

$E$ coinciding with $N$  $Z_1$ coinciding with $E$

Figure 5. Point $E$ coinciding with $N$  Point $Z_1$ coinciding with $E$

Figure 6. Generating the quadrilateral from its associated harmonic one
Lemma 3. Let \( p = ABCD \) be a convex quadrilateral with associated harmonic one \( q = NZ_1MZ_2 \), such that \( Z_1 \) is the similarity center of triangles \( ABZ_1, CDZ_1 \). Then triangle \( NMZ_1 \) is also similar to the above triangles.

In fact, by the Theorem 2, triangles \( ACZ_1, BDZ_1 \) are also similar, and \( N \), being the midpoint of side \( AC \), maps, by the similarity sending \( ACZ_1 \) to \( BDZ_1 \), to the corresponding midpoint \( M \) of \( CD \) (see Figure 6). This implies that triangles \( Z_1AN, Z_1BM \) are also similar, hence \( \frac{|Z_1N|}{|Z_1M|} = \frac{|Z_1A|}{|Z_1B|} \). Since the rotation angle, involved in the similarity mapping \( ACZ_1 \) to \( BDZ_1 \), is the angle \( AZ_1B \), this angle will be also equal to angle \( NZ_1M \), thereby proving the similarity of triangles \( ABZ_1 \) and \( NMZ_1 \).

Lemma 3 implies that all quadrangles \( p = ABCD \), having the given quadrangle \( q = NZ_1MZ_2 \) as their associated harmonic, are parameterized by the similarities \( f \) with center at \( Z_1 \). For, each such similarity produces a triangle \( ABZ_1 = f(NMZ_1) \) and defines through it the two vertices \( A, B \). The other two vertices \( C, D \) of the quadrilateral \( p \) are found by taking, correspondingly, the symmetrics of \( A, B \) with respect to \( N \) and \( M \). Note, that, by reversing the argument in Lemma 3, the diagonals \( AC, BD \) of the resulting quadrilateral intersect at a point \( E \) of the circumcircle of the harmonic quadrilateral. Hence their angle is the same with angle \( NZ_1M \). Also the ratio of the diagonals of \( ABCD \) is equal to the ratio \( \frac{|Z_2N|}{|Z_1M|} \), thus it is determined by the harmonic quadrilateral \( q = NZ_1MZ_2 \).

We have proved the following theorem.

**Theorem 4.** Given a harmonic quadrilateral \( q = NZ_1MZ_2 \), there is a double infinity of quadrilaterals \( p = ABCD \) having \( q \) as their harmonic associate with similarity centers at \( Z_1 \) and \( Z_2 \) and midpoints of diagonals at \( M \) and \( N \). All these quadrilaterals have their diagonals intersecting at the same angle \( NZ_1M \), the same ratio \( \frac{|AC|}{|BD|} = \frac{|Z_2N|}{|Z_2M|} \) and their Newton lines coinciding with \( MN \). Each of these quadrilaterals is characterized by a similarity \( f \) with center at \( Z_1 \), mapping \( f(Z_1NM) = Z_1AB \).
An alternative way to generate all quadrilaterals with given harmonic associate $q = NZ_1MZ_2$ and similarity centers at $Z_1, Z_2$, is to use a point $E$ on the circumcircle $\kappa$ of $q$, draw lines $EM, EN$, and consider their intersections $A, C$ with the circles passing through $E$ and $Z_1$. Equivalently, construct all triangles $Z_1AB$ similar to $Z_1NM$ and having the vertex $A$ on line $EN$. Then the other vertex $B$ moves on line $EN$ ([11, II, p.68]) and $C, D$ are again, respectively, the symmetrics of $A, B$ with respect to $N$ and $M$. A fourth method is described in §7.

4. Two related similar quadrilaterals

In order to prove some additional properties of our configuration, the following lemma is needed, which, though elementary in character, I could not locate a proof of it in the literature. For the completeness of the exposition I outline a short proof of it.

![Figure 8. Quadrilateral from angles and angle of diagonals](image)

**Lemma 5.** Two quadrilaterals having equal corresponding angles and equal angles between diagonals are similar.

In fact, let $ABCD$ be a quadrilateral with given angles and the angle $\omega$ between its diagonals. The two triangles $ECD, FAD$, where $E, F$ are the intersection points of opposite sides, have known angles and are constructible up to similarity. Thus, we can fix triangle $ECD$ and move a line parallel to $AF$ intersecting the sides $EC, ED$ correspondingly at $B', A'$. The quadrilateral with the required data must have the angle formed at the intersection point $P = (A'C, B'D)$ equal to $\omega$. This position $K$ for $P$ is found as follows (see Figure 8). As $A'B'$ moves parallel to itself it creates a homographic correspondence $B' \mapsto A'$ between the points of the lines $EC$ and $ED$ and induces a corresponding homography between the pencils of lines at $C$ and $D$. Then, according to the Chasles-Steiner theorem, the intersection point $P$ of corresponding rays $CA', DB'$ describes a conic ([9, p.109]). It is easily seen that this conic is a hyperbola passing through the vertices of triangle $ECD$, whose tangents at $C, D$ are parallel to $A'B'$ and its center is the midpoint $M$ of $CD$. The intersection point $K$ of the conic with a circular arc $c$ of points viewing $CD$ under the angle $\omega$ determines the quadrilateral with the required properties and shows that it is unique, up to similarity.
Theorem 6. (1) The circles $\alpha = (Z_1 NA), \beta = (Z_1 MB)$ pass through the intersection point $A'$ of the Newton line with side $AB$. Analogously, the circles $\gamma = (Z_1 NC), \delta = (Z_1 MD)$ pass through the intersection point $C'$ of the Newton line with side $CD$.

(2) Circles $\beta$ and $\gamma$ intersect at a point $B'$ of $BC$. Analogously circles $\alpha$ and $\delta$ intersect at a point $D'$ of $AD$.

(3) The centers $A''', B''', C''', D'''$ of corresponding circles $\alpha, \beta, \gamma, \delta$ build a quadrilateral $A''B''C''D''$ similar to $ABCD$, whose diagonals pass through $O$.

(4) Analogous to the above properties hold by replacing $Z_1$ with $Z_2$ and defining $A', B', C', D'$ and circles $\alpha, \beta, \gamma, \delta$ properly.

In fact, (1) and (2) result by a simple angle chasing argument (see Figure 9). (3) follows from the Lemma 5 and the fact that $A''B''C''D''$ has the same angles with $ABCD$ and also the same angle of diagonals, which intersect at $O$. (4) is proved by the same arguments.

5. The case of cyclic quadrilaterals

The location of the similarity centers $Z_1, Z_2$ in the case of a cyclic quadrilateral is, in most cases, immediate according to the following.

Theorem 7. In the case of a cyclic quadrilateral $p = ABCD$, whose opposite sides intersect at points $F, G$, the similarity centers $Z_1, Z_2$ are the intersections of the circumcircle $\kappa$ of the associated harmonic quadrilateral with the circle $\mu$ on diameter $FG$. 
In [7] it is proved that a quadrilateral $p$ is cyclic if and only if the circle $\mu$, with diameter $FG$, is orthogonal to the corresponding circle $\kappa = (MNE)$. Thus, in this case there are indeed two intersection points $Z_1, Z_2$ on $\kappa$ (see Figure 10). There is also proved, that in this case line $FG$ is the polar of $E$ and coincides with the radical axis of the pencil of circles generated by $\kappa$ and the circumcircle $\lambda$ of $ABCD$. Since angle $FZ_1G$ is a right one and points $(B, C, N, F)$ make a harmonic division, the two lines $Z_1G, Z_1F$ are the bisectors of the angle $BZ_1C$ as well as of angle $AZ_1D$. Thus, angles $AZ_1B$ and $CZ_1D$ are equal and angles $AZ_1C, BZ_1D$ are also equal. Since $G$ is on the radical axis of $\kappa$ and $\lambda$ the quadrilateral $CDZ_1E$ is cyclic, hence the angles $E CZ_1$ and $EDZ_1$ are equal. This implies that triangles $AZ_1C$ and $BZ_1C$ are similar and from this follows that triangles $AZ_1B, CZ_1D$ are also similar. This identifies point $Z_1$ with the center of similarity transforming $AB$ to $CD$. Analogously is proved the corresponding property for the other intersection point $Z_2$.

Next theorem explores the possibility to determine a generic cyclic quadrilateral $p = ABCD$ on the basis of its associated harmonic one.

**Theorem 8.** A convex cyclic quadrilateral $p$, whose opposite sides intersect, is uniquely determined from its associated harmonic quadrilateral $q$ and the location of the intersection $E$ of the diagonals of $p$ on the circumcircle $\kappa$ of $q$. Point $E$ can be taken arbitrarily on the arc defined by $Z_1Z_2$, which is less than half the circumference of $\kappa$. All cyclic quadrilaterals resulting by such a choice of $E$ have the angle between their diagonals equal to $\angle Z_1MZ_2$ or its complementary and the ratio of diagonal-lengths equal to $\frac{|Z_1M|}{|Z_2N|} = \frac{|Z_2M|}{|Z_1N|}$.

The first statement follows easily from two facts. The first is that, according to Theorem 7, the circle $\mu$ on diameter $FG$, where $F, G$ are the intersections of opposite sides of $p = ABCD$, is orthogonal to the circumcircle $\kappa$ of $q$ and its center is at the intersection $P$ of tangents to $\kappa$, respectively at $Z_1$ and $Z_2$ or the pole of $Z_1Z_2$ with respect to $\kappa$. Hence this circle is constructible from the data of the harmonic
The associated harmonic quadrilateral $q = MZ_1NZ_2$. The second fact, proved in the aforementioned reference, is that the circumcircle $\lambda$ of the quadrilateral $p$ is orthogonal to $\mu$ and its center is the diametral point $O$ of $E$ with respect to circle $\kappa$. This implies that $\lambda$ can be constructed as the circle, which is orthogonal to $\mu$ and has its center at $O$. Having this circle, we obtain the vertices of the quadrilateral $p$ by intersecting it with lines $EM$ and $EN$. The other statements follow from fundamental properties of the harmonic quadrilateral, such as, for example, the fact, that $M, N$ are separated by $Z_1, Z_2$ and that generic cyclic convex quadrilaterals have the intersection point $E$ always in the arc $Z_1Z_2$, which is less than half the circumference of $\kappa$.

Having excluded from the beginning the parallelogrammic quadrilaterals, which have both pairs of opposite sides intersecting at infinity, the case of cyclic quadrilaterals, not included in both theorems, is the one of equilateral trapezia, having one pair of sides intersecting at infinity. In this case the harmonic associated is found easily, having the similarity centers coinciding correspondingly with the intersection point $E = Z_1$ of the diagonals and the circumcenter $O = Z_2$ (see Figure 12). Theorem 8 is not valid in this case, since, then, there are infinite many cyclic quadrilaterals with the same harmonic associate. In fact, in this case, every circle
centered at $Z_2 = O$, with radius $r > |Z_1Z_2|$ defines, through its intersections with lines $Z_1M, Z_1N$, an equilateral trapezium having the given $q = NZ_1MZ_2$ for harmonic associated. Two other cases, in which the intersection point $E$ of the diagonals of $p = ABCD$ coincides with a particular point, are the quadrilaterals having $E = N$, i.e., coinciding with the midpoint of one diagonal (see Figure 13), and the quadrilaterals $p = ABCD$, which are also themselves harmonic. In this case $E$ is on the diameter of the circumcircle $\kappa$ of $q$, which contains the intersection point $H$ of the diagonals of $q$. Then the polar $\varepsilon$ of $H$ with respect to $\kappa$ coincides with the radical axis of the circle $\kappa$ and the circumcircle $\lambda$ of $p$ (see Figure 14).

6. The two lemniscates

Fixing the harmonic quadrilateral $q = NZ_1MZ_2$, as seen in the previous section, all cyclic quadrilaterals $p$, having $q$ as their associated, are parameterized by a point $E$ varying on an arc $Z_1Z_2$ of the circumcircle $\kappa$ of $q$. The following theorem shows that the vertices of the resulting quadrilaterals $p = ABCD$ vary on two lemniscates of Bernoulli ([10, p.13], [4, p.110]).
Theorem 9. The vertices of all convex cyclic quadrilaterals \( p = ABCD \), having the same harmonic associated quadrilateral \( q = NZ_1MZ_2 \) are on two Bernoulli lemniscates with nodes, respectively, at \( M \) and \( N \). Each pair of opposite vertices lies on the same lemniscate and is symmetric with respect to the corresponding node.

The proof of the theorem follows from a simple calculation, using cartesian coordinates centered at the vertex \( M \) of the given harmonic quadrilateral \( q = NZ_1MZ_2 \). Vertex \( N \) is set at \((n, 0)\) and the circumcircle \( \kappa \) of \( q \) intersects the \( y \)-axis at \((0, t)\). Point \( P(p, 0) \) is the center of the circle \( \mu \), which passes through \( Z_1, Z_2 \) and is orthogonal to \( \kappa \). The equations can be set in dependence of the parameters \( n, p \) and \( t \) by following the recipe of reconstruction of \( p \) from \( q \), described in Theorem 8. For a variable point \( E(u, v) \) on \( \kappa \), the intersection points of line \( EM \) and the circle \( \lambda \), centered at the diametral \( O \) of the point \( E \) and orthogonal to \( \mu \), are found by eliminating \((u, v)\) from the three equations representing the circle \( \lambda \), the line \( ME \) and the circle \( \kappa \). These are correspondingly:

\[
\begin{align*}
x^2 + y^2 &- 2x(n-u) - 2y(t-v) - 2pu + pn = 0, \\
vx - uy & = 0, \\
u^2 + v^2 &- nu - tv = 0.
\end{align*}
\]

Eliminating \((u, v)\) from these equations, leads to an equation of the 8-th degree, which splits into the two quadratics \((x - p)^2 = 0, (x - n)^2 + (y - t)^2 - (n^2 + t^2) + np = 0\) and the equation of the fourth degree

\[
(x^2 + y^2)^2 + np(y^2 - x^2) - 2ptxy = 0,
\]
for the coordinates \((x, y)\) of the points \(A\) and \(C\). The first equation represents the line \(x = p\) not satisfied by the points \(A, C\). The second represents the circle \(\lambda\) obtained when \(E = M\) and satisfied by \(A, C\) only when \(AC\) is tangent to \(\kappa\) at \(M\). Finally the last equation, by inverting on the unit circle, leads to
\[ np(y^2 - x^2) - 2ptxy + 1 = 0, \]
representing a rectangular hyperbola centered at the origin. By the well known property of Bernoulli’s lemniscates to be the inverses of such hyperbolas ([4, p.110]), this proves the theorem for the pair of opposite vertices \(A\) and \(C\). For the other pair of opposite vertices, \(B\) and \(D\), an analogous calculation, leads to a corresponding system of three equations
\[
\begin{align*}
x^2 + y^2 - 2x(n - u) - 2y(t - v) - 2pu + pn &= 0, \\
vx + (n - u)y - nv &= 0, \\
u^2 + v^2 - nu - tv &= 0.
\end{align*}
\]
Here again, elimination of \((u, v)\), transfer of the origin at \(N\), and inversion on the unit circle centered at \(N\), leads, through the factorization of an equation of the 8-th degree, to the equation of the rectangular hyperbola
\[ (n^2 - np)(y^2 - x^2) + 2t(n - p)xy + 1 = 0. \]
This, using the aforementioned property of Bernoulli’s lemniscate, proves the theorem for the vertices \(B\) and \(D\).

**Remarks.** (1) Using, for convenience, the corresponding equations of the rectangular hyperbolas, one can easily compute the symmetry axes of the lemniscates and

Figure 16. The similarities of the two lemniscates

see that they are obtained, respectively, by lines \(AC, BD\), when their intersection
The associated harmonic quadrilateral

$E$ is such that $EO$ is parallel to line $MN$ (see Figure 16). A simple computation shows also that the two lemniscates are similar with respect to two similarities. The first one $P' = f_1(P)$ has its center at $Z_1$, its oriented rotation-angle equals $\angle MZ_1N$ and its ratio is $r_1 = \frac{|Z_1M|}{|Z_1N|}$. The second similarity $S' = f_2(S)$ has its center at $Z_2$, its oriented rotation-angle equals $\angle NZ_2M$ and its ratio is $r_2 = \frac{|Z_2N|}{|Z_2M|} = r_1^{-1}$.

(2) Fixing a certain lemniscate $\xi$, one can use the above results to give a parametrization of all cyclic quadrilaterals, up to similarity, by three points $Z_1, Z_2, P$ properly chosen on the lemniscate. In fact, select first two points $Z_1, Z_2$, each on a different loop and on the same side of the axis $AC$ of $\xi$ (see Figure 16). This, together with the node $M$ of $\xi$ creates a triangle $Z_1MZ_2$ with the angle at $M$ greater than a right one. This triangle defines also a unique point $N$, such that $q = NZ_1MZ_2$ is a harmonic quadrilateral. Excepting the squares, all other harmonic quadrilaterals, up to similarity, are obtained in this way. Having $q$, one can define the similarity $f_1$ of the previous remark. Then, every point $A$ on the arc $\eta = Z_1Z^*$, where $Z^*$ the symmetric of $Z_2$ with respect to $M$, defines a cyclic quadrilateral $p = ABCD$.

Point $B = f_1(A)$, point $C$ is the symmetric of $A$ with respect to $M$ and point $D$ is the symmetric of $B$ with respect to $N$.

(3) The symbol $q = NZ_1MZ_2$ for the harmonic quadrilateral sets a certain order on its vertices. In the resulting construction of the cyclic quadrilateral $p = ABCD$ it is assumed that $Z_1, Z_2$ play the role of the similarity centers and $M, N$ are the midpoints of the diagonals. Interchanging these roles, changes also the related cyclic quadrilaterals. Thus, giving $q$ without an ordering for its vertices, produces two families of cyclic quadrilaterals, depending on how we interpret its two pairs of opposite vertices. Figure 17 shows the two pairs of lemniscates.
corresponding to the two interpretations of the opposite vertices of the harmonic quadrilateral \( q = ABCD \). All cyclic quadrilaterals having \( q \) for their associated harmonic, have their vertices on these lemniscates.

7. The associated cyclic quadrilateral

Starting with an arbitrary convex quadrilateral \( p = ABCD \) with intersections of opposite sides \( F \) and \( G \), we can, through the intermediate construction of its associated harmonic, pass to a natural associated cyclic quadrilateral \( p' = A'B'C'D' \). In fact, consider the associated harmonic \( q = Z_1NZ_2M \) of \( p \) and from this, construct, following the recipe of Theorem 8, the corresponding cyclic \( p' = A'B'C'D' \) (see Figure 18). From its definition, \( p' \) has the same harmonic associated \( q \) with \( p \). Further it is easy to see that \( |AA'| = |CC'|, |BB'| = |DD'| \) and the ratio \( \frac{|AA'|}{|BB'|} = \frac{|AC'|}{|BD'|} \) (see Figure 18). If one of the intersection points \( F, G \) of the opposite sides is at infinity then \( p \) is a trapezium and the corresponding harmonic quadrilateral has one of the similarity centers \((Z_1)\) coinciding with the intersection \( E \) of its diagonals. Excluding this case, the procedure described above can be reversed. Starting from the convex cyclic quadrilateral \( p' = A'B'C'D' \) and taking on its diagonals segments

\[
|AA'| = |CC'|, |BB'| = |DD'|
\]

we obtain quadrilaterals \( p = ABCD \) with the same associated harmonic quadrilateral. This gives an alternative construction of the one exposed in §3. In the excluded case of trapezia \( p = ABCD \), the result is different and the procedure must be slightly modified. In fact, in this case there is no proper associated cyclic quadrilateral, the corresponding construction leading to a degenerate cyclic quadrilateral, which coincides with a triangle \( Z_1C'D' \) (see Figure 19). In this case the quadrilaterals \( p' = A'B'C'D' \), having the same associated harmonic quadrilateral \( q = NZ_1MZ_2 \) with \( p \) are also trapezia and are obtained by taking an arbitrary point \( A' \) on \( Z_1N \), on the other halfline than \( N \) and projecting it parallel to \( MN \)
The associated harmonic quadrilateral

onto $B'$ on $Z_1 M$. Then taking, respectively, the symmetrics, $C', D'$ with respect to $N$ and $M$.

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Dynamics of the Nested Triangles Formed by the Tops of the Perpendicular Bisectors

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Abstract. Given a triangle, we construct a new triangle by taking as vertices the tops of the interior perpendicular bisectors. We describe the dynamics of this transformation exhaustively up to similarity. An acute initial triangle generates a sequence with constant largest angle: except for the equilateral case, the transformation is then ergodic and amounts to a surjective tent map of the interval. An obtuse initial triangle either becomes acute or degenerates by reaching a right-angled state.

1. Introduction

The midpoints of the sides of a triangle \(ABC\) are the vertices of the medial triangle, which is obtained from \(ABC\) by a homothety of ratio \(-1/2\) about the centroid. By iterating this transformation, one obtains a sequence of directly similar nested triangles that converges to the common centroid. The feet of the medians and of the perpendicular bisectors generate thus a boring sequence! The feet of the angle bisectors are more interesting: the iterated transformation produces a sequence of nested triangles that always converges to an equilateral shape, as shown in [8] for isosceles initial triangles and in [2] for the general case (in 2006). In the 1990s, four papers [3, 4, 9, 1] analyzed the pedal or orthic sequence defined by the feet of the altitudes: Peter Lax [4] proved the ergodicity of the construction. We considered in [5] reflection triangles and their iterates: the vertices of the new triangle are obtained by reflecting each vertex of \(ABC\) in the opposite side. We were able to decrypt the complex fractal structure of this mapping completely: the sequences generated by acute or right-angled triangles behave nicely, as they always converge to an equilateral shape.

The tops of the medians, angle bisectors, and altitudes are the vertices of the original triangle. But what about the tops of the interior perpendicular bisectors of \(ABC\) as new vertices? We found no trace of this problem in the literature. The solution presented here offers an elementary and concrete approach to chaos and requires almost no calculations. Note that another kind of triangle and polygon transformations have a long and rich history, those given by circulant linear combinations of the old vertices, like Napoleon’s configuration (see the references in

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2. Triangle of the tops of the perpendicular bisectors

Let $\Delta = ABC$ be a triangle with corresponding angles $\alpha$, $\beta$, $\gamma$ and opposite sides $a$, $b$, $c$. The interior perpendicular bisectors issued from the side midpoints $M_a$, $M_b$, and $M_c$ end at $A_1$, $B_1$, and $C_1$, respectively (Figure 1). The triangle $\Delta_1 = A_1B_1C_1 = T(\Delta)$ is the child of $\Delta$, and $\Delta$ a parent of $\Delta_1$. We denote the $m$th iterate of the transformation $T$ by $T^m$, $m \in \mathbb{Z}$. We are interested in the shape of the descendants and ancestors of $\Delta$, i.e., in their angles. In this paper, we provide an exhaustive description of the dynamics of $T$ with respect to shape.

We consider only two types of degenerate triangles, the “isosceles” ones: we assign angles $0^\circ$, $0^\circ$, $180^\circ$ to every nontrivial segment with midpoint and angles $90^\circ$, $90^\circ$, $0^\circ$ to every nontrivial segment with one double endpoint. A nondegenerate triangle is proper. We identify the shape of a triangle $\Delta$ with the point of the set

$$\mathcal{S} = \{(\alpha, \beta) \mid 0^\circ < \beta \leq \alpha \leq 90^\circ - \beta/2\} \cup \{(0^\circ, 0^\circ), (90^\circ, 0^\circ)\}$$

given by the two smallest angles of $\Delta$ (Figure 2). $\mathcal{S}$ is the disjoint union of the subsets $\mathcal{O}$, $\mathcal{R}$, and $\mathcal{A}$ of the obtuse, right-angled, and acute shapes, respectively. We denote the shape of an isosceles triangle with equal angles $\alpha$ by $I_\alpha$, $0^\circ \leq \alpha \leq 90^\circ$. The shapes of the isosceles triangles form the roof of $\mathcal{S}$, whose top is the equilateral shape $I_{90^\circ}$. The transformation $T$ induces a transformation $\tau$ of $\mathcal{S}$. We set $\tau(I_{90^\circ}) = I_{90^\circ}$ by definition. The children of right-angled triangles are degenerate with two vertices at the midpoint of the hypotenuse: $\tau$ maps the whole segment $\mathcal{R}$ to $I_{90^\circ}$. There are no other proper triangles with a degenerate child.
Dynamics of the nested triangles formed by the tops of the perpendicular bisectors

Figure 2. Set $\mathcal{S}$ of the triangle shapes and a curve $\Gamma_\lambda$ with corresponding $\Gamma_{\lambda}^{\text{bent}}$

Figure 3. Isosceles parent triangles with angles $\alpha = \beta$ for $\alpha > 60^\circ$, $45^\circ < \alpha < 60^\circ$, and $\alpha < 45^\circ$

The roof is invariant under $\tau$ (Figure 3). $I_0^\circ$ and the points of the right roof side are fixed points, and

$$\tau(I_\alpha) = \begin{cases} I_{2\alpha} & \text{if } 0^\circ \leq \alpha \leq 45^\circ \\ I_{180^\circ - 2\alpha} & \text{if } 45^\circ \leq \alpha \leq 60^\circ \\ I_\alpha & \text{if } 60^\circ \leq \alpha \leq 90^\circ \end{cases}$$
When $\alpha$ grows, $\tau(I_\alpha)$ travels on the roof as follows: from $I_{90^\circ}$ to $I_{90^\circ}$ for $0^\circ \leq \alpha \leq 45^\circ$, then back to $I_{60^\circ}$ for $45^\circ \leq \alpha \leq 60^\circ$ before descending the right roof side for $60^\circ \leq \alpha \leq 90^\circ$. Each shape $I_\alpha$ has thus one, two, or three isosceles parents according as it lies on the left roof side, at $I_{60^\circ}$ or $I_{90^\circ}$, or on the rest of the right roof side, respectively. For $60^\circ < \alpha < 90^\circ$, the three isosceles parents of $I_\alpha$ are itself, the point where the parallel to $R$ through $I_\alpha$ cuts the left roof side, and the reflection of this parent in the line of $R$. An equilateral triangle has four isosceles parents: itself and three of shape $I_{30^\circ}$. $I_\alpha$ degenerates eventually to $I_{90^\circ}$ (and this after $m$ steps) if and only if $\alpha = 90^\circ / 2^m$ for some integer $m \geq 0$. Otherwise, $I_\alpha \neq I_{90^\circ}$ reaches its final nondegenerate state $I_{2^m \alpha}$ or $I_{180^\circ - 2^m \alpha}$ after $n$ steps, where $n \geq 0$ is given by $30^\circ / 2^{n-1} \leq \alpha < 45^\circ / 2^{n-1}$ in the first and by $45^\circ / 2^{n-1} < \alpha < 60^\circ / 2^{n-1}$ in the second case.

We consider the tangents of the angles of $\Delta$:

$$u = \tan \alpha, \quad v = \tan \beta, \quad w = \tan \gamma = \frac{u + v}{uv - 1}.$$  

We write $\tan(\alpha, \beta)$ for $(\tan \alpha, \tan \beta)$. In the $(u, v)$-plane, the transformation induced by $\tau$ is essentially the radial stretch $(u, v) \mapsto \frac{|w|}{u} (u, v)$. More precisely, we have the following result.

**Theorem 1.** When $(\alpha, \beta)$ is the shape of the proper obtuse or acute triangle $\Delta$, the angles $\alpha_1$ and $\beta_1$ of $T(\Delta)$ are acute with $\alpha_1 \geq \beta_1$ and $\tan(\alpha_1, \beta_1) = \frac{|w|}{u} (u, v)$. According as the parent is obtuse or acute, its child has a smaller or the same largest angle.

**Proof.** We first look at the acute case $90^\circ > \gamma > \alpha \geq \beta > 0^\circ$ (Figure 1). Let $D$ be the intersection of side $b$ with the perpendicular bisector of $c$. By Thales’ theorem, $A_1M_cM_aC_1$ and $M_cB_1DM_b$ are convex cyclic quadrilaterals with circumcircles of diameters $A_1C_1$ and $B_1D$, respectively. One has thus $\alpha_1 = \angle M_cM_aB = \gamma = \angle M_cB_1D$ and

$$\tan \beta_1 = \frac{C_1M_c}{M_cB_1} = \frac{C_1M_c}{c/2} \cdot \frac{DM_c}{DM_c} = \tan \gamma \frac{DM_c}{M_cB_1} = \frac{\tan \beta}{\tan \alpha} \tan \gamma.$$  

Since $\alpha_1 = \gamma$, one has $\gamma_1 \leq \angle A_1C_1B = \alpha \leq \gamma$.

In the obtuse case $\gamma > 90^\circ > \alpha \geq \beta > 0^\circ$, $A_1$ lies on the right of $M_c$ and $B_1$ on the left. One considers the quadrilaterals $M_cA_1M_aC_1$ and $B_1M_cD$ to get $\alpha_1 = \angle M_cM_aC_1 = 180^\circ - \gamma = \angle M_cB_1D$ and $\tan \beta_1 = \tan \beta \cdot \tan(180^\circ - \gamma) / \tan \alpha$. One has $\gamma_1 = 180^\circ - (\alpha_1 + \beta_1) = \gamma - \beta_1 < \gamma$. \hfill \Box

An acute shape and its descendants lie thus on a parallel to the line of the right-angled shapes: for every $\varphi \in (0^\circ, 15^\circ]$, the segment of acute shapes $P_{\varphi}^{ac}$ parallel to $R$ from $I_{45^\circ + \varphi}$ to $I_{90^\circ - 2\varphi}$ is invariant under $\tau$ (Figure 4).

The shape of an obtuse or right-angled child $\tau(\alpha, \beta)$ is $(\alpha_1, \beta_1)$ since $\gamma_1 > \alpha_1$ (with equality for $(\alpha, \beta) = I_{90^\circ}$). The shape of an acute child is $(\alpha_1, \beta_1)$, $(\gamma_1, \beta_1)$, or $(\beta_1, \gamma_1)$ since $\alpha_1 \geq \beta_1$. We describe below the conditions of each occurrence.
Note that $(\gamma_1, \beta_1)$ is the horizontal reflection of $(\alpha_1, \beta_1)$ in the line of the right roof side and that $(\beta_1, \gamma_1)$ is the reflection of $(\gamma_1, \beta_1)$ in the line of the left roof side.

**Theorem 2.** The acute shape $(\alpha, \beta)$ and its reflection in the line of $R$, the obtuse shape $(90^\circ - \beta, 90^\circ - \alpha)$, share the same child.

**Proof.** The tangents of the shapes are $(u, v)$ and $(u', v') = (1/v, 1/u)$. Since $u' = -w$ and $v'/u' = v/u$, the children have the same angles. \qed

As a consequence, the segment $Q$ of obtuse shapes joining $I_{30^\circ}$ and $I_{90^\circ}$ is reflected in $R$ by $\tau$ to the right roof side consisting of fixed points (Figures 2 and 4). (When an isosceles child triangle has a scalene parent, note that a reflection in the child’s axis gives a second parent triangle.) And for $\varphi \in (0^\circ, 15^\circ]$, the action of $\varphi$ on the segment $P_{\varphi}^{\text{obt}}$ of obtuse shapes parallel to $R$ from $I_{45^\circ - \varphi}$ to $(90^\circ - 4\varphi, 2\varphi) \in Q$ is the reflection to $P_{\varphi}^{\text{ac}}$ followed by $\tau$: since $P_{\varphi}^{\text{ac}}$ is invariant under $\tau$, $\tau$ maps $P_{\varphi}^{\text{obt}}$ to $P_{\varphi}^{\text{ac}}$.

### 3. Dynamics of the transformation

Consider a slope $\lambda$ with $0 < \lambda \leq 1$. On the segment $v = \lambda u$, $0 \leq u < \sqrt{1/\lambda}$, $|w|$ is given by $(1 + \lambda)u/(1 - \lambda u^2)$ and is a strictly growing convex function of $u$ with image $[0, +\infty)$. The segment $v = \lambda u$, $0 \leq u < \sqrt{1/\lambda}$, is thus stretched bijectively and continuously to the whole half-line $v = \lambda u$, $u \geq 0$, by the map $(u, v) \mapsto |w|/(u, v)$, whose only fixed point is the origin. The strictly growing curves $\Gamma_\lambda$ corresponding to these segments are given by

$$\Gamma_\lambda: \beta = \arctan(\lambda \tan \alpha), \quad 0^\circ \leq \alpha \leq 90^\circ, \quad 0 < \lambda \leq 1,$$

which is $\beta = \alpha$ for $\lambda = 1$: they link the origin $I_{90^\circ}$ and the point $(90^\circ, 90^\circ)$, are symmetric in the line of $R$, and provide a simple covering of $\{(\alpha, \beta) \mid 0^\circ < \beta \leq \alpha < 90^\circ\}$ (Figures 2 and 4). The lower parts $\Gamma_\lambda^{\text{obt}}$ of the $\Gamma_\lambda$’s for $0^\circ \leq \alpha \leq \arctan \sqrt{1/\lambda}$ cover the set of the obtuse and proper right-angled shapes. The middle parts $\Gamma_\lambda^{\text{ac}}$ for $\arctan \sqrt{1/\lambda} < \alpha \leq \arctan \sqrt{1 + 2/\lambda}$ cover the set of the acute shapes. By Theorem 1, the transformation $\tau$ stretches each $\Gamma_\lambda^{\text{obt}}$ to the doubly bent curve $\Gamma_\lambda^{\text{bent}}$ linking $I_{90^\circ}$ and $I_{90^\circ}$ (Figure 2): $\Gamma_\lambda^{\text{bent}}$ is obtained from $\Gamma_\lambda$ by a horizontal reflection of its upper part in the line of the right roof side followed by a partial reflection in the left roof side. By Theorem 2, $\tau$ maps each path $\Gamma_\lambda^{\text{ac}}$ (joining a right-angled shape and an isosceles fixed point of $\tau$) to the way back along $\Gamma_\lambda^{\text{bent}}$ from $I_{90^\circ}$ to the right roof side.

$\tau$ provides thus a simple covering of $O \cup R \setminus \{I_{90^\circ}\}$ (by obtuse shapes exclusively), a fivefold covering of $A$ without roof, a triple covering of the section $\{I_\alpha \mid 45^\circ < \alpha < 60^\circ\}$ of the left roof side, a double covering of $I_{60^\circ}$ and a quadruple covering of the right roof side without endpoints. The covering of $A$ without roof consists of three layers of obtuse and two layers of acute shapes. More precisely, each region $O_n \subset O$ bordered by the curve $\tau^{-n}(R)$ and its parent curve $\tau^{n-1}(R)$ is mapped by $\tau$ bijectively and upwardly (along the curves $\Gamma_\lambda$) to the next less obtuse region $O_{n-1}$ for all integers $n \geq 1$ (Figure 2). The curve
\( \tau^{-n}(\mathcal{R}) \), \( n \geq 0 \), joins \( I_{45^\circ}/2^\circ \) and \( I_{90^\circ} \). Every neighborhood of the \( \alpha \)-axis contains all but finitely many ancestor regions of \( \mathcal{O}_0 \), the region of the slightly obtuse shapes. \( \mathcal{O}_0 \setminus \{I_{30^\circ}\} \) is the disjoint union of three layers of the fivefold covering of \( \mathcal{A} \) without roof and of a double covering of the acute roof section without \( I_{60^\circ} \) and \( I_{90^\circ} \). Each of the following three subregions of \( \mathcal{O}_0 \) is mapped bijectively to \( \mathcal{A} \) without roof (along the three smooth sections of the curves \( \Gamma_{\text{bent}} \)) : the lower subregion \( \mathcal{O}_0^{\text{low}} \), the middle lens \( \mathcal{O}_0^{\text{mid}} \), and the upper subregion \( \mathcal{O}_0^{\text{up}} \) delimited by the curves \( \tau^{-1}(\mathcal{R}) \), \( \mathcal{Q}, \mathcal{L}_{\text{obt}} \), and \( \mathcal{R} \). The curve \( \mathcal{L}_{\text{obt}} \) consists of the obtuse parent shapes mapped to a bend point of some curve \( \Gamma_{\text{bent}} \) on the left roof side. The reflections of \( \mathcal{O}_0^{\text{mid}} \) and \( \mathcal{O}_0^{\text{up}} \) in the line of \( \mathcal{R} \), separated by the curve \( \mathcal{L}_{\text{ac}} \), constitute the remaining two layers of acute shapes in the covering of \( \mathcal{A} \).

The shape of an acute child \( \tau(\alpha, \beta) \) is \((\alpha_1, \beta_1)\) if the parent \((\alpha, \beta)\) lies in \( \mathcal{O}_0^{\text{low}} \), \((\gamma_1, \beta_1)\) if the parent is in \( \mathcal{O}_0^{\text{mid}} \) or \( \mathcal{A}_{\text{tight}} \), and \((\beta_1, \gamma_1)\) if the parent is located between \( \mathcal{L}_{\text{obt}} \) and \( \mathcal{L}_{\text{ac}} \) (Figure 2).

The bijective stretch

\[
p^{-1}: \{(u, v) \mid u \geq v > 0, uv < 1\} \rightarrow \{(u, v) \mid u \geq v > 0\}, \quad (u, v) \mapsto \frac{|w|}{u} (u, v)
\]
is the inverse function of
\[ p(u, v) = \frac{\sqrt{(u + v)^2 + 4uv^2} - (u + v)}{2u^2} (u, v). \]

One parent \((\alpha, \beta)\) of the proper shape \((\alpha_1, \beta_1)\) is thus the obtuse shape given by \(\tan(\alpha, \beta) = p(\tan(\alpha_1, \beta_1))\) (Figure 4). When \((\alpha_1, \beta_1)\) is an acute shape, the parent \((\alpha, \beta)\) lies in \(O^{\text{low}}\) and the other parents are the obtuse shapes \((\alpha', \beta')\), \((\alpha'', \beta'')\) given by \(\tan(\alpha', \beta') = p(\tan(\gamma_1, \beta_1))\), \(\tan(\alpha'', \beta'') = p(\tan(\gamma_1, \alpha_1))\) and their acute reflections \((90^\circ - \beta', 90^\circ - \alpha')\) and \((90^\circ - \beta'', 90^\circ - \alpha'')\) on \(P^{\text{ac}}_{(\alpha_1 + \beta_1 - 90^\circ)/2}\). Except \((\alpha, \beta)\), the parents are the vertices of a rectangle. The parent \((\alpha, \beta)\) and \((\alpha_1, \beta_1)\) lie on the same curve \(\Gamma_\lambda (\lambda = \tan \beta_1 / \tan \alpha_1)\), whereas \((\alpha', \beta')\) and \((\alpha'', \beta'')\) reach their child by traveling on bent curves \(\Gamma^{\text{bent}}_\lambda\) and \(\Gamma^{\text{bent}}_{\lambda'}\) \((\lambda' = \tan \beta_1 / \tan \gamma_1, \lambda'' = \tan \alpha_1 / \tan \gamma_1)\), after the first bend for \((\alpha', \beta')\) in \(C^{\text{mid}}_0\) and the second for \((\alpha'', \beta'')\) in \(C^{\text{mid}}_0\).

Each acute nonequilateral triangle has exactly five differently placed parents: by axial symmetry, an isosceles child has one or two pairs of inversely congruent scalene parents according as its equal angles are larger or smaller than 60°. We already mentioned the four parents of an equilateral triangle: itself and three of shape \(I_{30^\circ}\). The formula for \(p(u, v)\) shows that the parents of a given proper triangle are all constructible by straightedge and compass.

The parents \((\alpha, \beta)\) of the proper right-angled shapes \((\alpha_1, \beta_1)\) with \(\tan(\alpha_1, \beta_1) = (u_1, v_1)\) are the obtuse solutions of \(u_1v_1 = 1\) and constitute thus the curve
\[ \tau^{-1}(\mathcal{R}) \setminus \{I_{90^\circ}\} : \quad \tan^2(\alpha + \beta) = \tan \alpha / \tan \beta, \quad 0^\circ < \beta \leq \alpha, \alpha + \beta < 90^\circ \]
(Figure 2). Since \(u_1/v_1 = u/v\), the proper shapes of \(\tau^{-2}(\mathcal{R})\) are the shapes \((\alpha, \beta)\) below \(\tau^{-1}(\mathcal{R})\) with \(\tan^2(\alpha_1 + \beta_1) = \tan \alpha / \tan \beta\), and so on. For each integer \(n \geq 1\), a parametric representation of the curve \(\tau^{-n}(\mathcal{R})\) without \(I_{90^\circ}\) is given by \(\tan(\alpha, \beta) = p^n(\hat{u}, 1/\hat{u})\) with \(\tan \beta = \hat{u}^{-2} \tan \alpha\) and \((\hat{u}, 1/\hat{u}) = \tan(\hat{\alpha}, 90^\circ - \hat{\alpha})\), \(45^\circ \leq \hat{\alpha} < 90^\circ\), where \(p^n\) denotes the \(n\)th iterate of \(p\). The maximal elevation of \(\tau^{-n}(\mathcal{R})\) seems to be approximately \(23^\circ / n\).

Because the origin is the unique fixed point of the stretch \((u, v) \mapsto \frac{|w|}{u}(u, v)\) in the set \{(u, v) | u \geq v > 0, wv < 1\}, the only possible fixed point of \(\tau\) on \(\Gamma_\lambda \cap \mathcal{S}\) (besides the endpoints) is the double point of \(\Gamma^{\text{bent}}_\lambda\) (Figure 2). But this acute double point is the unique point of \(\Gamma^{\text{bent}}_\lambda\) on its parallel to \(\mathcal{R}\) and is forced to be its own child. The curve \(C_1\) of the nonisosceles fixed shapes joins \(I_{60^\circ}\) and \(I_{90^\circ}\) in \(\mathcal{A}\) and lies below \(\mathcal{L}^{\text{ac}}\): the shapes \((\alpha, \beta) \in C_1\) are thus the solutions of the equation \((\alpha, \beta) = \tau(\alpha, \beta) = (\beta_1, \gamma_1)\), i.e., \(u = v_1\), which can be transformed into \(v^2 + (u - u^3)v + u^2 = 0\). The solution that corresponds to shapes is
\[ C_1 : \quad v = \frac{u}{2} \left( u^2 - 1 - \sqrt{(u^2 - 1)^2 - 4} \right), \quad u = \tan \alpha, v = \tan \beta, u \geq \sqrt{3}. \]

When the shape \((\alpha, \beta)\) of \(\Delta\) lies on \(C_1\), \(T(\Delta)\) is directly similar to \(\Delta\), since \(\Delta = ABC\) and \(A_1B_1C_1\) are then equally oriented with \(\gamma = \alpha_1 \geq \alpha = \beta_1\geq \beta = \gamma_1\).
In general, one finds the child of the acute shape \((\alpha, \beta)\) as follows. Draw the curve \(\Gamma_\lambda\) through \((\alpha, \beta)\) by taking \(\lambda = \tan \beta / \tan \alpha\). The child is the point where the parallel to \(R\) through \((\alpha, \beta)\) cuts one of the two bent sections of \(\Gamma_{\lambda}^{\text{bent}}\) (Figure 2).

A proper scalene shape \((\alpha, \beta)\) on the curve \(L^{\text{obt}}\), which links \(I_{90\beta}\) and \(I_{30\beta}\), is the obtuse parent of some \(I_{\hat{\alpha}}\), \(45^\circ < \hat{\alpha} \leq 60^\circ\). \(I_{\hat{\alpha}} = (\hat{\alpha}, \hat{\alpha})\) is the bend of \(\Gamma_{\lambda}^{\text{bent}}\) on the left roof side. Before bending, \(I_{\hat{\alpha}}\) was the point \((180^\circ - 2\hat{\alpha}, \hat{\alpha})\) located on the curve \(\Gamma_{\lambda}^{\text{bent}}\), like its parent \((\alpha, \beta)\). One has thus

\[
\hat{\lambda} = \frac{\tan \hat{\alpha}}{\tan(180^\circ - 2\hat{\alpha})} = \frac{(\tan^2 \hat{\alpha} - 1)/2}{\tan(180^\circ - 2\hat{\alpha})}
\]

and \(\tan(\alpha, \beta) = (u, v) = (u, \hat{\lambda}u)\) with \(w < 0\). The condition \(\tau(\alpha, \beta) = I_{\hat{\alpha}}\) is equivalent to \(w_1 = v_1\), since the two smaller angles of \(T(\Delta)\) have to be equal. The equation \(w_1 = v_1\) can be transformed into \(1 + u/v = \sqrt{1 + w^2}\) and further, since \(v = \hat{\lambda}u\) and \(uv < 1\), into

\[
\hat{\lambda} \tan \hat{\alpha} u^2 + \hat{\lambda}(\hat{\lambda} + 1)u - \tan \hat{\alpha} = 0.
\]

The positive solution \(u\) gives

\[
L^{\text{obt}}:\quad u = \tan \alpha = \sqrt{\frac{(\tan \hat{\alpha} + \cot \hat{\alpha})^2}{16} + \frac{2}{\tan^2 \hat{\alpha} - 1} - \frac{\tan \hat{\alpha} + \cot \hat{\alpha}}{4}},
\]

\[
\beta = 180^\circ - \alpha - 2\hat{\alpha}, \quad 45^\circ < \hat{\alpha} \leq 60^\circ.
\]

One obtains the parametric representation of \(L^{\text{ac}}\) by reflection in \(R\) or (since now \(uv > 1\)) from the equation \(\hat{\lambda} \tan \hat{\alpha} u^2 - \hat{\lambda}(\hat{\lambda} + 1)u - \tan \hat{\alpha} = 0\), which leads to

\[
L^{\text{ac}}:\quad \tan \alpha = \sqrt{\frac{(\tan \hat{\alpha} + \cot \hat{\alpha})^2}{16} + \frac{2}{\tan^2 \hat{\alpha} - 1} + \frac{\tan \hat{\alpha} + \cot \hat{\alpha}}{4}},
\]

\[
\beta = 2\hat{\alpha} - \alpha, \quad 45^\circ < \hat{\alpha} \leq 60^\circ.
\]

The segment \(P_{\phi}^{\text{ac}}\) parallel to \(R\) from \(I_{145^\circ + \varphi}\) to \(I_{90^\circ - 2\varphi}\), \(0^\circ < \varphi < 15^\circ\), has five parent curves (Figure 4): \(P_{\phi}^{\text{ac}}\) is doubly covered by itself under \(\tau\), once by the portion between the left roof side and \(L^{\text{ac}}\), once by the rest. \(P_{\phi}^{\text{ac}}\) is covered twice by \(P_{\phi}^{\text{obt}}\) through reflection followed by \(\tau\). And the curve

\[
K_{\phi}:\quad \tan(\alpha, \beta) = p(\tan(\tilde{\alpha}, 90^\circ + 2\varphi - \tilde{\alpha})), \quad 45^\circ + \varphi \leq \tilde{\alpha} \leq 90^\circ - 2\varphi,
\]

formed by the parents in \(O_{0}^{\text{low}}\) of some shape of \(P_{\phi}^{\text{ac}}\) is mapped bijectively onto \(P_{\phi}^{\text{ac}}\) (along the \(\Gamma_{\lambda}'s\)). \(K_{\phi}\) links \(I_{22.5^\circ + \varphi/2}\) and the end shape \((90^\circ - 4\varphi, 2\varphi)\) of \(P_{\phi}^{\text{obt}}\). The region \(O_{0} \setminus \{I_{30^\circ}\}\) is the disjoint union of the nested pointed arches \(K_{\phi} \cup P_{\phi}^{\text{ac}}\), \(0^\circ < \varphi < 15^\circ\). Such an arch covers \(P_{\phi}^{\text{ac}}\) three times under \(\tau\): down, and up, and down again.

When restricted to \(P_{\phi}^{\text{ac}}\), the transformation \(\tau\) amounts to the surjective tent map \(t \mapsto 2 \min(t, 1 - t)\) of the unit interval and is thus ergodic once the appropriate measure has been defined (see Section 4). \(P_{\phi}^{\text{ac}}\) is first folded by \(\tau\) about the point on \(L^{\text{ac}}\) in such a way that its left endpoint coincides with the fixed right endpoint (Figure 4). Both halves, being held firmly at their common right end, are then stretched to the left until each of them covers simply \(P_{\phi}^{\text{ac}}\). The subregions of acute shapes \(A^{\text{left}}, A^{\text{mid}}\), and \(A^{\text{right}}\) delimited in \(A\) by the curves \(R, C_1\), and \(L^{\text{ac}}\) (Figure 2)
are thus transformed in the following way along the parallels to $R$: $A_{\text{left}}$ is mapped bijectively to $A_{\text{mid}} \cup A_{\text{right}}$, $A_{\text{mid}}$ to $A_{\text{left}}$, and $A_{\text{right}}$ to $A$, giving two acute parents to every acute shape not lying on the left roof side. According as the acute shape is on the right or left of $C_1$, it is located between its acute parents or on their left, respectively (Figure 4).

Take a unit circle and draw a horizontal chord $AB$ below the diameter at distance $\sin 2\varphi$, $0^\circ < \varphi < 15^\circ$. The acute shapes $(\alpha, \beta)$ of the segment $P_{\varphi}^{\text{ac}}$ are then represented by the inscribed triangles $ABC$ for which $C$ lies between the north pole and the limit position $C_{\text{lim}}$ where $\alpha = \gamma = 90^\circ - 2\varphi$. $\tau$ moves $C$ on this arc.

The 120 integer-angled shapes $I_{\varphi}$ on the roof and $(2m^\circ, (45 - m)^\circ)$ on $Q$ have integer-angled children. A systematic search shows that there are only eight further such cases: $\tau(36^\circ, 12^\circ) = (48^\circ, 18^\circ)$, $\tau(42^\circ, 12^\circ) = (54^\circ, 18^\circ)$, $\tau(60^\circ, 18^\circ) = \tau(72^\circ, 24^\circ) = (54^\circ, 42^\circ)$, $\tau(50^\circ, 30^\circ) = \tau(60^\circ, 40^\circ) = (70^\circ, 30^\circ)$, $\tau(70^\circ, 30^\circ) = \tau(60^\circ, 20^\circ) = I_{50^\circ}$. These cases verify the identity

$$\tan(60^\circ - \delta) \tan(60^\circ + \delta) \tan \delta = \tan 3\delta$$

for $\delta = 12^\circ$, $18^\circ$, $6^\circ$, $20^\circ$, and $10^\circ$ in order: $\tan 12^\circ \tan 48^\circ / \tan 36^\circ = \tan 18^\circ$ and so on.

4. The transformation as symbolic dynamics

We use symbolic dynamics to give a short and elementary proof of the ergodicity of $\tau$ on every segment $P_{\varphi}^{\text{ac}}$. We identify $P_{\varphi}^{\text{ac}}$, $0^\circ < \varphi < 15^\circ$, with the interval $[0, 1]$, where $0 = I_{60^\circ - 2\varphi}$ and $1 = I_{45^\circ + \varphi}$ are the isosceles endpoints on the right and left roof sides, respectively (Figure 4). We represent each shape $s \in P_{\varphi}^{\text{ac}}$ by its infinite binary address $x = x_1x_2x_3 \ldots$ giving the position of $s$ with respect to the fractal binary address $P_{\varphi}^{\text{ac}}$ induced by the monotonicity intervals of $\tau$ and its iterates. The $k$th digit of the address is 0 or 1 according as $\tau^k$ restricted to $P_{\varphi}^{\text{ac}}$ is direction-preserving or reversing in a neighborhood of the shape (and addresses of turning points end in a constant sequence of 0s or 1s). If $x$ is eventually periodic, we overline the period’s digits. We identify the ends $0\overline{1}$ and $1\overline{0}$. The child $\tau(s)$ is then given by a left shift when $x_1 = 0$ and a left shift with permutation 0 $\leftrightarrow$ 1 in $x$ when $x_1 = 1$. Note that $\tau^n(x) = x_{n+1} \ldots$ or $\tau^n(x) = (x_{n+1} \ldots)_{0 \leftrightarrow 1}$ according as $x_1 \ldots x_n$ contains an even or odd number of 1s.

The shape on $L_{\varphi}^{\text{ac}}$ is $1/2 = 0\overline{1}$ and the nonisosceles fixed point is $2/3 = \overline{10}$. The parents of $x$ are $0x$ and $1x_{0 \leftrightarrow 1}$ (these are the acute parents of the shape). The only ancestors of $\overline{1}$ are the addresses ending in $\overline{1}$ or $T$, i.e., integer multiples of some $2^{-n}$.

We prove that the orbit of $x$ becomes eventually periodic if and only if the digits of $x$ are eventually periodic. Consider first an eventually periodic sequence $x = x_1 \ldots x_kp_1 \ldots p_n$: since $\{p_1 \ldots p_n, (p_1 \cdots p_n)_{0 \leftrightarrow 1}\}$ contains both $\tau^{k+n}(x)$ and $\tau^{k+2n}(x)$, one of them is $\tau^k(x)$, whose orbit is thus periodic. Conversely, if the orbit of $x = x_1x_2 \ldots$ is eventually periodic with $\tau(x) = \tau^{k+n}(x)$, $\tau^k(x)$ is a periodic sequence given by $x_{k+1}x_{k+2} \ldots$ or $(x_{k+1}x_{k+2} \ldots)_{0 \leftrightarrow 1}$, i.e., the sequence $x$ is eventually periodic.
The only 2-cycle is $2/5 = \underline{0110} \leftrightarrow \underline{1100} = 4/5$. Figure 5 shows the two curves $C_2$ of these 2-cycles in $A$. When the shape $(\alpha, \beta)$ of $\Delta$ lies on $C_2$, $T^2(\Delta)$ is inversely similar to $\Delta$, since $\Delta = ABC$ and $A_2B_2C_2$ are then equally oriented with $\gamma = \beta_2 \geq \alpha = \alpha_2 \geq \beta = \gamma_2$ if $(\alpha, \beta)$ is on the left of $\mathcal{L}^{ac}$ and $\gamma = \gamma_2 \geq \alpha = \beta_2 \geq \beta = \alpha_2$ if $(\alpha, \beta)$ is on the right of $\mathcal{L}^{ac}$. The two 3-cycles are $010 = 2/7 \rightarrow 100 = 4/7 \rightarrow 110 = 6/7 \rightarrow 010$ and $0011000 = 2/9 \rightarrow 4/9 \rightarrow 8/9 \rightarrow 2/9$. The iterated tent map $\tau^n$, $n \geq 1$, has $2^n$ fixed points in $\mathcal{P}^{ac}_{\varphi}$: since $2^n > 2 + 2^2 + \cdots + 2^{n-1}$ for $n \geq 2$, $\tau$ has $n$-cycles (of fundamental period $n$) in $\mathcal{P}^{ac}_{\varphi}$ for all integers $n \geq 1$. It is easy to see that the $2^n$ fixed points are the fractions $2k/(2^n - 1)$, $2k/(2^n + 1)$ in $[0, 1]$ and that for $n \geq 3$ such a fixed point generates an $n$-cycle if and only if it cannot be written with a smaller denominator $2^m \pm 1$. One can construct addresses with almost any behavior under iteration of $\tau$. We design for example an address $x_{dense}$ whose forward orbit is dense in $\mathcal{P}^{ac}_{\varphi}$: concatenate successively all binary words of length 1, 2, 3, and so on to an infinite sequence, and submit if necessary each of them in order to a permutation $0 \leftrightarrow 1$ in such a way that the original word appears as head of the corresponding descendant of $x_{dense}$.

The measure of an interval of $\mathcal{P}^{ac}_{\varphi}$ of $k$th generation, i.e., with an address of length $k$, is $2^{-k}$ by definition. $\tau$ is then measure-preserving on $\mathcal{P}^{ac}_{\varphi}$. By using the binomial distribution, it is easy to see that almost all shapes of $\mathcal{P}^{ac}_{\varphi}$ have a normal address [9]. An address $x$ is normal if, for every fixed integer $k \geq 1$, all binary words of $k$ digits appear with the same asymptotic frequency $2^{-k}$ as heads of the successive $\tau^n(x)$, $n \in \mathbb{N}$. The descendants of a normal address visit thus all intervals of $k$th generation with equal asymptotic frequency, and this for every $k$:
the forward orbit of almost every shape of $P^\text{ac}_\phi$ “goes everywhere in $P^\text{ac}_\phi$ equally often”, $\tau$ is an ergodic transformation of $P^\text{ac}_\phi$.

We conclude by formulating a condensed version of our results.

**Theorem 3.** Consider the map that transforms a triangle into the triangle of the tops of its interior perpendicular bisectors.

1. An acute initial triangle generates a sequence with constant largest angle: except for the equilateral case, the transformation is then ergodic in shape and amounts to a surjective tent map of the interval. A proper obtuse initial triangle either becomes acute or degenerates by reaching a right-angled state.

2. A proper triangle is the transform of exactly one, four or five triangles according as it is nonacute, equilateral, or acute but not equilateral, respectively.

**References**


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Kitta’s Double-Locked Problem

J. Marshall Unger

Abstract. I present strongly contrasting solutions of a remarkable sangaku. The general solution requires Fujita’s celebrated mixtilinear circle theorem. The data-specific solution shows the premodern Japanese preoccupation with Pythagorean triples.

1. The Problem

Shinpeki sanpō [2] is a collection of problems that FUJITA Sadasuke ordered his son Yoshitoki to compile to show off the prowess of his disciples, although problems of enthusiasts from other schools were also included. The problem in question ([2, 1.42-3]), which was not discussed in [3], [4] or [5], is reproduced in Figure 1 followed by my translation.

Figure 1.

A circular segment is split by a line such that its resulting parts contain two congruent circles as shown in the figure. The diameter of the large circle is 697 inches; the diameters of the congruent circles are 272 inches each. What is the length of the chord of the segment?

The answer is 672 inches.
The method is as follows: Subtract twice the small diameter \((d)\) from the large diameter \((D)\). Call the difference heaven \((h)\). Add this to \(d\). Call this earth \((e)\). Add 10 times the square root of \(eh\) to \(4h + 6e\). Call this man \((m)\). Now divide \(3(m + D) + e\) by \(d\), take the square root, and divide it into \(m\). You get the chord asked for.

By KITTA Yasohachi Motokatsu, a disciple of Fujita Sadasuke of the Seki School, at Kōjimachi in the Eastern capital, in the 9th year of Tenmei (1789), first (lunar) month.

As we shall see, one can analyze the figure and verify that the numerical solution is correct for the given data. Yet the analysis that emerges does not lead to the stated solution procedure, which may be paraphrased in modern notation as follows:

\[
h = D - 2d, \quad e = h + d, \quad m = 10\sqrt{eh} + 4h + 6e, \quad w = \sqrt{\frac{3(m + D) + e}{d}}, \quad x = \frac{m}{w}.
\]

2. A data-driven solution

Given \(D = 697\) and \(d = 272\), we notice that all but the chord length are divisible by 17 and that \((\sqrt{h}, \sqrt{d}, \sqrt{e})\) is a Pythagorean triple.

<table>
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<td>16</td>
<td>9</td>
<td>25</td>
<td>336</td>
<td>39.52941</td>
</tr>
</tbody>
</table>

Since \(w = \frac{17}{2}\) whether one uses the first or second row of numbers, Kitta evidently scaled up the data to ensure that \(x\) would be an integer.

Because segment-inscribed circles in many but not all other sangaku problems touch the segment’s chord at its midpoint, and since that is (approximately) what one sees in Figure 1, Kitta’s readers were likely to take this for granted even though it was not included in the statement of the problem. In fact, this is a necessary condition for Kitta’s solution, but it would have led readers acquainted with other sangaku results into a less than general line of reasoning such as the following.

Lemma 1 ([3, 2.2.7]). The radius of the mixtilinear incircle touching the legs of a right triangle with sides \(a, b, c\) (hypotenuse) and its circumcircle is \(\rho = a + b - c\).

Proof. In Figure 2, triangle \(ABC\) has a right angle at \(C\), and the circle \((J)\) touches \(BC, AC\) at \(T_a, T_b\) respectively. \(P\) is the orthogonal projection of the circumcenter \(O\) on \(JT_A\). In triangle \(POJ\), we have

\[
\left(\frac{a}{2} - \rho\right)^2 + \left(\rho - \frac{b}{2}\right)^2 = \left(\frac{c}{2} - \rho\right)^2
\]

or

\[
\frac{a^2}{4} + \frac{b^2}{4} - a\rho - b\rho + 2\rho^2 = \frac{c^2}{4} - c\rho + \rho^2.
\]

Since \(a^2 + b^2 = c^2\), this quickly reduces to the result. \(\square\)
Corollary 2. \( \rho = 2r \), where \( r \) is the inradius of the right triangle \( ABC \).

Corollary 3 ((5, 254, 275-76)). The difference between each leg and the radius of the mixtilinear circle is twice the sagitta on that leg, i.e.,

\[ 2k_a = a - \rho \quad \text{and} \quad 2k_b = b - \rho. \]

Proof. \( a - \rho = a - (a + b - c) = c - b = 2 \left( \frac{c}{2} - \frac{k}{2} \right) = 2(R - OM_a) = 2k_a \). The proof for \( b \) is analogous. \( \square \)

If \( 2\rho = k_a \), then \( 2(a + b - c) = \frac{1}{2}(c - b) \), \( 4a + 4b - 4c = c - b \), \( 4a + 5b = 5c \).

From this, \( (4a + 5b)^2 = (5c)^2 = 25(a^2 + b^2) \), \( 16a^2 + 40ab + 25b^2 = 25a^2 + 25b^2 \), \( 9a^2 = 40ab \), and \( a : b : c : \rho = 40 : 9 : 41 : 8. \)

Assume \( BC = 40 \), \( CA = 9 \), and \( AB = 41 \). Now add \( E \) such that \( BE \) touches \((OJ)\) at \( T_c \). We seek the length of \( BE \) (see Figure 3). Let \( F \) be the intersection...
of $CA$ and $BE$ extended. By equal tangents on $(J)$, let $F T_b = F T_c = x$. Since $B T_c = B T_a = 40 - 8 = 32$, by the Pythagorean relation for the triangle $F B C$,

$$(8 + x)^2 + 40^2 = (32 + x)^2 \implies x = \frac{40}{3}.$$ 

This means that $C F = 8 + \frac{40}{3} = \frac{64}{3}$ and $B F = 32 + \frac{40}{3} = \frac{136}{3}$.

By the intersecting chords theorem, $E F \cdot B F = A F \cdot C F$, and $E F = \frac{A F \cdot C F}{B F} = \frac{37}{3} \cdot \frac{64}{3} \cdot \frac{3}{136} = \frac{296}{51}$. This gives $B E = \frac{136}{3} - \frac{296}{51} = \frac{672}{17}$. Scaling up by 17 gives the answer.

In summary, if one focuses on the data given and applies other well-known *sangaku* results, one can verify the value of the given answer. Kitta does not state that $A C$ touches $(J)$ in Figure 3, although this is true for the given data. But neither does Kitta state that the circle inscribed in the segment touches $B C$ at its midpoint, which is in fact a necessary condition, so it would not be unreasonable for the reader to think that both pieces of information were to be inferred from the data. Yet a little experimentation shows that Kitta’s method is valid even when $A C$ does not touch $(J)$.

### 3. The general case

Let us start afresh with the Figure 4, in which $Y J = \frac{d}{2}$. By an elementary *sangaku* theorem ([3, 1.1]), $E Y = 2 \sqrt{\frac{d}{2}(\frac{D}{2} - d)} = \sqrt{d(D - 2d)} = \sqrt{dh}$.

By the intersecting chords theorem, $E A = \sqrt{d(D - d)} = \sqrt{de}$. Hence $b = 2E A = 2\sqrt{de}$. Let $r$ be the inradius of $A B C$, $p = \sqrt{de} + \sqrt{dh} = Y A$, and $q = Y'A$. Notice that all seven right triangles

are similar. In particular, \( \frac{KJ}{IK} = \frac{Y'I}{Y'I} \) and \( \frac{IK}{KY} = \frac{IY'}{Y'A} \). That is, \( \frac{d}{2} - r = \frac{r \cdot IK}{q} \) and \( IK = \frac{r^2}{q} \). Hence, \( \frac{d}{2} - r = r \cdot \frac{r^2}{q} \). But by applying Fujita’s mixtilinear circle theorem ([3, 2.2.8], also [7, 8]) with \( A \) on \( (O) \), \( \frac{e}{q} = \frac{d}{2p} \). Therefore, \( \frac{d}{2} - r = r \cdot \frac{d^2}{4p^2} \).

Solving for \( r \), \( r = \frac{2dp^2}{d^2 + 4p^2} \).

Likewise, since the triangles \( AEP \) and \( BZ'I \) are similar, \( \frac{d}{2d} = \frac{c}{r} - \frac{2p}{d} \). Solving for \( c \), \( c = \frac{2dp^2}{d^2 + 4p^2} \cdot \frac{b + 4p}{2d} = \frac{p^2(b + 4p)}{d^2 + 4p^2} \).

Now, \( b + 4p = 6\sqrt{de} + 4\sqrt{dh} \), and, because \( d = (\sqrt{e} + \sqrt{h})(\sqrt{e} - \sqrt{h}) \), we see that \( p^2 = (\sqrt{e} + \sqrt{h})^3(\sqrt{e} - \sqrt{h}) \) and

\[
d^2 + 4p^2 = (\sqrt{e} + \sqrt{h})^2(\sqrt{e} - \sqrt{h})^2 + 4(\sqrt{e} + \sqrt{h})(\sqrt{e} - \sqrt{h})(\sqrt{e} + \sqrt{h})^2
= (\sqrt{e} + \sqrt{h})^2(\sqrt{e} - \sqrt{h})[\sqrt{e} - \sqrt{h} + 4(\sqrt{e} + \sqrt{h})]
= (\sqrt{e} + \sqrt{h})^2(\sqrt{e} - \sqrt{h})(5\sqrt{e} + 3\sqrt{h}).
\]

Hence, \( \frac{p^2}{d^2 + 4p^2} = \frac{\sqrt{e} + \sqrt{h}}{5\sqrt{e} + 3\sqrt{h}} \) and \( c = \frac{2\sqrt{d}(3\sqrt{e} + 2\sqrt{h})(\sqrt{e} + \sqrt{h})}{5\sqrt{e} + 3\sqrt{h}} \).

This is Kitta’s formula because \( m = 10\sqrt{eh} + 4h + 6e = 2(3\sqrt{e} + 2\sqrt{h})(\sqrt{e} + \sqrt{h}), \)

and

\[
3(m+D)+e = 3(m+2e-h)+e = 3m+7e-3h = 25e+30\sqrt{eh}+9h = (5\sqrt{e}+3\sqrt{h})^2.
\]

But we are not quite done since Kitta defined \( w \) as \( \frac{\sqrt{(5\sqrt{e}+3\sqrt{h})^2}}{d} \) and used \( m \) to define the numerator of \( c = \frac{m_0}{w} \). Why not set \( w_0 = \frac{5\sqrt{e}+3\sqrt{h}}{\sqrt{d}} \) and \( m_0 = \frac{dw_0^2 - e}{3} - D \) and have \( c = \frac{m_0}{w_0} \)? I suspect the reason is that Kitta relied on two other sangaku results, which took him on a circuitous path to \( c \).

One result was an expression for the diameter of the circumcircle in terms of two sagittae and the inradius of a circumscribed triangle (see Figure 5), \( D = \frac{4d(4d+2\ell+r)}{4d^2-\ell^2} \). This follows from the equivalent of Carnot’s Theorem stated in terms of the sagittae on the sides of a triangle rather than the signed perpendicular distances from the circumcenter to the sides. This elegant version of the theorem was cited as if common knowledge, with no mention of Carnot, in a Meiji period paper by Y. Sawayama ([6, 153]), and we can reasonably conclude that it was known in premodern Japan.

Proof: If \( n \) is the third sagitta in Figure 5, Sawayama’s form of Carnot’s Theorem states that \( d + \ell + n = D - r \). Hence \( 2D - 2n = 2(d + \ell + r) \). Since the right triangles \( CPQ \) and \( AIY \) are similar, \( \frac{d}{2n} = \frac{d}{r} \); but by the intersecting chords

\[1\] The proof in [7] was flawed and is superseded by [8].
theorem, \(n(D - n) = \left(\frac{a}{2}\right)^2\). Hence \(D - n = \frac{a^2}{r^2}\cdot n\), or, \(n = \frac{Dr^2}{q^2 + r^2}\). Now, another sangaku theorem ([3, 2.2]) states that the square of the distance from the vertex of a triangle to its incenter is four times the product of the sagittae on the adjacent sides. In the present case, \(4d\ell = x^2\), where \(x^2 = q^2 + r^2\). Therefore, \(2D - 2n = 2D - \frac{D(4d\ell^2 - r^2)}{2d\ell}\) and so \(2(d + \ell + r) = \frac{D(4d\ell^2 - r^2)}{2d\ell}\), or \(D = \frac{4d\ell(d + \ell + r)}{4d\ell - r^2}\). \(\square\)

Armed with \(D = \frac{4d\ell(d + \ell + r)}{4d\ell - r^2}\) and \(\ell = \frac{x^2}{4d}\), one calculates \(c\) by the intersecting chords theorem without much difficulty because \(D - \ell\) factors nicely into

\[
\frac{4d^2\ell + 4d\ell^2 + 4d\ell r}{4d\ell - r^2} - \ell = \frac{4d^2\ell + 4d\ell r + \ell r^2}{4d\ell - r^2} = \frac{\ell(2d + r)^2}{4d\ell - r^2}.
\]

Therefore,

\[
\left(\frac{c}{2}\right)^2 = \ell(D - \ell) = \frac{\ell^2(2d + r)^2}{4d\ell - r^2} = \frac{(2d + r)^2x^4}{16d^2(x^2 - r^2)} = \frac{(2d + r)^2(q^2 + r^2)^2}{16d^2q^2}
\]

\[
= \frac{(d^2 + 4p^2)(2d + r)}{64d^2p^2} = \frac{\sqrt{d}(5\sqrt{e} + 3\sqrt{h}) + p}{4d^2(5\sqrt{e} + 3\sqrt{h})^2}
\]

\[
= \frac{d(d + 4(\sqrt{e} + \sqrt{h})^2)(3\sqrt{e} + 2\sqrt{h})^2}{(5\sqrt{e} + 3\sqrt{h})^4}
\]

\[
= \frac{(\sqrt{e} - \sqrt{h})(\sqrt{e} + \sqrt{h})^3(3\sqrt{e} + 2\sqrt{h})^2}{(5\sqrt{e} + 3\sqrt{h})^2}.
\]

Hence,

\[
c = 2\sqrt{\frac{(\sqrt{e} - \sqrt{h})(\sqrt{e} + \sqrt{h})^3(3\sqrt{e} + 2\sqrt{h})^2}{(5\sqrt{e} + 3\sqrt{h})^2}} = 2\sqrt{\frac{((\sqrt{e} + \sqrt{h})^2(3\sqrt{e} + 2\sqrt{h})^2}{(5\sqrt{e} + 3\sqrt{h})^2}}.
\]

(We get to the same expression if we eliminate \(D\) using \(D = 2e - h\), but the algebra is lengthier.) Thus we obtain a quadratic in \(c\) with no linear term and
therefore with symmetric roots, one negative definite and one positive. From this perspective, \( c = \sqrt{\frac{e^2}{m}} \) and Kitta’s definitions of \( m \) and \( w \) make sense.

I have recently discovered that this problem is the first one discussed by AIDA Yasuaki in a manuscript criticizing various solutions in Shinpeki sanpō ([1, 1.5-6]). Aida’s solution is equivalent to the equation \( c = \frac{2\sqrt{d(3\sqrt{e}+2\sqrt{h})(\sqrt{e}+\sqrt{h})}}{5\sqrt{e}+3\sqrt{h}} \) derived above. In terms of the notation we have been using, Aida sets \( h_0 = \sqrt{d(D-d)} \) and \( e_0 = \sqrt{h_0^2 - d^2} + h_0 \) (that is, \( e_0 = \sqrt{d(D-2d)} + \sqrt{d(D-d)} \)), and writes \( c = \frac{e_0^2}{2h_0+3e_0} + e_0 \), avoiding Kitta’s \( m \) and \( w \) placeholders altogether.

4. Discussion

Kitta evidently took pains to write up this problem in such a way that even a reader who could verify the numerical result might still be baffled by the general procedure. In this way, it is like a double-locked box. For the sophisticated solver, the unanswered question is why neat Pythagorean relationships emerge in the data.

A connection between the general and special cases of the problem lies in the fact that, if \( d \) and \( p \) are integers, then \( h, d, \) and \( e \) are perfect squares and \((\sqrt{h}, \sqrt{d}, \sqrt{e})\) is a Pythagorean triple. To prove this, recall that \( dp^2 = (\sqrt{e} + \sqrt{h})^4(\sqrt{e} - \sqrt{h})^2 \). Replacing \( h \) with \( e - d \) and solving for \( e \), \( e = \frac{(d^2+p^2)^2}{4dp^2} \); that is, \( e = \frac{(d^2+p^2)^2}{2dp} \).

Thus, provided \( d \) and \( p \) are integers, \( e \) and \( d \) are perfect squares, and, since \( h + d = e \) implies \( h = (p^2 - d^2)^2 \), \( h \) is a perfect square too.

References

Appendix (by the editor): Kitta’s configurations with integer diameters

Given a triangle $ABC$ with sidelengths $BC = a$, $CA = b$, $AB = c$, semiperimeter $s$, and area $\Delta$, the radius of the mixtilinear incircle in angle $A$ is

$$ρ_a = \frac{r}{\cos^2 \frac{A}{2}} = \frac{bc}{s(s-a)} \cdot \frac{\Delta}{s} = \frac{bc\Delta}{s^2(s-a)},$$

where $r = \frac{\Delta}{s}$ is the inradius of the triangle. With reference to Figure 5, $d = R(1 - \cos B) = 2R \sin^2 \frac{B}{2}$. The condition $2ρ_a = d$ becomes $\frac{2bc\Delta}{s^2(s-a)} = \frac{2ab}{\Delta} \cdot \frac{(s-c)(s-a)}{ca}$. Since $\Delta^2 = s(s-a)(s-b)(s-c)$, this reduces to $4c(s-b) = s(s-a)$, and $a^2 - b^2 + 7c^2 - 10bc + 8ca = 0$. By completing squares, we rewrite this as $(a + 4c)^2 = (b + c)(b + 9c)$. It follows that

$$a = -4c + \sqrt{(b + c)(b + 9c)} = -4c + \sqrt{(b + 5c)^2 - (4c)^2}.$$

To obtain integer solutions we put $b + 5c = p^2 + q^2$, $4c = 2pq$ for relatively prime integers $p$ and $q$. This leads to

$$b = 2p^2 + 2q^2 - 5pq, \quad c = pq, \quad a = 2(p^2 - q^2 - 2pq).$$

These satisfy the triangle inequality if and only if $p > 5q^2$. The circumradius of the triangle is rational if and only if the area is an integer. Since $Δ = 2pq(p - 2q)\sqrt{(2p - 5q)^q}$, we choose $p$ and $q$ such that $2p - 5q : q = u^2 : v^2$ for relatively prime integers $u, v$. Equivalently, $(p, q) = \left(\frac{u^2 + 5v^2}{9}, \frac{4u^2}{9}\right)$ with $g = \gcd(u^2 + 5v^2, 2u^2)$. The following table shows that with small values of $u$ and $v$, we exhaust all examples in which $a, b, c,$ after reduction by their gcd, are all integers below 1000.

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<td>9</td>
<td>8</td>
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<td>$\frac{441}{2}$</td>
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<tr>
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<td>2</td>
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<td>$\frac{353}{2}$</td>
<td>$\frac{353}{2}$</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>15</td>
<td>1</td>
<td>388</td>
<td>377</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
</tr>
</tbody>
</table>

Kitta’s example is the one obtained from $(u, v) = (1, 2)$, magnified by a factor 8 to make the circumradius an integer. Here are the only configurations with integer values of $a, b, c, D,$ and $2p_a = d$, all below 1000:

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$D = 2p_a$</th>
<th>$\ell$</th>
<th>$AT_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>10</td>
<td>6</td>
<td>10</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>41</td>
<td>40</td>
<td>9</td>
<td>41</td>
<td>17</td>
<td>$\frac{17}{2}$</td>
</tr>
<tr>
<td>328</td>
<td>680</td>
<td>672</td>
<td>697</td>
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<td>256</td>
</tr>
<tr>
<td>408</td>
<td>390</td>
<td>42</td>
<td>442</td>
<td>117</td>
<td>1</td>
</tr>
</tbody>
</table>

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Distances Between the Circumcenter of the Extouch Triangle and the Classical Centers of a Triangle

Marie-Nicole Gras

Abstract. We compute, in a triangle, the distances between the circumcenter of the extouch triangle and the circumcenter, the incenter, and the orthocenter, respectively. For this calculation, we use the absolute barycentric coordinates and obtain relatively simple formulas which seem unknown. To conclude, we compute the barycentric coordinates of the incenter of the extouch triangle.

1. Introduction

We consider a triangle $ABC$ and we denote by $O$ the circumcenter, $I$ the in-center, $H$ the orthocenter, $G$ the centroid, and $N$ the nine-point center. We denote the side-lengths by $a$, $b$, $c$, the semiperimeter by $s$, $R$ the circumradius, and $r$ the inradius. The distances between the classical centers of the triangle $ABC$ are well known. We recall that

$$O \overline{I}^2 = R^2 - 2Rr,$$

$$O \overline{H}^2 = R^2 - 8R^2 \cos A \cos B \cos C,$$

$$H \overline{I}^2 = 2r^2 - 4R^2 \cos A \cos B \cos C.$$

It is well known that the circle through the excenters of triangle $ABC$ has center $I'$, the reflection of $I$ in $O$, and that the radii through the excenters are perpendicular to the corresponding sides of $ABC$. It follows that the extouch triangle $XYZ$ is the pedal triangle of $I'$, and its circumcircle is the common pedal circle of $I'$ and

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its isogonal conjugate $I^*$. The circumcenter $\Omega$ is the midpoint between $I'$ and $I^*$.

In this note we compute the distances between $\Omega$ and the above classical triangle centers.

**Theorem 1.**
(a) $\Omega O^2 = R^2 - \frac{4R^3(R-r)}{r^2} \cos A \cos B \cos C$.
(b) $\Omega I^2 = 2R^2 - 4Rr - \frac{4R^3(R-2r)}{r^2} \cos A \cos B \cos C$.
(c) $\Omega H^2 = 2R^2 - 4Rr - 2r^2 - \frac{4R^3(R-r)(R-3r)}{r^2} \cos A \cos B \cos C$.

We collect a number of useful formulas for cyclic sums of trigonometrical expressions involving the angles of a triangle.

**Lemma 2.**
(a) $\cos A + \cos B + \cos C = \frac{R+r}{R}$.
(b) $\sum_cyclic \cos B \cos C = \frac{(2R+r)^2}{2R^2} + \cos A \cos B \cos C$.
(c) $\sum_cyclic \sin A \cos A = \frac{r^2}{2R^2}$.
(d) $\sum_cyclic (\cos B + \cos C) \sin A = \sin A + \sin B + \sin C = s \frac{r}{R}$.
(e) $\sin A \sin B \sin C = \sum_cyclic (\cos B \cos C) \sin A = s \frac{rs}{2R^2}$.

2. Homogeneous barycentric coordinates of some centers

In the *Encyclopedia of triangles centers* [1], henceforth referred to as ETC, Kimberling publishes a list of more than 5600 triangle centers with homogeneous trilinear and barycentric coordinates. In this paper we consider barycentric coordinates exclusively. An introduction to barycentric coordinates can be found in [3]. Sometimes it is useful to work with absolute barycentric coordinates. For a finite point, the absolute barycentric coordinates can be found from a set of homogeneous barycentric coordinates by dividing by its coordinate sum. If the triangle center $X(n)$ in ETC is a finite point, we denote by $(\alpha_n, \beta_n, \gamma_n)$ its absolute barycentric coordinates.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$X(n)$</th>
<th>$\alpha_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$I$</td>
<td>$\frac{R \sin A}{r}$</td>
</tr>
<tr>
<td>3</td>
<td>$O$</td>
<td>$\frac{R \sin A}{r} (R \cos A)$</td>
</tr>
<tr>
<td>4</td>
<td>$H$</td>
<td>$\frac{R \sin A}{r} (2R \cos B \cos C)$</td>
</tr>
<tr>
<td>8</td>
<td>$N_a$</td>
<td>$\frac{R \sin A}{r} (2R(\cos A + \cos B \cos C) - 2r)$</td>
</tr>
<tr>
<td>20</td>
<td>$L$</td>
<td>$\frac{R \sin A}{r} \cdot 2R(\cos A - \cos B \cos C)$</td>
</tr>
<tr>
<td>40</td>
<td>$I'$</td>
<td>$\frac{R \sin A}{r} (2R \cos A - r)$</td>
</tr>
</tbody>
</table>

The isogonal conjugate of $I' := X(40)$ is the triangle center $X(84)$.

**Proposition 3.** $\alpha_{84} = \frac{R \sin A}{r} \cdot \frac{(2R \cos B - r)(2R \cos C - r)}{r}$.

**Proof.** Since, in homogeneous barycentric coordinates,

$X(40) = (\sin A(2R \cos A - r) : \sin B(2R \cos B - r) : \sin C(2R \cos C - r))$, 

we have
\[ X(84) = \left( \frac{\sin A}{2R \cos A - r} : \frac{\sin B}{2R \cos B - r} : \frac{\sin C}{2R \cos C - r} \right). \]

Therefore,
\[ \alpha_{84} = \frac{\sin A}{2R \cos A - r} \cdot \frac{\sin B}{2R \cos B - r} + \frac{\sin C}{2R \cos C - r} = \frac{\sin A(2R \cos B - r)(2R \cos C - r)}{\sum \text{cyclic} \sin A(2R \cos B - r)(2R \cos C - r)}. \]

Using the formulas in Lemma 2, we have
\[ \sum \text{cyclic} \sin A(2R \cos B - r)(2R \cos C - r) = 4R^2 \sum \text{cyclic} \sin A \cos B \cos C - 2Rr \sum \text{cyclic} \sin A \cos B \cos C + r^2 \sum \text{cyclic} \sin A = 4R^2 \cdot \frac{rs}{2R^2} - 2Rr \cdot \frac{s}{R} + r^2 \cdot \frac{s}{R} = \frac{r^2 s}{R}. \]

From this the result follows. \( \square \)

Lemma 4. The line joining \( X(40) \) and \( X(84) \) contains the Nagel point \( X(8) \).

Proof. With \( t = \frac{r}{2R} \), we have
\[ (1 - t)\alpha_{40} + t\alpha_{84} = \frac{R \sin A}{rs} \left( \left( 1 - \frac{r}{2R} \right)(2R \cos A - r) + \frac{r}{2R} \cdot \frac{(2R \cos B - r)(2R \cos C - r)}{r} \right) = \frac{R \sin A}{rs} \left( 2R \cos A - r \cos A - r + \frac{r^2}{2R} + 2R \cos B \cos C - r \cos B \cos C + \frac{r^2}{2R} \right) = \frac{R \sin A}{rs} \left( 2R \cos B \cos C + 2R \cos A - r \cos A - r + \frac{r^2}{R} \right) = \frac{R \sin A}{rs} \left( 2R \cos B \cos C + 2R \cos A - r \cdot \frac{R + r}{R} - r + \frac{r^2}{R} \right) = \alpha_{84}. \]

\( \square \)

Proposition 5. The circumcenter \( \Omega \) of the extouch triangle lies on the line joining \( X(40) \) and \( X(8) \). It has first absolute barycentric coordinate
\[ \alpha = \frac{R \sin A}{rs} \left( \frac{2R^2}{r} \cos B \cos C + 2R \cos A - (R + r) \right). \]
Proof. Since $\Omega$ is the midpoint of $X(40)$ and $X(84)$, it follows from Lemma 4 that it lies on the line $X(40)X(8)$. Furthermore,

$$\alpha = \frac{1}{2}(\alpha_{40} + \alpha_{84})$$

$$= \frac{R \sin A}{rs} \left( \frac{2R \cos A - r}{2} + \frac{(2R \cos B - r)(2R \cos C - r)}{2r} \right)$$

$$= \frac{R \sin A}{rs} \cdot \frac{(2R \cos A - r)r + (2R \cos B - r)(2R \cos C - r)}{2r}$$

$$= \frac{R \sin A}{rs} \cdot \frac{4R^2 \cos B \cos C + 4Rr \cos A - 2Rr(\cos A + \cos B + \cos C)}{2r}$$

$$= \frac{R \sin A}{rs} \cdot \frac{4R^2 \cos B \cos C + 4Rr \cos A - 2(R + r)r}{2r}$$

$$= \frac{R \sin A}{rs} \left( \frac{2R^2}{r} \cos B \cos C + 2R \cos A - (R + r) \right).$$

$$\square$$

Remark. In ETC, $\Omega$ is the triangle center $X(1158)$.

Figure 2 shows $\Omega$ on the line joining $I'$ to $N_a$. Since the deLongchamps point $L = X(20)$ is the reflection of $H$ in $O$, $O$ is the common midpoint of $II'$ and $HL$. From this, $IH$ is parallel to $I'L$. Also, the centroid $G$ divides both segments $IN_a$ and $HL$ in the ratio $1:2$, $HI$ is also parallel to $N_a$. It follows that $L$ lies on the line $I'N_a$ and $I'$ is the midpoint of $LN_a$.

Lemma 6. $\Omega I' = \frac{R}{r} \cdot HI$. 
Distances between circumcenter of extouch triangle and classical centers

Proof. By Proposition 5,
\[ \alpha - \alpha_40 = \frac{R \sin A}{rs} \left( \frac{2R^2}{r} \cos B \cos C + 2R \cos (R + r) - (2R \cos A - r) \right) \]
\[ = \frac{R \sin A}{rs} \left( \frac{2R^2}{r} \cos B \cos C - R \right) \]
\[ = \frac{R}{r} \cdot \frac{R \sin A}{rs} (2R \cos B \cos C - r) \]
\[ = \frac{R}{r} (\alpha_4 - \alpha_1). \]

\[ \square \]

3. Proof of Theorem 1

Lemma 7. (a) \( 2\Omega O^2 - \Omega I^2 = 4Rr - \frac{4R^4}{r^2} \cos A \cos B \cos C \).
(b) \( 2\Omega O^2 - \Omega H^2 = 4Rr + 2r^2 - \frac{4R^2}{r^2} (R^2 + 2Rr - 3r^2) \cos A \cos B \cos C. \)

Proof. (a) Applying Apollonius to the median \( \Omega O \) of triangle \( \Omega II' \), we have
\[ \Omega I^2 + \Omega I'^2 = 2(\Omega O^2 + OI^2). \]
From this,
\[ 2\Omega O^2 - \Omega I^2 = \Omega I^2 - 2OI^2 \]
\[ = \frac{R^2}{r^2} HI^2 - 2OI^2 \]
\[ = \frac{R^2}{r^2} (2r^2 - 4R^2 \cos A \cos B \cos C) - 2R(R - 2r) \]
\[ = 4Rr - \frac{4R^4}{r^2} \cos A \cos B \cos C. \]
(b) Applying Apollonius to the median \( \Omega O \) of triangle \( \Omega HL \), we have
\[ \Omega H^2 + \Omega L^2 = 2(\Omega O^2 + OH^2). \]
From this,
\[2\Omega O^2 - \Omega H^2 = \Omega L^2 - 2\Omega H^2\]
\[= \frac{(R + r)^2}{r^2} H^2 - 2\Omega H^2\]
\[= \frac{(R + r)^2}{r^2} \left(2r^2 - 4R^2 \cos A \cos B \cos C\right) - 2R^2 \left(1 - 8 \cos A \cos B \cos C\right)\]
\[= 4Rr + 2r^2 - \frac{4R^2}{r^2} \left(R^2 + 2Rr - 3r^2\right) \cos A \cos B \cos C\]
\[\square\]

**Lemma 8.** \[\Omega O^2 - \Omega I^2 = -R^2 + 4Rr - \frac{4R^3}{r} \cos A \cos B \cos C\]

**Proof.** We begin with \(AI^2 = \frac{r^2 \sin A}{s} = \frac{r^2 bc}{s-b(s-c)} = \frac{\frac{s-a}{s}bc}{b-c}\), similarly for \(BI^2\) and \(CI^2\). Therefore,
\[\alpha AI^2 + \beta AB^2 + \gamma CI^2\]
\[= 4R^2 \sin A \sin B \sin C \left(\frac{\alpha}{\sin A} + \frac{\beta}{\sin B} + \frac{\gamma}{\sin C}\right) - 4Rr(\alpha + \beta + \gamma)\]
\[= 2rs \left(\frac{\alpha}{\sin A} + \frac{\beta}{\sin B} + \frac{\gamma}{\sin C}\right) - 4Rr\]
\[= 2R \sum_{\text{cyclic}} \left(2R^2 \cos B \cos C + 2R \cos A - (R + r)\right) - 4Rr\]
\[= 2R \left(2R^2 \cos B \cos C + 2R \cos A - (R + r)\right) - 4Rr\]
\[= 2R \left(\frac{2R^2}{r} \left(\frac{(2R + r)r}{2R} + \cos A \cos B \cos C\right) + 2R \cdot \frac{R + r}{R} - 3(R + r)\right) - 4Rr\]
\[= 2R^2 - 4Rr + \frac{4R^3}{r} \cos A \cos B \cos C\]

We make use of a formula of Scheer [2]. For an arbitrary point \(P\),
\[\Omega P^2 = \alpha AP^2 + \beta BP^2 + \gamma CP^2 - (\beta \gamma a^2 + \gamma \alpha b^2 + \alpha \beta c^2)\]. Applying this to \(P = O\) and \(P = I\) respectively, we have
\[\Omega O^2 - \Omega I^2 = (\alpha AO^2 + \beta BO^2 + \gamma CO^2) - (\alpha AI^2 + \beta BI^2 + \gamma CI^2)\]
\[= R^2 - (\alpha AI^2 + \beta BI^2 + \gamma CI^2)\]
\[= R^2 - (2R^2 - 4Rr + \frac{4R^3}{r} \cos A \cos B \cos C)\]
\[= -R^2 + 4Rr - \frac{4R^3}{r} \cos A \cos B \cos C\].
\[\square\]
Now we complete the proof of Theorem 1.

(a) For the distance from $\Omega$ to the circumcenter:
\[
\Omega O^2 = (2\Omega O^2 - \Omega I^2) - (\Omega O^2 - \Omega I^2)
\]
\[
= 4Rr - \frac{4R^3}{r^2} \cos A \cos B \cos C
\]
\[
- \left( -R^2 + 4Rr - \frac{4R^3}{r} \cos A \cos B \cos C \right)
\]
\[
= R^2 - \frac{4R^3}{r^2}(R - r) \cos A \cos B \cos C.
\]

(b) For the distance from $\Omega$ to the incenter:
\[
\Omega I^2 = \Omega O^2 - (\Omega O^2 - \Omega I^2)
\]
\[
= \left( R^2 - \frac{4R^3}{r^2}(R - r) \cos A \cos B \cos C \right)
\]
\[
- \left( -R^2 + 4Rr - \frac{4R^3}{r} \cos A \cos B \cos C \right)
\]
\[
= 2R^2 - 4Rr - \frac{4R^3}{r^2}(R - 2r) \cos A \cos B \cos C.
\]

(c) For the distance from $\Omega$ to the orthocenter:
\[
\Omega H^2 = 2\Omega O^2 - (2\Omega O^2 - \Omega H^2)
\]
\[
= 2 \left( R^2 - \frac{4R^3}{r^2}(R - r) \cos A \cos B \cos C \right)
\]
\[
- \left( 4Rr + 2r^2 - \frac{4R^2}{r^2} \left( R^2 + 2Rr - 3r^2 \right) \cos A \cos B \cos C \right)
\]
\[
= 2R^2 - 4Rr - 2r^2 - \frac{4R^2}{r^2} (R^2 - 4Rr + 3r^2) \cos A \cos B \cos C.
\]

The proof of Theorem 1 is now complete.

Since the centroid $G$ and the nine-point center $N$ divide the segment $OH$ in the ratio $OG : ON : OH = 2 : 3 : 6$, further applications of the Apollonius theorem yield the distances from $\Omega$ to $G$ and $N$.

**Corollary 9.**

(a) $\Omega G^2 = \frac{2}{9} (5R^2 - 6Rr - 3r^2) - \frac{4R^2}{9r^2} (9R^2 - 18Rr + 5r^2) \cos A \cos B \cos C$.

(b) $\Omega N^2 = \frac{5}{4} R^2 - 2Rr - r^2 - \frac{2R^2}{r^2} (2R^2 - 5Rr + 2r^2) \cos A \cos B \cos C$.

**Remarks.** (1) Since $4R^2 \cos A \cos B \cos C = s^2 - (r + 2R)^2$, these distances can all be expressed in terms of $R, r, s$.

(2) We also note the two simple relations:

(i) $\Omega O^2 + OI^2 = \frac{R(R-r)}{r^2} HI^2$;

(ii) $\Omega I = \frac{OI \times HI}{r}$.
4. The cyclocevian conjugate of the Nagel point

Since the circumcircle of the extouch triangle is the pedal circle of $I' = X(40)$, it is also the pedal circle of $I^* = X(84)$. The pedals of $X(84)$ are the vertices of the cyclocevian conjugate of the Nagel point $N_a = X(8)$. It is interesting to note that this is also a point on the line $I'N_a$. In ETC, this is $X(189)$ with homogeneous barycentric coordinates are

$$
\left( \frac{1}{\cos B + \cos C - \cos A - 1}, \frac{1}{\cos C + \cos A - \cos B - 1}, \frac{1}{\cos A + \cos B - \cos C - 1} \right).
$$

![Figure 4](image)

**Proposition 10.** The first absolute barycentric coordinate of the cyclocevian conjugate of the Nagel point is

$$
\alpha_{189} = \frac{R \sin A}{rs} \left( (2R \cos A - r) + k(2R \cos B \cos C - r) \right),
$$

where

$$
k = \frac{(4R + r)r + 4R^2 \cos A \cos B \cos C}{r^2 + 4R^2 \cos A \cos B \cos C}.
$$

**Proof.** The point $X(189)$ divides $I'N_a$ in the ratio

$$
N_aX(189) : X(189)I' = t : 1 - t
$$
for $t = \frac{-4Rr}{r^2 + 4R^2 \cos A \cos B \cos C}$. From this,
\[
\alpha_{189} = t\alpha_{40} + (1 - t)\alpha_8
\]
\[
= \frac{R \sin A}{rs} \left( t(2R \cos A - r) + (1 - t)(2R \cos B \cos C + 2R \cos A - 2r) \right)
\]
\[
= \frac{R \sin A}{rs} \left( (2R \cos A - r) + (1 - t)(2R \cos B \cos C - r) \right).
\]
The coefficient $1 - t$ is $k$ given in the statement of the proposition. \(\square\)

5. The centroid and orthocenter of the extouch triangle

The centroid of the extouch triangle is very easy to determine: It is the triangle center $X = Y + Z = \frac{1}{3} \left( \left( 0, c + a - b, a + b - c \right) + \left( b + c - a, 0, a + b - c \right) \right) \frac{2a}{2b} + \frac{(b + c - a, c + a - b, 0)}{2c} = \frac{(a(b + c)(b + c - a), b(c + a)(c + a - b), c(a + b)(a + b - c))}{6abc}$. This is the triangle center $X(210)$ in ETC. Clearly,
\[
\alpha_{210} = \frac{a(b + c)(b + c - a)}{6abc}.
\]
By expressing this in the form
\[
\alpha_{210} = \frac{R \sin A}{rs} \left( \frac{R}{3} \cdot (\sin B + \sin C)(\sin B + \sin C - \sin A) \right), \quad (1)
\]
we easily determine also the orthocenter of the extouch triangle:

**Proposition 11.** The orthocenter of the extouch triangle has first absolute barycentric coordinate
\[
\alpha' = \frac{R \sin A}{rs} \left( R((\sin B + \sin C)(\sin B + \sin C - \sin A) - 4 \cos A) \right.
\]
\[
- \frac{4R^2}{r} \cos B \cos C + 2(R + r) \right). \]

**Proof.** Since the orthocenter divides the centroid $X(210)$ and the circumcenter $\Omega$ in the ratio $3 : -2$, we have the first absolute barycentric coordinate equal to $\alpha' = 3\alpha_{210} - 2\alpha$. The result follows from Proposition 5 and (1) above. \(\square\)

**Remark.** In terms of $a$, $b$, $c$, the orthocenter of the extouch triangle has homogeneous barycentric coordinates
\[
(af(a, b, c)g(a, b, c) : bf(b, c, a)g(b, c, a) : cf(c, a, b)g(c, a, b))
\]
where
\[ f(a, b, c) = a^2(b + c) - a^2(b - c)^2 - a(b + c)(b - c)^2 + (b^2 - c^2)^2, \]
\[ g(a, b, c) = a^5 - a^4(b + c) - 2a^3(b^2 - c^2) + 2a^2(b + c)(b^2 + c^2) + a(b^4 - 4b^3c - 2b^2c^2 - 4bc^3 + c^4) - (b - c)^2(b + c)^3. \]

It is not in the current edition of ETC and has (6–9–13)-search number 52.7618273660 \cdots.

6. The incenter of the extouch triangle

Lemma 12. Let \( a', b', c' \) be the sidelengths of the extouch triangle \( XYZ \).
\[ a'^2 = a^2(1 \sin B \sin C), \quad b'^2 = b^2(1 \sin C \sin A), \quad c'^2 = c^2(1 \sin A \sin B). \]

Proof. It is enough to establish the expression for \( a'^2 \). Since \( AY = s - c \) and \( AZ = s - b \), applying the law of cosines to triangle \( AYZ \), we have
\[
a'^2 = (s - b)^2 + (s - c)^2 - 2(s - b)(s - c) \cos A
\]
\[
= (s - b)^2 + (s - c)^2 - 2(s - b)(s - c) \left( 2 \cos^2 \frac{A}{2} - 1 \right)
\]
\[
= (s - b)^2 + (s - c)^2 + 2(s - b)(s - c) - 4(s - b)(s - c) \cdot \frac{s(s - a)}{bc}
\]
\[
= ((s - b) + (s - c))^2 - \frac{4\Delta^2}{bc}
\]
\[
= a'^2 - \frac{4\Delta^2}{bc}
\]
\[
= a'^2(1 - \sin B \sin C)
\]
since \( \Delta = \frac{1}{2}ca \sin B = \frac{1}{2}ab \sin C \).

Proposition 13. The incenter of the extouch triangle has homogeneous barycentric coordinates
\[
(\sin B + \sin C - \sin A)(\sqrt{1 - \sin C \sin A} + \sqrt{1 - \sin A \sin B} : \cdots : \cdots).
\]

Proof. With reference to triangle \( XYZ \), this incenter has homogeneous barycentric coordinates \((a': b': c')\). The absolute barycentric with reference to \( ABC \) is therefore
\[
a'X + b'Y + c'Z
\]
\[
a' + b' + c'.
\]
In homogeneous coordinates, this can be taken as
\[
a'((0, c + a - b, a + b - c) + b'(b + c - a, 0, a + b - c) + c'(b + c - a, c + a - b, 0)
\]
\[
= \frac{1}{2} \left( (b + c - a) \left( \frac{b'}{b} + \frac{c'}{c} \right), (c + a - b) \left( \frac{c'}{c} + \frac{a'}{a} \right), (a + b - c) \left( \frac{a'}{a} + \frac{b'}{b} \right) \right).
\]
The result follows from the law of sines and an application of Lemma 12. \( \square \)
We conclude by giving the coordinates of the incenter of the extouch triangle in terms of $a, b, c$. Using the Heron formula
\[ \Delta^2 = \frac{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}{16}, \]
we have
\[ a'^2 = \frac{4a^2bc - 16\Delta^2}{4bc} \]
\[ = \frac{4a^2bc - 2b^2c^2 - 2c^2a^2 - 2a^2b^2 + a^4 + b^4 + c^4}{4bc} \]
\[ = \frac{a^4 - 2a^2(b - c)^2 + (b^2 - c^2)^2}{4bc}. \]
Therefore,
\[ \frac{a'}{a} = \sqrt{bc\left(a^4 - 2a^2(b - c)^2 + (b^2 - c^2)^2\right)}; \]
similarly for $\frac{b'}{b}$ and $\frac{c'}{c}$. This leads to
\[ \left((b + c - a) \left(\sqrt{ca(b^4 - 2b^2(c - a)^2 + (c^2 - a^2)^2)} + \sqrt{ab(c^4 - 2c^2(a - b)^2 + (a^2 - b^2)^2)}\right) \right) : \cdots : \cdots. \]

This is a triangle center not in the current edition of ETC. It has $(6 - 9 - 13)$-search number 4.662905022010···.

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The Touchpoints Triangles and the Feuerbach Hyperbolas

Sándor N. Kiss and Paul Yiu

Abstract. In this paper we generalize the famous Kariya theorem on the perspectivity of a given triangle with the homothetic images of the intouch triangle from the incenter to the touchpoints triangles of the excircles, leading to the triad of ex-Feuerbach hyperbolas. We also study in some details the triangle formed by the orthocenters of the touchpoints triangles. An elegant construction is given for the asymptotes of the Feuerbach hyperbolas.

1. Introduction

Consider a triangle $ABC$ with incircle $I(r)$ tangent to the sides $BC$, $CA$, $AB$ at $X$, $Y$, $Z$ respectively. These form the intouch triangle $T_i$ of $ABC$. For a real number $t$, let $T_i(t)$ be the image of the $T_i$ under the homothety $h(I, \frac{t}{r})$. Its vertices are the points $X(t)$, $Y(t)$, $Z(t)$ on the lines $IX$, $IY$, $IZ$ respectively, such that

$$IX(t) = IY(t) = IZ(t) = t.$$

The famous Kariya’s theorem asserts that the lines $AX(t)$, $BY(t)$, $CY(t)$ are concurrent, i.e., the triangles $ABC$ and $T_i(t)$ are perspective, and that the perspector is a point $Q(t)$ on the Feuerbach hyperbola $\mathcal{F}$, the rectangular circum-hyperbola which is the isogonal conjugate of the line $OI$ joining the circumcenter and the incenter of $ABC$. This fact was actually known earlier to J. Neuberg and H. Mandart; see [5] and the interesting note in [2, §1242]. We revisit in §3 this theorem with a proof leading to simple relations of the perspectors $Q(t)$ and $Q(-t)$ (Proposition 4 below), and their isogonal conjugates on the line $OI$. In §5 we obtain analogous results by replacing $T_i(t)$ by homothetic images of the touchpoints triangles of the excircles, leading to the triad of ex-Feuerbach hyperbolas. Some properties of the triad of ex-Feuerbach hyperbolas are established in §§7–10. Specifically, we give an elegant construction of the asymptotes of the Feuerbach hyperbolas in §10. The final section §11 is devoted to further properties of the touchpoints triangles, in particular, the loci of perspectors of the triangle $H^aH^bH^c$ formed by their orthocenters.

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2. Generalization of Kariya’s theorem

Let \( a, b, c \) be the sidelengths of triangle \( ABC \), \( s = \frac{1}{2}(a+b+c) \) the semiperimeter, and \( \Delta \) the area of the triangle. It is well known that

(i) \( \Delta = rs \),

(ii) \( abc = 4R\Delta \) for the circumradius \( R \), and

(iii) \( 16\Delta^2 = a^2(b^2 + c^2 - a^2) + b^2(c^2 + a^2 - b^2) + c^2(a^2 + b^2 - c^2) \).

We shall work with homogeneous barycentric coordinates with reference to triangle \( ABC \), and refer to [7] for basic results and formulas.

Let \( T \) be the pedal triangle of a point \( P = (u : v : w) \) in homogeneous barycentric coordinates. The vertices of \( T \) are the points

\[
X = (0 : (a^2 + b^2 - c^2)u + 2a^2v : (c^2 + a^2 - b^2)u + 2a^2w),
\]

\[
Y = ((a^2 + b^2 - c^2)v + 2b^2u : 0 : (b^2 + c^2 - a^2)v + 2b^2w),
\]

\[
Z = ((c^2 + a^2 - b^2)w + 2c^2u : (b^2 + c^2 - a^2)w + 2c^2v : 0).
\]

For a real number \( k \), let \( T_k \) be the image of \( T \) under the homothety \( h(P,k) \).

**Lemma 1.** The vertices of \( T_k \) are the points

\[
X_k = (2a^2(1-k)u : (a^2 + b^2 - c^2)ku + 2a^2v : (c^2 + a^2 - b^2)ku + 2a^2w), \quad (1)
\]

\[
Y_k = ((a^2 + b^2 - c^2)kv + 2b^2u : 2b^2(1-k)v : (b^2 + c^2 - a^2)kv + 2b^2w), \quad (2)
\]

\[
Z_k = ((c^2 + a^2 - b^2)kw + 2c^2u : (b^2 + c^2 - a^2)kw + 2c^2v : 2c^2(1-k)w). \quad (3)
\]

**Proof.** The point \( X_k \) divides the segment \( PX \) in the ratio \( PX_k : X_kX = k : 1-k \).

In absolute barycentric coordinates,

\[
X_k = (1-k)P + kX = \frac{(1-k)(u,v,w) + k(0, (a^2 + b^2 - c^2)u + 2a^2v, (c^2 + a^2 - b^2)u + 2a^2w)}{u + v + w}.
\]

Ignoring the denominator, we obtain the homogeneous barycentric coordinates of \( X_k \) given above; similarly for \( Y_k \) and \( Z_k \).

**Proposition 2.** The only points that satisfy Kariya’s theorem, as the incenter \( I \) does, are the orthocenter \( H \), the circumcenter \( O \), and the three excenters \( I_a, I_b, I_c \).

**Proof.** The equations of the lines \( AX_k, BY_k, CZ_k \) are

\[
((c^2 + a^2 - b^2)ku + 2a^2w)y - ((a^2 + b^2 - c^2)ku + 2a^2v)z = 0,
\]

\[
-(b^2 + c^2 - a^2)kv + 2b^2w)x + ((a^2 + b^2 - c^2)kv + 2b^2u)z = 0,
\]

\[
((b^2 + c^2 - a^2)kw + 2c^2v)x - ((c^2 + a^2 - b^2)kw + 2c^2u)y = 0.
\]

They are concurrent if and only if

\[
\begin{vmatrix}
0 & (c^2 + a^2 - b^2)ku + 2a^2w & ((a^2 + b^2 - c^2)ku + 2a^2v) \\
-(b^2 + c^2 - a^2)kv + 2b^2w & 0 & (a^2 + b^2 - c^2)kv + 2b^2u \\
(b^2 + c^2 - a^2)kw + 2c^2v & -(c^2 + a^2 - b^2)kw + 2c^2u & 0
\end{vmatrix} = 0.
\]
Equivalently, $2F_1(P) \cdot k - F_2(P) = 0$ for every $k$, where

$$F_1(P) = \sum_{\text{cyclic}} a^2(b^2 + c^2 - a^2)u(c^2v^2 - b^2w^2),$$

$$F_2(P) = \sum_{\text{cyclic}} (c^2 + a^2 - b^2)(a^2 + b^2 - c^2)u(c^2v^2 - b^2w^2).$$

This means that $F_1(P) = F_2(P) = 0$. The point $P$ is common to the McCay cubic $pK(X(6), X(3))$, and the orthocubics $pK(X(6), X(4))$. These appear in [3] as $K_{003}$ and $K_{006}$ respectively. It is known that the common points of these circum cubics are the vertices of $ABC$ and the points $H$, $O$, $I$, $I_a$, $I_b$, $I_c$, as can be readily verified. □

The case $P = H$ is trivial because the perspector is $H$ for every $k$.

The case $P = O$ is also trivial because the triangle $T_k$ is homothetic to $ABC$, and the perspector is obviously the point $Q_k$ on the Euler line dividing $OH$ in the ratio $k : 2$. This follows from

$$OQ_k : Q_kH = OX(k) : HA = k : 2.$$

In the remainder of this paper, we study the cases when $P$ is the incenter or an excenter.
3. Kariya’s theorem and the Feuerbach hyperbola

The pedal triangle of the incenter \( I = (a : b : c) \) has vertices

\[
X = (0 : a + b - c : c + a - b), \\
Y = (a + b - c : 0 : b + c - a), \\
Z = (c + a - b : b + c - a : 0).
\]

The coordinates of \( X(t), Y(t), Z(t) \) can be determined from equations (1), (2), (3) by putting \( k = \frac{t}{r} \).

Proposition 3. The lines \( AX(t), BY(t), CZ(t) \) are concurrent at the point

\[
Q(t) = \left( \frac{1}{2rbc + t(b^2 + c^2 - a^2)} : \frac{1}{2rca + t(c^2 + a^2 - b^2)} : \frac{1}{2rab + t(a^2 + b^2 - c^2)} \right),
\]

which is the isogonal conjugate of the point \( P(t) \) dividing \( OI \) in the ratio

\[
OP(t) : P(t)I = R : t.
\]

Proof. Writing the homogeneous barycentric coordinates of \( X(t), Y(t), Z(t) \) as

\[
X(t) = \left( **:** : \frac{1}{2rca + t(c^2 + a^2 - b^2)} : \frac{1}{2rab + t(a^2 + b^2 - c^2)} \right), \\
Y(t) = \left( \frac{1}{2rbc + t(b^2 + c^2 - a^2)} : **:** : \frac{1}{2rab + t(a^2 + b^2 - c^2)} \right), \\
Z(t) = \left( \frac{1}{2rbc + t(b^2 + c^2 - a^2)} : \frac{1}{2rca + t(c^2 + a^2 - b^2)} : **:** \right),
\]

we note that the lines \( AX(t), BY(t), CZ(t) \) are concurrent at a point \( Q(t) \) with coordinates given in (4) above. This is clearly the isogonal conjugate of the point

\[
P(t) = (2rabc \cdot a + ta^2(b^2 + c^2 - a^2), 2rabc \cdot b + tb^2(c^2 + a^2 - b^2), \\
2rabc \cdot c + tc^2(a^2 + b^2 - c^2))
\]

\[
= 2rabc(a + b + c) + t((a^2(b^2 + c^2 - a^2), b^2(c^2 + a^2 - b^2), c^2(a^2 + b^2 - c^2)) \\
= 2r \cdot 4Rrs(a + b + c) \cdot I + t \cdot 16\Delta^2 \cdot O \\
= 16r^2s^2 \cdot R \cdot I + 16\Delta^2 \cdot t \cdot O \\
= 16r^2s^2(R \cdot I + t \cdot O).
\]

In absolute barycentric coordinates, \( P(t) = \frac{R \cdot I + t \cdot O}{R + t} \). This is the point dividing \( OI \) in the ratio \( OP(t) : P(t)I = R : t \).

It follows that the locus of the point \( Q(t) \) is the Feuerbach hyperbola \( \mathcal{F} \). The center is the Feuerbach point \( F_n \), the point of tangency of the incircle and the nine-point circle (see Figure 2). Note that for each \( t \), the points \( P(t) \) and \( P(-t) \) divide \( OI \) harmonically.
Proposition 4. The line joining $Q(t)$ and $Q(-t)$ contains the triangle center

$H_t = \left( \frac{a(b+c)}{b+c-a} : \frac{b(c+a)}{c+a-b} : \frac{c(a+b)}{a+b-c} \right)$. \hfill (5)

Proof. The line joining $Q(t)$ to $Q(-t)$ has equation

$$\sum_{\text{cyclic}} a(b-c)(b+c-a)(4b^2c^2r^2 - t^2(b^2 + c^2 - a^2)^2)x = 0.$$ 

With $(x : y : z)$ given in (5) we have

$$\sum_{\text{cyclic}} a^2(b^2 - c^2)(4b^2c^2r^2 - t^2(b^2 + c^2 - a^2)^2)$$

$$= 4a^2b^2c^2r^2 \left( \sum_{\text{cyclic}} (b^2 - c^2) \right) - t^2 \left( \sum_{\text{cyclic}} a^2(b^2 - c^2)(b^2 + c^2 - a^2)^2 \right)$$

$$= 0.$$ 

While the first sum obviously is zero, the second sum vanishes because the Euler line

$$\sum_{\text{cyclic}} (b^2 - c^2)(b^2 + c^2 - a^2)x = 0 \hfill (6)$$

contains the circumcenter. \hfill \square
Remark. The triangle center $H_i$ is the orthocenter of the intouch triangle $T_i$; it appears as $X(65)$ in [4]. It divides $OI$ in the ratio $R : r = -r$. Its isogonal conjugate is the Schiffler point

$$S_c = \left( \frac{a(b + c - a)}{b + c} : \frac{b(c + a - b)}{c + a} : \frac{c(a + b - c)}{a + b} \right),$$

which is the point of concurrency of the Euler lines of the four triangles $ABC$, $IBC$, $ICA$, and $IAB$. Therefore, $H_i$ is a point on the Jerabek hyperbola $J$:

$$\frac{a^2(b^2 - c^2)(b^2 + c^2 - a^2)}{x} + \frac{b^2(c^2 - a^2)(c^2 + a^2 - b^2)}{y} + \frac{c^2(a^2 - b^2)(a^2 + b^2 - c^2)}{z} = 0,$$

the isogonal conjugate of the Euler line.

4. The touchpoints triangles

Consider the $A$-excircle $I_a(r_a)$ of triangle $ABC$, tangent to the sidelines $BC$ at $X_a$, $CA$ at $Y_a$, and $AB$ at $Z_a$ respectively. We call triangle $T_a := X_aY_aZ_a$ the $A$-touchpoints triangle. Clearly,

$$I_aX_a = I_aY_a = I_aZ_a = r_a = \frac{\Delta}{s - a}.$$

In homogeneous barycentric coordinates these are the points

$$I_a = (-a : b : c),$$
$$X_a = (0 : c + a - b : a + b - c),$$
$$Y_a = -(c + a - b) : 0 : a + b + c),$$
$$Z_a = -(a + b - c) : a + b + c : 0).$$

Similarly, we also have the $B$-touchpoints triangle $T_b := X_bY_bZ_b$ from the $B$-excircle $I_b(r_b)$ and the $C$-touchpoints triangles $T_c := X_cY_cZ_c$ from the $C$-excircle $I_c(r_c)$ (see Figure 3).

Consider the reflection $I_a'$ of $I_a$ in the line $Y_aZ_a$. Since $Y_aZ_a$ is perpendicular to the line $AI_a$, $I_a'$ lies on $AI_a$, and $I_aI_a' = 2r_a \sin \frac{A}{2}$. On the other hand, $I_aI$, being a diameter of the circle through $I$, $B$, $I_a$, $C$, has length $\frac{a}{\sin \frac{B + C}{2}} = \frac{a}{\cos \frac{A}{2}}$.

It follows that

$$I_aI_a' : I_aI = 2r_a \sin \frac{A}{2} : \frac{a}{\cos \frac{A}{2}} = r_a : \frac{a}{\sin A} = r_a : 2R = I_aX_a : I_aI'.$$

Therefore, $X_aI_a'$ and $I'I$ are parallel. Note that the midpoint of $X_aI_a'$ is the nine-point center of $T_a$.

The same conclusions apply to the other two touchpoints triangles $T_b := X_bY_bZ_b$ and $T_c := X_cY_cZ_c$ associated with the $B$- and $C$-excircles $I_b(r_b)$ and $I_c(r_c)$.

Corollary 5. (a) The Euler lines of the touchpoints triangles of the excircles are concurrent at $O$.
(b) The nine-point centers of the touchpoints triangles form a triangle perspective with the extouch triangle $X_aY_bZ_c$ at the infinite point of the $OI$ line.
Let $H^a$ be the orthocenter of $T_a$. Since the nine-point center $N_a$ is also the midpoint of its circumcenter $I_a$ and $H^a$, we obtain, from the parallelogram $H^a X_a I_a I'_a$,

$$H^a = X_a + I'_a - I_a$$

$$= X_a + \frac{r_a}{2R} (I - I_a)$$

$$= X_a + \frac{2s(s-b)(s-c)}{abc} (I - I_a)$$

$$= \frac{(s-b)B + (s-c)C}{a} + \frac{2s(s-b)(s-c)}{abc} \left( \frac{aA + bB + cC}{2s} - \frac{-aA + bB + cC}{2(s-a)} \right)$$

$$= \frac{(b+c)(s-b)(s-c)}{bc(s-a)} A + \frac{s(c-a)(s-b)}{ac(s-a)} B - \frac{s(a-b)(s-c)}{ab(s-a)} C.$$

**Proposition 6.** In homogeneous barycentric coordinates, the orthocenters of the touchpoints triangles are the points

$$H^a = \left( \frac{a(b+c)}{a+b+c} : \frac{b(c-a)}{a+b-c} : \frac{-c(a-b)}{c+a-b} \right),$$

$$H^b = \left( \frac{-a(b-c)}{a+b-c} : \frac{b(c+a)}{a+b+c} : \frac{c(a-b)}{b+c-a} \right),$$

$$H^c = \left( \frac{a(b-c)}{c+a-b} : \frac{-b(c-a)}{b+c-a} : \frac{c(a+b)}{a+b+c} \right).$$

**Remark.** The orthocenter $H^a$ of $T_a$ divides $OI_a$ in the ratio $R : r_a$; similarly for $H^b$ and $H^c$. These orthocenters lie on the Jerabek hyperbola $J$ (see Figure 4).
**Proposition 7.** The triangle $H^a H^b H^c$ is perspective with the orthic triangle $H_a H_b H_c$ (of $ABC$) at $H_i$.

5. Kariya’s theorem for the $A$-touchpoints triangle

Consider the image of the $A$-touchpoints triangle $T_a$ under the homothety $h \left( I_a, \frac{1}{r_a} \right)$ for a real number $t$. This is the triangle $T_a(t)$ with vertices $X_a(t), Y_a(t), Z_a(t)$ on the lines $I_a X_a, I_a Y_a, I_a Z_a$ respectively, such that

$$I_a X_a(t) = I_a Y_a(t) = I_a Z_a(t) = t.$$ 

These points can be determined from Lemma 1 by putting $k = \frac{t}{r_a}$. In homogeneous barycentric coordinates, they are

$$X_a(t) = (2(r_a - t)a^2 : -2r_a ab + t(a^2 + b^2 - c^2) : -2r_a ca + t(c^2 + a^2 - b^2)), \quad (8)$$

$$Y_a(t) = (-2r_a ab + t(a^2 + b^2 - c^2) : 2(r_a - t)b^2 : 2r_a bc + t(b^2 + c^2 - a^2)), \quad (9)$$

$$Z_a(t) = (-2r_a ca + t(c^2 + a^2 - b^2) : 2r_a bc + t(b^2 + c^2 - a^2) : 2(r_a - t)c^2). \quad (10)$$
Proposition 8. The lines $AX_a(t)$, $BY_a(t)$, $CZ_a(t)$ are concurrent at the point

$$Q_a(t) = \left( \frac{1}{2r_a bc + t(b^2 + c^2 - a^2)} : \frac{1}{-2r_a ca + t(c^2 + a^2 - b^2)} : \frac{1}{-2r_a ab + t(a^2 + b^2 - c^2)} \right),$$

which is the isogonal conjugate of the point $P_a(t)$ dividing $OI_a$ in the ratio

$$OP_a(t) : P_a(t)I_a = R : -t.$$

Proof: Rewrite the homogeneous barycentric coordinates of $X_a(t)$, $Y_a(t)$, $Z_a(t)$ as follows:

$$X_a(t) = \left( \ast \ast \ast \ast \ast : \frac{1}{-2r_a ca + t(c^2 + a^2 - b^2)} : \frac{1}{-2r_a ab + t(a^2 + b^2 - c^2)} \right),$$

$$Y_a(t) = \left( \frac{1}{2r_a bc + t(b^2 + c^2 - a^2)} : \ast \ast \ast \ast \ast : \frac{1}{-2r_a ab + t(a^2 + b^2 - c^2)} \right),$$

$$Z_a(t) = \left( \frac{1}{2r_a bc + t(b^2 + c^2 - a^2)} : \ast \ast \ast \ast \ast : \frac{1}{-2r_a ca + t(c^2 + a^2 - b^2)} \right).$$

It follows easily that the lines $AX_a(t)$, $BY_a(t)$, $CZ_a(t)$ are concurrent at a point with coordinates given in (11) above (see Figure 5). This is clearly the isogonal

Figure 5.
conjugate of the point

\[ P_a(t) = \left( a^2(2r_a bc + t(b^2 + c^2 - a^2)), b^2(-2r_a ca + t(c^2 + a^2 - b^2)), \\
    c^2(-2r_a ab + t(a^2 + b^2 - c^2)) \right) \]

\[ = -2r_a abc(-a, b, c) + t((a^2(b^2 + c^2 - a^2), b^2(c^2 + a^2 - b^2), c^2(a^2 + b^2 - c^2)) \]

\[ = -2r_a \cdot 4Rrs(b + c - a) \cdot I_a + t \cdot 16\Delta^2 c \cdot O \]

\[ = -16rr_{rs}(s - a) \cdot R \cdot I_a + 16\Delta^2 t \cdot O \]

\[ = -16\Delta^2 (R \cdot I_a - t \cdot O). \]

In absolute barycentric coordinates, \( P_a(t) = \frac{R \cdot I_a - t \cdot O}{R - t}. \) This is the point dividing \( OI_a \) in the ratio \( OP_a(t) : P_a(t)I_a = R : -t. \)

**Proposition 9.** The locus of the point \( Q_a(t) \) is the rectangular circum-hyperbola \( \mathcal{F}_a : \]

\[ a(b - c)(a + b + c)yz + b(c + a)(a + b - c)zx - c(a + b)(c + a - b)xy = 0 \]

with center

\[ F_a = (-b - c)^2(a + b + c) : (c + a)^2(a + b - c) : (a + b)^2(c + a - b), \]

the point of tangency of the nine-point circle with the A-excircle.

**Proof.** By Proposition 8, the locus of \( Q_a(t) \) is the isogonal conjugate of the line \( OI_a. \) It is a rectangular hyperbola since it contains the orthocenter \( H, \) the isogonal conjugate of \( O. \) The center of the hyperbola is a point on the nine-point circle. The equation of the line \( OI_a \) is

\[
\begin{vmatrix}
    x & y & z \\
    a^2(b^2 + c^2 - a^2) & b^2(c^2 + a^2 - b^2) & c^2(a^2 + b^2 - c^2) \\
    -a & b & c
\end{vmatrix} = 0.
\]

After simplification, this becomes

\[-bc(b - c)(a + b + c)(b + c - a)x \]
\[-ca(c + a)(a + b - c)(b + c - a)y \]
\[+ab(a + b)(b + c - a)(c + a - b)z = 0.\]

Replacing \((x, y, z)\) by \((a^2yz, b^2zx, c^2xy)\) we obtain the equation of the hyperbola \( \mathcal{F}_a \) given above. Since the center of the circumconic \( pyz + qzx + rxy = 0 \) is the point

\[ (p(q + r - p) : q(r + p - q) : r(p + q - r)), \]

with

\[ p = bc(b - c)(a + b + c), \quad q = ca(c + a)(a + b - c), \quad r = -ab(a + b)(c + a - b), \]

we obtain the center of the hyperbola as the point

\[ F_a = p(q + r - p) : q(r + p - q) : r(p + q - r) \]

\[ = -2abc(b - c)^2(a + b + c) : 2abc(c + a)^2(a + b - c) : 2abc(a + b)^2(c + a - b) \]

\[ = -(b - c)^2(a + b + c) : (c + a)^2(a + b - c) : (a + b)^2(c + a - b). \]
This is indeed a point on the line joining $I_a$ to the nine-point center
$$N = (a^2(b^2+c^2)-(b^2-c^2)^2 : b^2(c^2+a^2)-(c^2-a^2)^2 : c^2(a^2+b^2)-(a^2-b^2)^2).$$

It is routine to verify that
$$2abc(-a, b, c) + (a^2(b^2+c^2)-(b^2-c^2)^2, b^2(c^2+a^2)-(c^2-a^2)^2, c^2(a^2+b^2)-(a^2-b^2)^2)$$

$$= (b + c - a)(-b(c + a), c + a, a + b)(a + b - c).$$

Since the coordinate sum in (12) is
$$2(2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4)$$
$$= 2(a + b + c)(b + c - a)(c + a - b)(a + b - c)$$
$$= 32 \Delta^2,$$

this is the point dividing $NI_a$ in the ratio
$$2abc(b + c - a) : 32 \Delta^2 = 2 \cdot 4R\Delta(b + c - a) : 32 \Delta^2 = \frac{R}{2} : \frac{2\Delta}{b + c - a} = \frac{R}{2} : r_a,$$

i.e., the point of tangency of the nine-point circle and the $A$-excircle. \qed

**Proposition 10.** The line joining $Q_a(t)$ and $Q_a(-t)$ contains the orthocenter $H^a$ of the $A$-touchpoints triangle $T_a$.

**Proof.** The equation of the line $Q_a(t)Q_a(-t)$ is
$$a(b - c)(a + b + c)(-4b^2c^2r_a^2 + t^2(b^2 + c^2 - a^2)^2)x$$
$$+ b(c + a)(a + b - c)(-4c^2a^2r_a^2 + t^2(c^2 + a^2 - b^2)^2)y$$
$$- c(a + b)(c + a - b)(-4a^2b^2r_a^2 + t^2(a^2 + b^2 - c^2)^2)z$$
$$= 0.$$

Substituting the coordinates of the point $H^a$ given in Proposition 6, we obtain
$$- 4a^2b^2c^2r_a^2 ((b^2 - c^2) + (c^2 - a^2) + (a^2 - b^2))$$
$$+ t^2 (a^2(b^2 - c^2)(b^2 + c^2 - a^2)^2 + b^2(c^2 - a^2)(c^2 + a^2 - b^2)^2$$
$$+ c^2(a^2 - b^2)(a^2 + b^2 - c^2)^2)$$
$$= 0,$$

as in the proof of Proposition 4. \qed

**6. The triad of ex-Feuerbach hyperbolas**

We call the hyperbola $\mathcal{F}_a$ in Proposition 9 the $A$-ex-Feuerbach hyperbola. We also consider the triangles $T_b(t) := X_b(t)Y_b(t)Z_b(t)$ and $T_c(t) := X_c(t)Y_c(t)Z_c(t)$. These vertices are the points on the lines $I_bX_b, I_bY_b, I_bZ_b$ satisfying
$$I_bX_b(t) = I_bY_b(t) = I_bZ_b(t) = t,$$
and \(X_c(t), Y_c(t), Z_c(t)\) on \(I_cX_c, I_cY_c, I_cZ_c\) satisfying
\[I_cX_c(t) = I_cY_c(t) = I_cZ_c(t) = t.\]

In homogeneous barycentric coordinates,
\[
\begin{align*}
X_b(t) &= (2(r_b - t)a^2 : -2r_bab + t(a^2 + b^2 - c^2) : 2r_bca + t(c^2 + a^2 - b^2)), \\
Y_b(t) &= (-2r_bab + t(a^2 + b^2 - c^2) : 2(r_b - t)b^2 : -2r_bbc + t(b^2 + c^2 - a^2)), \\
Z_b(t) &= (2r_bca + t(c^2 + a^2 - b^2) : -2r_bbc + t(b^2 + c^2 - a^2) : 2(r_b - t)c^2), \\
X_c(t) &= (2(r_c - t)a^2 : 2r_cab + t(a^2 + b^2 - c^2) : -2r_cca + t(c^2 + a^2 - b^2)), \\
Y_c(t) &= (2r_cab + t(a^2 + b^2 - c^2) : 2(r_c - t)b^2 : -2r_cbc + t(b^2 + c^2 - a^2)), \\
Z_c(t) &= (-2r_cca + t(c^2 + a^2 - b^2) : -2r_cbc + t(b^2 + c^2 - a^2) : 2(r_c - t)c^2).
\end{align*}
\]

Clearly there are analogous hyperbolas \(\mathcal{F}_b\) and \(\mathcal{F}_c\) which are isogonal conjugates of the lines \(OI_b\) and \(OI_c\). These hyperbolas have centers
\[
\begin{align*}
F_b &= ((b + c)^2(a + b - c) : -(c - a)^2(a + b + c) : (a + b)^2(b + c - a)), \\
F_c &= ((b + c)^2(c + a - b) : (c + a)^2(b + c - a) : -(a - b)^2(a + b + c)).
\end{align*}
\]

**Remark.** The centers of the triad of ex-Feuerbach hyperbolas, being the points of tangency of the nine-point circle with the excircles, are perspective with \(ABC\) at the outer Feuerbach point
\[
X(12) = \begin{pmatrix}
\frac{(b + c)^2}{b + c - a} & \frac{(c + a)^2}{c + a - b} & \frac{(a + b)^2}{a + b - c}
\end{pmatrix}.
\]
Proposition 11. The triangle \(X_a(t)Y_b(t)Z_c(t)\) is homothetic to the intouch triangle at the point \((a(r_a - t) : b(r_b - t) : c(r_c - t))\), which divides \(OI\) in the ratio \(2R + r - t : -2r\) (see Figure 6).

7. Some special cases

7.1. \(t = R\). By Corollary 5(b), the point \(P_a(R)\) is the infinite point on the line \(OI_a\). It follows that \(Q_a(R)\) is on the circumcircle. It is the (fourth) intersection of the hyperbola \(F_a\) with the circumcircle. These points are the reflections of \(H\) in \(F_a, F_b, F_c\) respectively (see Figure 7).

7.2. \(t = 2R\). It is well known that the circumcenter of the excentral triangle is the reflection \(I'\) of \(I\) in \(O\), and is the point of concurrency of the perpendiculars from the excenters to the respective sidelines of triangle \(ABC\) (see Figure 6), and the circumradius is \(2R\). It follows that the points \(X_a(2R), Y_b(2R), Z_c(2R)\) all coincide with this circumcenter. It follows that the lines \(AQ_a(2R), BQ_b(2R), CQ_c(2R)\) are concurrent at this point. \(t = 2R\) is the only nonzero value of \(t\) for which the triangle \(Q_a(t)Q_b(t)Q_c(t)\) is perspective with \(ABC\).

In this case, both \(Y_c(2R)\) and \(Z_b(2R)\) are the reflection of \(I'\) in the line \(I_bI_c\). We call this \(X'\). The line \(AX'\) is the reflection of \(AI'\) in \(I_bI_c\). Since \(I_bI_c\) is the external bisector of angle \(A\) of triangle \(ABC\), \(AX'\) and \(AI'\) are isogonal lines with respect to this angle. Likewise, we have \(Y' = Z_a(2R) = X_c(2R)\) with \(BY'\), \(BI'\) isogonal with respect to \(B\), and \(Z' = Y_b(2R) = X_b(2R)\) with \(CZ'\), \(CI'\) isogonal with respect to \(C\). It follows that \(AX', BY', CZ'\) are concurrent at a point \(P\), which is the isogonal conjugate of \(I'\), and lies on the Feuerbach hyperbola \(F\) (see Figure 8).
8. Second tangents from $O$ to the ex-Feuerbach hyperbolas

The hyperbolas $\mathcal{F}_a$, $\mathcal{F}_b$, $\mathcal{F}_c$ also appear in [1], where they are called the ex-central Feuerbach hyperbolas. Neuberg [5] also mentioned these hyperbolas. The $A$-ex-Feuerbach hyperbola $\mathcal{F}_a$, being the isogonal conjugate of the line $OI_a$, is tangent to the line at $I_a$ (see Figure 9). If the second tangent from $O$ to $\mathcal{F}_a$ touches it at $T_a$, then the line $I_aT_a$ is the polar of $O$ with respect to the hyperbola $\mathcal{F}_a$. This is the line

$$bc(b - c)(a + b + c)(a(b^2 + c^2 - a^2) + (b + c)(c + a - b)(a + b - c))x$$
$$+ ca(c + a)(a + b - c)(c(a^2 + b^2 - c^2) - (a + b)(b + c - a)(c + a - b))y$$
$$- ab(a + b)(c + a - b)(b(c^2 + a^2 - b^2) - (c + a)(a + b - c)(b + c - a))z$$
$$= 0.$$

Apart from the excenter $I_a$, this line intersects the hyperbola $\mathcal{F}_a$ again at

$$T_a = \left( \frac{a}{a(b^2 + c^2 - a^2) + (b + c)(c + a - b)(a + b - c)}, \frac{b}{c(a^2 + b^2 - c^2) - (a + b)(b + c - a)(c + a - b)}, \frac{c}{b(c^2 + a^2 - b^2) - (c + a)(a + b - c)(b + c - a)} \right).$$
Similarly, the second tangents from $O$ to $\mathcal{F}_b$ and $\mathcal{F}_c$ (apart from $OI_b$ and $OI_c$) touch these hyperbolas at

$$T_b = \left( \begin{array}{c}
\frac{a}{c(a^2 + b^2 - c^2) - (a + b)(b + c - a)(c + a - b)} \\
\frac{b}{b(c^2 + a^2 - b^2) + (c + a)(a + b - c)(b + c - a)} \\
\frac{c}{a(b^2 + c^2 - a^2) - (b + c)(c + a - b)(a + b - c)}
\end{array} \right),$$

and

$$T_c = \left( \begin{array}{c}
\frac{a}{b(c^2 + a^2 - b^2) - (c + a)(a + b - c)(b + c - a)} \\
\frac{b}{a(b^2 + c^2 - a^2) - (b + c)(c + a - b)(a + b - c)} \\
\frac{c}{c(a^2 + b^2 - c^2) + (a + b)(b + c - a)(c + a - b)}
\end{array} \right).$$

These three points of tangency form a triangle perspective with $ABC$ at

$$T = \left( \begin{array}{c}
(a(a(b^2 + c^2 - a^2) - (b + c)(c + a - b)(a + b - c)) \\
\frac{b(b(c^2 + a^2 - b^2) - (c + a)(a + b - c)(b + c - a))}{c(c(a^2 + b^2 - c^2) - (a + b)(b + c - a)(c + a - b))}
\end{array} \right).$$

Figure 9.
This is the triangle center $X(46)$ in [4]. It has a number of interesting properties. It divides $OI$ externally in the ratio $R + r : -2r$, and can be constructed as the cevian quotient $H/I$. In other words, it is the perspector of the orthic triangle and the excentral triangle. Therefore, the point $T_a$, and similarly $T_b$ and $T_c$, can be easily constructed as follows.

1. Join $I_a$ and $H_a$ to intersect the line $OI$ at $T$.
2. Join $A$ and $T$ to intersect the hyperbola $I_a$ at $T_a$ (see Figure 9).

9. A correspondence between the Euler line and the Feuerbach hyperbola

Let $P = (u : v : w)$ be an arbitrary point. The lines $PI_a, PI_b, PI_c$ intersect the respective ex-Feuerbach hyperbolas at

$$W_a = \left( \frac{(b - c)(a + b + c)}{cv - bw} : \frac{c + a)(a + b - c)}{aw + cu} : \frac{(a + b)(c + a - b)}{bu + av} \right),$$

$$W_b = \left( \frac{(b + c)(a + b - c)}{cv + bw} : \frac{c - a)(a + b + c)}{aw - cu} : \frac{(a + b)(b + c - a)}{bu + av} \right),$$

$$W_c = \left( \frac{(b + c)(b + c - a)}{cv + bw} : \frac{c + a)(b + c - a)}{aw + cu} : \frac{(a - b)(a + b - c)}{bu - av} \right).$$

Figure 10.
These form a triangle perspective with $ABC$. The perspector is the point
\[ W = \left( \frac{b + c}{(b + c - a)(cv + bw)} : \frac{c + a}{(c + a - b)(aw + cu)} : \frac{a + b}{(a + b - c)(bu + av)} \right). \]

**Proposition 12.** The perspector $W$ is on the Feuerbach hyperbola if and only if $P$ lies on the Euler line.

*Proof.* The perspector $W$ is on the Feuerbach hyperbola if and only if its isogonal conjugate
\[ W^* = \left( \frac{a^2(b + c - a)(cv + bw)}{b + c} : \frac{b^2(c + a - b)(aw + cu)}{c + a} : \frac{c^2(a + b - c)(bu + av)}{a + b} \right) \]
lies on the line $OI$ with equation
\[ \sum_{\text{cyclic}} bc(b - c)(b + c - a)x = 0. \]

By substitution, we have
\[ 0 = \sum_{\text{cyclic}} bc(b - c)(b + c - a) \cdot \frac{a^2(b + c - a)(cv + bw)}{b + c} \]
\[ = abc \sum_{\text{cyclic}} \frac{a(b - c)(b + c - a)^2(cv + bw)}{b + c} \]
\[ = \frac{abc}{(b + c)(c + a)(a + b)} \sum_{\text{cyclic}} a(b - c)(c + a)(a + b)(b + c - a)^2(cv + bw). \]

Ignoring the nonzero factor, we have
\[ 0 = \sum_{\text{cyclic}} a(b - c)(c + a)(a + b)(b + c - a)^2(cv + bw) \]
\[ = \sum_{\text{cyclic}} (b(c - a)(a + b)(b + c)(c + a - b)^2 \cdot cu \]
\[ + c(a - b)(b + c)(c + a)(a + b - c)^2 \cdot bu) \]
\[ = \sum_{\text{cyclic}} bc(b + c)((c - a)(a + b)(c + a - b)^2 + (a - b)(c + a)(a + b - c)^2)u \]
\[ = \sum_{\text{cyclic}} bc(b + c) \cdot (-2a(b - c)(b^2 + c^2 - a^2))u \]
\[ = -2abc \sum_{\text{cyclic}} (b + c)(b - c)(b^2 + c^2 - a^2)u. \]

This means that $P = (u : v : w)$ lies on the Euler line (with equation given in (6)). \qed

If $P$ divides $OH$ in the ratio $OP : PH = t : 1 - t$, then the isogonal conjugate of $W$ is the point dividing $OI$ in the ratio $OW^* : W^*I = R \frac{t}{2}(1 - t) : rt$. A simple
application of Menelaus’ theorem yields the following construction of $W^*$. Let $P'$ be the inferior of $P$. Then the line $F_e P'$ intersects $OI$ at $W^*$ (see Figure 11).

![Figure 11](image-url)

If we put $W = (x : y : z)$, then $P$ is the point with coordinates

$$
(u : v : w) = \left( a \left( -\frac{a(b + c)}{(b + c - a)x + \frac{b(c + a)}{(c + a - b)y + \frac{c(a + b)}{(a + b - c)z}} \right)
\right).
$$

10. The asymptotes of the Feuerbach hyperbolas

As is well known, the asymptotes of a rectangular circum-hyperbola which is the isogonal conjugate of a line through $O$ are the Simson lines of the intersections of the line with the circumcircle. For the Feuerbach hyperbola and the ex-Feuerbach hyperbolas, we give an easier construction based on the fact that the lines joining the circumcenter to the incenter and the excenters are tangent to the respective Feuerbach hyperbolas.

**Lemma 13.** Let $P$ be a point on a rectangular hyperbola with center $O$. The tangent to the hyperbola at $P$ intersects the asymptotes at two points on the circle with center $P$, passing through $O$.

**Proof.** Set up a Cartesian coordinate system with the asymptotes as axes. The equation of the rectangular hyperbola is $xy = c^2$ for some $c$. If $P \left( ct, \frac{c}{t} \right)$ is a point on the hyperbola, the tangent at $P$ is the line $\frac{1}{2} \left( \frac{c}{t} x + cty \right) = c^2$, or $\frac{c}{t} + yt = 2c$. It intersects the asymptotes (axes) at $X(2ct, 0)$ and $Y \left( 0, \frac{2c}{t} \right)$. Since $P$ is the midpoint of $XY$, $PO = PX = PY$. □
**Proposition 14.** (a) The lines joining the Feuerbach point $F_e$ to the intersections of the incircle with the line $OI$ are the asymptotes of the Feuerbach hyperbola $\mathcal{F}$. 
(b) The lines joining the point $F_a$ to the intersections of the $A$-excircle with the line $OI_a$ are the asymptotes of the $A$-ex-Feuerbach hyperbola $\mathcal{F}_a$; similarly for the hyperbolas $\mathcal{F}_b$ and $\mathcal{F}_c$ (see Figure 12).

![Figure 12](image_url)

11. More on the touchpoints triangles

11.1. The symmedian points of the touchpoints triangles. Since the $A$-excircle is the circumcircle of the touchpoints triangle $T_a$, and the lines $BC, CA, AB$ are the tangents at its vertices, the symmedian point of $T_a$ is the point of concurrency of $BY_a, CZ_a, and AX_a$, i.e.,

$$K_a = \frac{-(c + a - b)(a + b - c) : (a + b + c)(c + a - b) : (a + b + c)(a + b - c)}{(a + b + c)(a + b - c)}.$$ 

Note that $K_a$ is a point on the $A$-ex-Feuerbach hyperbola $\mathcal{F}_a$.

The line joining $K_a$ to $I_a$ is the Brocard axis of $T_a$. It has equation

$$(b - c)(a + b + c)^2 x + (c + a)(a + b - c)^2 y - (a + b)(c + a - b)^2 z = 0.$$
**Proposition 15.** (a) The Brocard axes of the touchpoints triangles and the intouch triangle are concurrent at the deLongchamps point \( L \), which is the point on the Euler line of triangle \( ABC \) dividing \( OH \) in the ratio \(-1:2\).

(b) The van Aubel lines (joining the orthocenter and the symmedian point) of the touchpoints triangles and the intouch triangle are concurrent at \( H^* \), the isotomic conjugate of the orthocenter of \( ABC \) (see Figure 13).

![Figure 13](image)

**Remark.** The intersection of the Euler line with the line \( IG_e \) at the deLongchamps point \( L \) is a well known fact. See [6].

**Proposition 16.** The triangle \( H^aH^bH^c \) is perspective with the cevian triangle of \( Q \) if and only if \( Q \) lies on the line

\[
\mathcal{L} : \frac{(b+c)(b^2 + c^2 - a^2)}{b+c-a} x + \frac{(c+a)(c^2 + a^2 - b^2)}{c+a-b} y + \frac{(a+b)(a^2 + b^2 - c^2)}{a+b-c} z = 0
\]

or the Feuerbach hyperbola \( \mathcal{F} \).

(a) If \( Q \) traverses \( \mathcal{L} \), the perspector traverses the line \( \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \).

(b) If \( Q \) is on the Feuerbach hyperbola, the perspector \( P \) lies on the Jerabek hyperbola \( \mathcal{J} \). The line joining \( QP \) passes through the orthocenter \( H \) (see Figure 14).
Proof. Let \( Q = (u : v : w) \) with cevian triangle \( Q_aQ_bQ_c \) where \( Q_a = (0 : v : w) \), \( Q_b = (u : 0 : w) \), \( Q_c = (u : v : 0) \). The equations of the lines \( H^aQ_a \), \( H^bQ_b \), \( H^cQ_c \) are

\[
(a + b + c)(c(a - b)(a + b - c)v + b(c - a)(c + a - b)w)x - a(c + a - b)(a + b - c)(b + c)(wy - vz) = 0, ~ (13)
\]
\[
(a + b + c)(a(b - c)(b + c - a)w + c(a - b)(a + b - c)u)y - b(a + b - c)(b + c - a)(c + a)(uz - wx) = 0, ~ (14)
\]
\[
(a + b + c)(b(c - a)(c + a - b)u + a(b - c)(b + c - a)v)z - c(b + c - a)(c + a - b)(a + b)(vx - uy) = 0. ~ (15)
\]

Eliminating \( x, y, z \) from equations (13), (14), (15), we have

\[
2abc(a + b + c) \left( \sum_{cyclic} (b + c)(c + a - b)(a + b - c)(b^2 + c^2 - a^2)u \right) \left( \sum_{cyclic} a(b - c)(b + c - a)vw \right) = 0.
\]

Therefore the lines \( H^aQ_a, H^bQ_b, H^cQ_c \) are concurrent if and only if \( Q = (u : v : w) \) lies on the line \( \mathcal{L} \) or the Feuerbach hyperbola \( \mathbb{F} \).
Eliminating \( u, v, w \) from equations (13), (14), (15), we have

\[
32\Delta^2(bc x + cay + abz) \left( \sum_{\text{cyclic}} a^2(b^2 - c^2)(b^2 + c^2 - a^2)yz \right) = 0.
\]

Therefore the locus of the point of concurrency is the union of the line \( \mathcal{L}(I) : bc x + cay + abz = 0 \) (the trilinear polar of the incenter) and the Jerabek hyperbola \( \mathcal{J} \).

Now the line \( \mathcal{L} \) contains the point

\[
Q_0 = \left( \frac{(a-b)(c+b-a)^2}{b+c} : \frac{b(c-a)(c+a-b)^2}{c+a} : \frac{c(a-b)(a+b-c)^2}{a+b} \right)
\]

as is easily verified. Choosing \( Q = (u : v : w) \) to be this point, and solving equations (13), (14), (15), we have the perspector

\[
P_0 = (a(b^2 - c^2) : b(c^2 - a^2) : c(a^2 - b^2))
\]

on the line \( \mathcal{L}(I) \). Therefore, by continuity, when \( Q \) traverses the line \( \mathcal{L} \), \( P \) traverses \( \mathcal{L}(I) \).

On the other hand, if \( Q \) lies on the Feuerbach hyperbola, then \( P \) lies on the Jerabek hyperbola. If we take \( Q \) to be the point

\[
\left( \frac{1}{bc + t(b^2 + c^2 - a^2)} : \frac{1}{ca + t(c^2 + a^2 - b^2)} : \frac{1}{ab + t(a^2 + b^2 - c^2)} \right)
\]
on the Feuerbach hyperbola, then \( P \) is the point

\[
\left( \frac{a(b+c)(b+c-a)}{(b^2 + c^2 - a^2)(2rbc + t(b^2 + c^2 - a^2))} : \cdots : \cdots \right)
\]
on the Jerabek hyperbola. The line joining \( Q \) and \( P \) contains the orthocenter \( H \).

□

Remarks. (1) The triangle center \( Q_0 \) appears in \([4]\) as \( X(1021) \).

(2) The triangle center \( P_0 \) is the intersection of the lines \( bc x + cay + abz = 0 \) and \( ax + by + cz = 0 \). It appears in \([4]\) as \( X(661) \).

(3) The line \( \mathcal{L} \) can be constructed as the line containing the harmonic conjugates of \( I_9H \cap BC \) in \( BC \), \( I_9H \cap CA \) in \( CA \), and \( I_cH \cap AB \) in \( AB \). It is the trilinear polar of the triangle center \( X(29) \).

Proposition 17. The triangle \( H^aH^bH^c \) is perspective with the anticevian triangle of \( Q \) if and only if \( Q \) lies on the orthic axis

\[
\mathcal{L}(H) : (b^2 + c^2 - a^2)x + (c^2 + a^2 - b^2)y + (a^2 + b^2 - c^2)z = 0
\]
or the circumconic

\[
\mathcal{C} : a(b^2 - c^2)yz + b(c^2 - a^2)zx + c(a^2 - b^2)xy = 0
\]

passing through \( I \) and its isotomic conjugate \( I^\ast = (bc : ca : ab) \).

(a) If \( Q \) traverses the orthic axis, the perspector traverses the line \( \frac{z}{a} + \frac{y}{b} + \frac{z}{c} = 1 \) (again).
(b) If $Q$ is on the circumconic $\mathcal{C}$, the perspector $P$ lies on the Jerabek hyperbola (again). The line $QP$ passes through $H_i = \left( \frac{a(b+c)}{b+c-a} : \cdots : \cdots \right)$, the common point of the two conics (see Figure 15).

Proof. Let $Q = (u : v : w)$ with anticevian triangle $Q^aQ^bQ^c$ where $Q^a = (-u : v : w), Q^b = (u : -v : w), Q^c = (u : v : -w)$. The equations of the lines $H^aQ^a, H^bQ^b, H^cQ^c$ are

\[(a + b + c)(c(a - b)(a + b - c)v + b(c - a)(c + a - b)w)x + (a + b - c)(-a(b + c)(c + a - b)w + c(a - b)(a + b + c)u)y + (c + a - b)(b(c - a)(a + b + c)u + a(b + c)(a + b - c)v)z = 0, \]

\[(a + b - c)(c(a - b)(a + b + c)v + b(c + a)(b + c - a)w)x + (a + b + c)(a(b - c)(b + c - a)w + c(a - b)(a + b - c)u)y + (b + c - a)(-b(c + a)(a + b - c)u + a(b - c)(a + b + c)v)z = 0, \]

\[(c + a - b)(-c(a + b)(b + c - a)v + b(c - a)(a + b + c)w)x + (b + c - a)(a(b - c)(a + b - c)w + c(a + b)(c + a - b)u)y + (a + b + c)(b(c - a)(c + a - b)u + a(b - c)(b + c - a)v)z = 0. \]
Eliminating $x, y, z$, we have

$$64abc\Delta^2 \left( \sum_{\text{cyclic}} (b^2 + c^2 - a^2)v \right) \left( \sum_{\text{cyclic}} a(b^2 - c^2)vw \right) = 0.$$ 

Therefore the lines $H^aQ^a, H^bQ^b, H^cQ^c$ are concurrent if and only if $Q = (u:v:w)$ lies on the orthic axis $\mathcal{L}(H)$ or the circumconic $\mathcal{C}$.

Eliminating $u, v, w$ from equations (16), (17), (18), we have

$$64\Delta^2 (bcx + cay + abz) \left( \sum_{\text{cyclic}} a^2(b^2 - c^2)(b^2 + c^2 - a^2)yz \right) = 0.$$ 

Therefore the locus of the point of concurrency is again the union of the line $\mathcal{L}(I) : bcx + cay + abz = 0$ (the trilinear polar of the incenter) and the Jerabek hyperbola $\mathcal{J}$.

Now the circumconic $\mathcal{C}$ is the circum-hyperbola which is the isotomic conjugate of the line joining the incenter $I$ to its isotomic conjugate. Its center is the point

$$(a(b - c)^2 : b(c - a)^2 : c(a - b)^2).$$

If we choose $Q$ to be the point $\left(\frac{1}{at+bc} : \frac{1}{bt+ca} : \frac{1}{ct+ab}\right)$, then the perspector is the point

$$\left(\frac{a(b^2 + c^2 - a^2)}{at + bc} : \frac{b(c^2 + a^2 - b^2)}{bt + ca} : \frac{c(a^2 + b^2 - c^2)}{ct + ab}\right)$$

on the Jerabek hyperbola. In fact, $\mathcal{C}$ intersects $\mathcal{J}$ at $H_i = \left(\frac{a(b+c)}{b+c-a} : \ldots : \ldots\right)$, and the line joining $Q$ and $P$ passes through $H_i$.

□

References


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Semi-Similar Complete Quadrangles
Benedetto Scimemi

Abstract. Let $A = A_1A_2A_3A_4$ and $B = B_1B_2B_3B_4$ be complete quadrangles and assume that each side $A_iA_j$ is parallel to $B_iB_j$ ($i, j, h, k$ is a permutation of $1, 2, 3, 4$). Then $A$ and $B$, in general, are not homothetic; they are linked by another strong geometric relation, which we study in this paper. Our main result states that, modulo similarities, the mapping $A_i \rightarrow B_i$ is induced by an involutory affinity (an oblique reflection). $A$ and $B$ may have quite different aspects, but they share a great number of geometric features and turn out to be similar when $A$ belongs to the most popular families of quadrangles: cyclic and trapezoids.

1. Introduction

Two triangles whose sides are parallel in pairs are homothetic. A similar statement, trivially, does not apply to quadrilaterals. How about two complete quadrangles with six pairs of parallel sides? An answer cannot be given unless the question is better posed: let $A = A_1A_2A_3A_4$, $B = B_1B_2B_3B_4$ be complete quadrangles and assume that each side $A_iA_j$ is parallel to $B_iB_j$; then $A$ and $B$ are indeed homothetic. In fact, the two triangle homotheties, say $A_1A_2A_3 \rightarrow B_1B_2B_3$, and $A_4A_2A_3 \rightarrow B_4B_2B_3$, must be the same mappings, as they have the same effect on two points. There is, however, another interesting way to relate the six directions. Assume $A_iA_j$ is parallel to $B_hB_k$ ($i, j, h, k$ will always denote a permutation of $1, 2, 3, 4$). Then $A$ and $B$ in general are not homothetic. Given any $A$, here is an elementary construction (Figure 1) producing such a $B$. Let $B_1B_2$ be any segment parallel to $A_3A_4$. Let $B_3$ be the intersection of the line through $B_1$ parallel to $A_2A_4$ with the line through $B_2$ parallel to $A_1A_4$; likewise, let $B_4$ be the intersection of the line through $B_1$ parallel to $A_2A_3$ with the line through $B_2$ parallel to $A_1A_3$. We claim that the sixth side $B_2B_4$ is also parallel to $A_1A_2$. Consider, in fact, the intersections $R$ of $A_1A_3$ with $A_2A_4$, and $S$ of $B_1B_3$ with $B_2B_4$. The following pairs of triangles have parallel sides: $A_1RA_4$ and $B_2SB_3$, $A_4RA_3$ and $B_1SB_2$, $A_3RA_2$ and $B_4SB_1$. Hence, they are homothetic in pairs. Since a translation $R \rightarrow S$ does not affect our statement, we can assume, for simplicity, $R = S$. Now two pairs of sides are on the same lines: $A_1A_3$, $B_2B_4$ and $A_2A_4$, $B_1B_3$. Consider the action of the three homotheties: $\lambda$: $A_1 \rightarrow B_2$, $A_4 \rightarrow B_3$, $\mu$: $A_3 \rightarrow B_2$, $A_4 \rightarrow B_1$.

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\(^1\) In fact, five pairs of parallel sides $A_iA_j$, $B_iB_j$ suffice to make the sixth pair parallel.
\[ \nu: A_2 \rightarrow B_1, A_3 \rightarrow B_4. \] Here \( \nu \mu^{-1} \lambda: A_1 \rightarrow B_4 \) and \( \lambda \mu^{-1} \nu: A_2 \rightarrow B_3. \) But these products are the same mapping, as all factors have the same fixed point \( R. \) Thus the triangles \( A_1RA_2, B_4RB_3 \) are homothetic and \( B_3B_4, A_1A_2 \) are parallel, as we wanted.

Figure 1. Five pairs of parallel sides imply sixth pair parallel

The purpose of this paper is to examine some geometric relations and invariances connecting \( \mathcal{A} \) and \( \mathcal{B}; \) in particular, \( \mathcal{A} \) and \( \mathcal{B} \) will have the same pair of asymptotic directions, a sort of central points at infinity. Some statements, as the somehow unexpected Theorem 1, will be proved by synthetic arguments only; other statements, which involve circumscribed conics, will be proved analytically. Our main result (Theorem 6) states that, modulo similarities, such a mapping \( A_i \rightarrow B_i \) is induced by an oblique reflection. This involutory affinity depends on \( \mathcal{A} \) and can be constructed from \( \mathcal{A} \) by ruler and compass. We shall also see that, when \( \mathcal{A} \) belongs to the most popular families of quadrangles, namely cyclic quadrangles and trapezoids (including parallelograms), this mapping leaves the shape of the quadrangle unchanged. This may be a reason why this subject, to our knowledge, has not raised previous attention.

2. Notation and terminology

If \( A, B \) are points, \( AB \) will denote, according to different contexts, the segment or the line through \( A, B. \) \( |AB| \) is a length. \( \overrightarrow{AB} = \overrightarrow{C} \) means that a half-turn about \( B \) maps \( A \) onto \( C; \) equivalently, we write \( B = \frac{1}{2}(A + C). \)

If \( r, s \) are lines, \( \angle r, s \) denotes the directed angle from \( r \) to \( s, \) to be measured mod \( \pi. \) \( \angle ABC \) means \( \angle AB, BC. \) We shall use the basic properties of directed angles, as described in [3, pp.11–15], for example, \( \angle ABC = 0 \) is equivalent to \( A, B, C \) being collinear, \( \angle ABC = \angle ADC \) to \( A, B, C, D \) being on a circle. \( A - B \) is a
vector; \((A - B) \cdot (C - D)\) is a scalar product; \((A - B) \land (C - D)\) is a vector product.

If two vectors are parallel, we write \(\frac{A - B}{C - D} = r\) to mean \(A - B = r(C - D)\).

In a complete quadrangle \(\mathcal{A} = A_1A_2A_3A_4\) the order of the vertices is irrelevant. We shall always assume that three of them are not collinear, so that all the angles \(\angle A_iA_jA_k\) are defined and do not vanish. \(A_1A_j\) and \(A_hA_k\) are a pair of opposite sides of \(\mathcal{A}\); they meet at the diagonal point \(A_{ij,hk}\). The diagonal points are vertices of the diagonal triangle of \(\mathcal{A}\). A quadrangle is a trapezoid if there is a pair of parallel opposite sides; then a diagonal point is at infinity. Parallelograms have two diagonal points at infinity. \(A_{ij} = \frac{1}{2}(A_i + A_j)\) is the midpoint of the side \(A_iA_j\). A bimedian of \(\mathcal{A}\) is the line \(A_{ij}A_{hk}\) (or the segment) through the midpoints of a pair of opposite sides. A complementary triangle \(A_jA_kA_h\) is obtained from \(\mathcal{A}\) by ignoring the vertex \(A_i\). A complementary quadrilateral is obtained from \(\mathcal{A}\) by ignoring a pair of opposite sides. An area is not defined for a complete quadrangle, but its complementary triangles and quadrilaterals do have oriented areas, which are mutually related (see §4).

3. Semi-similar and semi-homothetic complete quadrangles

**Definition.** Two complete quadrangles \(\mathcal{A} = A_1A_2A_3A_4\) and \(\mathcal{B} = B_1B_2B_3B_4\) will be called directly ( inversely) semi-similar if there is a mapping \(A_i \rightarrow B_i\) such that \(\angle A_iA_jA_k = \angle B_iB_2B_3\) ( \(\angle A_iA_jA_k = \angle B_iB_kB_h\), respectively).

In particular, \(\mathcal{A}\) and \(\mathcal{B}\) will be called semi-homothetic if each side \(A_iA_j\) is parallel to \(B_hB_k\).

The property of being semi-similar is obviously symmetric but not reflexive (only special classes of quadrangles will be self-semi-similar). Let \(\mathcal{A}\) and \(\mathcal{B}\) be semi-similar. If \(\mathcal{B}\) is similar to \(\mathcal{C}\), then \(\mathcal{A}\) is semi-similar to \(\mathcal{C}\). On the other hand, if \(\mathcal{B}\) is semi-similar to \(\mathcal{D}\), then \(\mathcal{A}\) and \(\mathcal{D}\) are similar. This explains the word semi (or half). Direct and inverse also follow the usual product rules. Although the relation of semi-similarity is not explicitly defined in the literature, semi-similar quadrangles do appear in classical textbooks; for example, some statements in [3, §39] regard the following case. Given a complete quadrangle \(\mathcal{A} = A_1A_2A_3A_4\), let \(O_i\) denote the circum-center of the complementary triangle \(A_jA_kA_h\). Then \(O_iO_j\) is orthogonal to \(A_hA_k\), and this clearly implies that \(\mathcal{A}\) is directly semi-similar to the quadrangle \(o(\mathcal{A}) = O_1O_2O_3O_4\).\(^2\) It can also be proved that \(\mathcal{A}\) is inversely semi-similar to \(n(\mathcal{A}) = N_1N_2N_3N_4\), where \(N_i\) denotes the nine-point center of the triangle \(A_jA_hA_k\).

Semi-homotheties are direct semi-similarities. The construction we gave in §1 confirms that, modulo homotheties, there is a unique \(\mathcal{B}\) which is semi-homothetic to a given quadrangle \(\mathcal{A}\).

A cyclic quadrangle is semi-similar to itself. More precisely, the mapping \(A_i \rightarrow A_i\) defines an inversely semi-similar quadrangle if and only if all the angle equalities \(\angle A_iA_jA_h = \angle A_iA_kA_h\) hold, and this is equivalent for the four points \(A_i\) to lie on a circle (see §9). We shall see, however, that, if different mappings \(A_i \rightarrow A_j\)

\(^2\)We shall reconsider the mapping \(A_i \rightarrow O_i\) in the footnote at the end of §10.
are considered, there exist other families of quadrangles for which semi-similarity implies similarity (see §§10 and 12).

**Theorem 1.** If \( A \) and \( B \) are semi-homothetic quadrangles, then the bimedians of \( A \) are parallel to the sides of the diagonal triangle of \( B \).

![Figure 2. Semi-homothetic quadrangles: the bimedians of \( A \) are parallel to the sides of the diagonal triangle of \( B \)](image)

**Proof.** It is well-known ([3, §91]) that a bimedian of \( A \), say \( A_{12}A_{34} \), meets a side of the diagonal triangle in its midpoint \( N_{12,34} = \frac{1}{2} (A_{13,24} + A_{14,23}) \). Let \( C \) denote the intersection of the line through \( A_1 \) parallel to \( A_2A_4 \) with the line through \( A_2 \) parallel to \( A_1A_3 \). Then \( A_{12} = \frac{1}{2} (C + A_{13,24}) \), hence \( CA_{14,23} \) and \( A_{12}N_{12,34} \) are parallel. Now let \( B = B_1B_2B_3B_4 \) be semi-homothetic to \( A \) so that each \( A_iA_j \) is parallel to \( B_kB_k \). Then in the quadrangles \( A_1CA_2A_{14,23} \) and \( B_3B_{13,24}B_4B_{14,23} \) (see the striped areas in Figure 2), five pairs of sides are parallel, either by assumption or by construction. Hence the same holds for the sixth pair \( A_{14,23}C \), \( B_{14,23}B_{13,24} \). But \( A_{14,23}C \) is trivially parallel to \( A_{34}A_{12} \). Therefore, the side \( B_{14,23}B_{13,24} \) of the diagonal triangle of \( B \) is parallel to the bimedian \( A_{12}A_{34} \) of \( A \), as we wanted. \( \square \)

Notice that, by symmetry, the sides of the diagonal triangle of \( A \) are parallel to the bimedians of \( B \).
4. Semi-isometric quadrangles

We shall now introduce a relationship between semi-similar quadrangles which replaces isometry. This notion is based on the following

Theorem 2. Let \( A = A_1A_2A_3A_4 \) and \( B = B_1B_2B_3B_4 \) be semi-homothetic quadrangles. Then the product \( \mu = \frac{B_i - B_j}{A_h - A_k} \cdot \frac{B_h - B_k}{A_i - A_j} \) is invariant under all permutations of indices.

Proof. Notice that the factors in the definition of \( \mu \) are ratios of parallel vectors, hence scalars with their own sign. Now consider, for example, the following triangles: \( A_1A_4A_{12,34} \), \( A_2A_3A_{12,34} \), \( A_1A_3A_{12,34} \), \( A_2A_4A_{12,34} \) which are homothetic, respectively, to the triangles \( B_3B_2B_{12,34} \), \( B_4B_1B_{12,34} \), \( B_4B_2B_{12,34} \), \( B_3B_1B_{12,34} \). Each of these homotheties implies an equal ratio of parallel vectors:

\[
\begin{align*}
\frac{B_3 - B_2}{A_1 - A_4} &= \frac{B_2 - B_{12,34}}{A_4 - A_{12,34}}, \\
\frac{B_4 - B_1}{A_2 - A_3} &= \frac{B_4 - B_{12,34}}{A_3 - A_{12,34}}, \\
\frac{B_4 - B_2}{A_1 - A_3} &= \frac{B_2 - B_{12,34}}{A_3 - A_{12,34}}, \\
\frac{B_3 - B_1}{A_2 - A_4} &= \frac{B_1 - B_{12,34}}{A_4 - A_{12,34}}.
\end{align*}
\]

By multiplication one finds

\[
\mu = \frac{B_3 - B_2}{A_1 - A_4} \cdot \frac{B_4 - B_1}{A_2 - A_3} = \frac{B_2 - B_{12,34}}{A_4 - A_{12,34}} \cdot \frac{B_1 - B_{12,34}}{A_3 - A_{12,34}} = \frac{B_3 - B_1}{A_2 - A_4} \cdot \frac{B_4 - B_2}{A_1 - A_3}.
\]

Likewise, \( \mu = \frac{B_3 - B_4}{A_1 - A_2} \cdot \frac{B_1 - B_2}{A_3 - A_4} \), as we wanted. \( \square \)

Thus a pair \( A, B \) of semi-similar quadrangles defines a scale factor

\[
|\mu| = \frac{|B_1B_2||B_3B_4|}{|A_1A_2||A_3A_4|} = \frac{|B_1B_3||B_2B_4|}{|A_1A_3||A_2A_4|} = \frac{|B_1B_4||B_2B_3|}{|A_1A_4||A_2A_3|}.
\]

A geometric meaning for the sign of \( \mu \) will be seen later (§7).
**Corollary 3.** Let $\mathcal{A} = A_1A_2A_3A_4$ and $\mathcal{B} = B_1B_2B_3B_4$ be semi-similar quadrangles. Then the products of the lengths of the pairs of opposite sides are proportional: 3

\[
|A_1A_2||A_3A_4| : |A_1A_3||A_2A_4| : |A_1A_4||A_2A_3| = |B_1B_2||B_3B_4| : |B_1B_3||B_2B_4| : |B_1B_4||B_2B_3|.
\]

A sort of isometry takes place when $|\mu|=1$:

**Definition.** Two quadrangles $\mathcal{A}$ and $\mathcal{B}$ will be called semi-isometric if the they are semi-similar and the lengths of two corresponding opposite sides have the same product: $|A_iA_j||A_hA_k| = |B_iB_j||B_hB_k|$.

We have just seen that if this equality holds for a pair of opposite sides of semi-similar quadrangles, then $|\mu|=1$ and therefore the same happens to the other two pairs.

We shall now derive a number of further relations between semi-isometric quadrangles which are almost immediate consequences of the definition. Some of them are better described if referred to the three complementary quadrilaterals. The oriented areas of these quadrilaterals are given by one half of the cross products $(A_1 - A_2) \land (A_3 - A_4), (A_1 - A_4) \land (A_2 - A_3), (A_1 - A_3) \land (A_2 - A_4)$. Therefore the defining equalities $|A_iA_j||A_hA_k| = |B_iB_j||B_hB_k|$, if combined with the angle equalities $\angle A_iA_jA_hA_k = -\angle B_iB_jB_hB_k$, imply that the three areas only change sign when passing from $\mathcal{A}$ to $\mathcal{B}$. On the other hand, it is well-known that the three cross products above, if added or subtracted in the four essentially different ways, produce 4 times the oriented area of the complementary triangles $A_iA_jA_k$. For example, by applying standard properties of vector calculus, one finds

\[
(A_1 - A_2) \land (A_3 - A_4) + (A_1 - A_4) \land (A_2 - A_3) + (A_1 - A_3) \land (A_2 - A_4)
\]

\[
= 2(A_1 - A_4) \land (A_1 - A_2),
\]

\[
(A_1 - A_2) \land (A_3 - A_4) + (A_1 - A_4) \land (A_2 - A_3) - (A_1 - A_3) \land (A_2 - A_4)
\]

\[
= 2(A_3 - A_4) \land (A_2 - A_3),
\]

etc. Therefore we have

**Theorem 4.** If $\mathcal{A}$ and $\mathcal{B}$ are semi-isometric quadrangles, the four pairs of corresponding complementary triangles $A_iA_jA_k$ and $B_iB_jB_k$ have the same (absolute) areas.

This insures, incidentally, that, when semi-similarity implies similarity, then semi-isometry implies isometry.

Other invariants can be written in terms of perimeters: if one first adds, then subtracts the lengths of the four contiguous sides in a complementary quadrilateral,

\[3\text{The following example shows that the inverse statement does not hold: let } A_1A_2 \text{ be the diameter of a circle; then, for any choice of a chord } A_3A_4 \text{ orthogonal to } A_1A_2, \text{ the three products above are proportional to } 2 : 1 : 1.\]
then the product is invariant; for example, the product

\[(|A_1 A_2| + |A_2 A_3| + |A_3 A_4| + |A_4 A_1|) (|A_1 A_2| - |A_2 A_3| + |A_3 A_4| - |A_4 A_1|)\]

is the same if all \(A_i\) are changed into \(B_i\). In particular,

\[|A_1 A_2| - |A_2 A_3| + |A_3 A_4| - |A_4 A_1| = 0\]

if and only if

\[|B_1 B_2| - |B_2 B_3| + |B_3 B_4| - |B_4 B_1| = 0.\]

Since these equalities are well-known to be equivalent to the fact that two pairs of opposite sides are tangent to a same circle, we conclude that semi-similarity of complete quadrangles preserves \textit{inscribability} for a complementary quadrilateral. Another invariance takes place if we subtract the squares of the lengths of two bimedians; for example,

\[|A_{12} A_{34}|^2 - |A_{14} A_{23}|^2 = |B_{12} B_{34}|^2 - |B_{14} B_{23}|^2.\]

These and other similar equalities can be derived by applying the classical formulas for the area of a quadrilateral (Bretschneider’s formula etc., see [5]).

Something more intriguing happens if one considers the circumcircles of the complementary triangles.

\textbf{Theorem 5.} If two quadrangles are semi-similar, the circumradii of corresponding complementary triangles are inversely proportional.

\textit{Proof.} Let \(R_i\) and \(S_i\) be, respectively, the circumradii of \(A_j A_k A_l A_m\) and \(B_j B_k B_l B_m\). We claim that \(R_1, R_2, R_3, R_4\) are inversely proportional to \(S_1, S_2, S_3, S_4\). In fact, by the law of sines, we can write, for example, the product \(|A_3 A_4||B_3 B_4|\) in two ways:

\[(2R_1 \sin A_3 A_2 A_4)(2S_1 \sin B_3 B_2 B_4) = (2R_2 \sin A_4 A_1 A_3)(2S_2 \sin B_2 B_1 B_3).\]

Since \(\angle A_3 A_2 A_4 = \angle B_4 B_1 B_3\) and \(\angle A_4 A_1 A_3 = \angle B_3 B_2 B_1\), all the sines can be canceled to conclude \(R_1 S_1 = R_2 S_2\). Thus the product \(R_i S_i\) is the same for all indices \(i\). \(\square\)

\textbf{5. The principal reference of a quadrangle}

The invariants we met in the last section suggest the possible presence of an affinity. In fact, our main result (Theorem 6) will state that for each quadrangle \(A\) there exists an involutory affinity (depending on \(A\)) that maps \(A\) into a semi-homothetic, semi-isometric quadrangle \(B\). It will then appear that any semi-similarity of quadrangles \(A \to C\) is a product of an oblique reflection \(A \to B\) by a similarity \(B \to C\). Since an oblique reflection is determined, modulo translations, by a pair of directions, in order to identify which reflection properly works for \(A\), we shall describe in the next section how to associate to a quadrangle \(A\) a pair of characteristic directions, that we shall call the \textit{asymptotic directions} of \(A\). As we
shall see, these directions also play a basic role in connection with some conics that are canonically defined by four points.\(^4\)

It is well-known that a quadrangle \(\mathcal{A} = A_1A_2A_3A_4\) has a unique circumscribed rectangular hyperbola \(\Psi = \Psi_A\). The center of \(\Psi\) is a central (synonym: notable) point of \(\mathcal{A}\) that we denote by \(H = H_A\). Several properties and various geometric constructions of \(H\) from the points \(A_i\) are described in [4] (but this point is defined also in [3, §396-8], and [2, Problem 46]). For example, \(H\) is the intersection of the nine-point circles of the four complementary triangles \(A_jA_k\). The directions of the asymptotes of \(\Psi = \Psi_A\) will be called the principal directions of \(\mathcal{A}\). The hyperbola \(\Psi\) essentially defines what we call the principal reference of \(\mathcal{A}\), namely an orthogonal Cartesian \(xy\)-frame such that the equation for \(\Psi\) is \(xy = 1\), the ambiguity between \(x\) and \(y\) not creating substantial difficulties (Figures 3 and 4). Our next proofs will be based on this reference (an approach which was first used by Wood in [6]). The principal reference is not defined when two opposite sides of \(\mathcal{A}\) are perpendicular. This class of quadrangles (orthogonal quadrangles) requires a different analytic treatment and will be discussed separately in §11.

Figure 3. A concave quadrangle and its principal reference. Asymptotic directions derived from the medial ellipse \(\Gamma\)

Within the principal reference of \(\mathcal{A}\), the origin is the central point \(H_A = [0,0]\) and the vertices will be denoted by \(A_i = [a_i, \frac{1}{a_i}],\ i = 1, 2, 3, 4\). In view of the next

\(^4\)These directions may be thought of as a pair of central points of \(\mathcal{A}\) at infinity. In this respect, the present paper may be considered a complement of [4]. Our treatment, however, will be self-contained, not requiring the knowledge of [4].

\(^5\)The only exceptions will be considered in §12.
calculations, it is convenient to introduce the elementary symmetric polynomials:

\[ s_1 = a_1 + a_2 + a_3 + a_4, \]
\[ s_2 = a_1 a_2 + a_1 a_3 + a_1 a_4 + a_2 a_3 + a_2 a_4 + a_3 a_4, \]
\[ s_3 = a_1 a_2 a_3 + a_1 a_2 a_4 + a_1 a_3 a_4 + a_2 a_3 a_4, \]
\[ s_4 = a_1 a_2 a_3 a_4. \]

Notice that the restriction for \( A \) not to be orthogonal not only implies \( s_4 \neq 0 \) but also excludes \( s_4 = -1 \), as the scalar product of two opposite sides turns out to be

\[ A_i A_j \cdot A_h A_k = (a_i - a_j)(a_h - a_k) \left( 1 + \frac{1}{s_4} \right). \]

The sign of \( s_4 = a_1 a_2 a_3 a_4 \) has the following relevant geometric meaning: \( s_4 \) is positive if and only if the number of vertices \( A_i \) which lie on a branch of \( \Psi \) is even: 4, 2 or 0. By applying standard arguments to the real convex function \( f(x) = \frac{1}{x} \), this condition is found to be equivalent to \( A \) being convex. On the other hand, if the branches of \( \Psi \) contain 1 and 3 vertices, then \( s_4 < 0 \) and \( A \) is concave, namely, there is a vertex \( A_i \) which lies inside the complementary triangle \( A_j A_h A_k \).

6. Central conics and asymptotic directions

It is well-known that, within the family of the conic sections circumscribed to a given quadrangle \( A \), the locus of the centers is itself a conic \( \Gamma = \Gamma_A \). We call it the
medial or the nine-points conic of $\mathcal{A}$, as $\Gamma$ contains the six midpoints $A_{ij}$ and the three diagonal points $A_{ij,hk}$ ([1, §16.7.5]). The equation for $\Gamma$ is calculated to be

$$x^2 - s_4 y^2 - \frac{1}{2} s_1 x + \frac{1}{2} s_3 y = 0,$$

or

$$(x - x_G)^2 - s_4 (y - y_G)^2 = \frac{1}{16 s_4} (s_4 s_1^2 - s_3^2),$$

which confirms that $H = [0, 0]$, the center of $\Psi$, lies on $\Gamma$. On the other hand, the center of $\Gamma$ is

$$G = G_A = \frac{1}{4} \left[ s_1, \frac{s_3}{s_4} \right] = \frac{1}{4} (A_1 + A_2 + A_3 + A_4).$$

This is clearly the centroid or center of gravity, another central point of $\mathcal{A}$. Two cases must be now distinguished:

(i) $\mathcal{A}$ is convex: $s_4 > 0$ and $\Gamma$ is a hyperbola. By definition, the points at infinity of $\Gamma$ will be called the asymptotic directions of $\mathcal{A}$; it appears from the equation that the slopes of the asymptotes are $\pm \frac{1}{\sqrt{s_4}}$ (Figure 5). This proves that the principal directions bisect the asymptotic directions. In particular, when $s_4 = 1$, $\Gamma$ is a rectangular hyperbola; as we shall soon see, this happens if and only if $\mathcal{A}$ is cyclic.

(ii) $\mathcal{A}$ is concave: $s_4 < 0$ and $\Gamma$ is an ellipse. The asymptotic directions of $\mathcal{A}$ will be defined by connecting contiguous vertices of the ellipse $\Gamma$. Equivalently, we can inscribe $\Gamma$ in a minimal rectangle and consider the directions of its diagonals.  

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They can also be defined as the directions of the only pair of conjugate diameters of the ellipse $\Gamma_A$ which have equal lengths; see [2, Problem 54].
Their slopes turn out to be \( \pm \frac{1}{\sqrt{-s_4}} \) (Figure 3). Again, the principal directions bisect the asymptotic directions. Notice that \( \Gamma \) cannot be a parabola, as \( s_4 \neq 0 \). We have also seen in §5 that \( s_4 \neq -1 \), so that \( \Gamma \) cannot even be a circle.\(^7\)

Next we want to introduce a new central point \( J = J_A \), defined as the reflection of \( H \) in \( G \):

\[
J = J_A = \frac{1}{2} \left[ s_1, \frac{s_3}{s_4} \right].
\]

Since \( G \) is the center of \( \Gamma \) and \( H \) lies on \( \Gamma \), the point \( J \) also lies on \( \Gamma \). Therefore, there exists a conic \( \Theta = \Theta_A \) circumscribed to \( A \) and centered at \( J \). The equation for \( \Theta \) is found to be

\[
x^2 + s_4 y^2 - s_1 x - s_3 y + s_2 = 0
\]
or

\[
(x - x_J)^2 + s_4 (y - y_J)^2 = \frac{1}{4s_4} (s_4 s_1^2 + s_3^2) - s_2.
\]

Looking at the roles of \( s_4 \) and \( -s_4 \) in the equations for \( \Gamma \) and \( \Theta \), it appears that \( \Gamma \) is an ellipse when \( \Theta \) is a hyperbola and conversely.\(^8\)

Moreover, the hyperbola asymptotes are parallel to the ellipse diagonals. We have thus produced two alternative ways for defining the asymptotic directions of any quadrangle \( A \):

- if \( A \) is concave, by the asymptotes of \( \Theta \) or by the vertices of \( \Gamma \) (Figure 3);
- if \( A \) is convex, by the asymptotes of \( \Gamma \) or by the vertices of \( \Theta \) (Figure 5).

A third equivalent definition only applies to the latter case: it is well-known (see, for example, [2, Problem 45]) that a convex quadrangle has two circumscribed parabolas, say \( \Pi^+ \) and \( \Pi^- \). Their equations are\(^9\)

\[
\Pi^+ : \quad (x + \sqrt{s_4}y)^2 - s_1 x - s_3 y + s_2 - 2\sqrt{s_4} = 0,
\]

\[
\Pi^- : \quad (x - \sqrt{s_4}y)^2 - s_1 x - s_3 y + s_2 + 2\sqrt{s_4} = 0.
\]

Therefore the asymptotic directions of a convex quadrangle may be also defined by the axes of symmetry of the two circumscribed parabolas.

If \( s_4 = 1 \) then \( \Theta \) is a circle and \( A \) is cyclic. In this case the diagonals of \( \Theta \) are not defined; but \( \Gamma, \Pi^+, \Pi^- \) define the asymptotic directions, which are just the bisectors \( y = \pm x \) of the principal directions.\(^{10}\)

---

\(^7\)See §10 for exceptions.

\(^8\)When \( A \) is convex, \( \Theta_A \) turns out to be the ellipse that deviates least from a circle among all the ellipses circumscribed to \( A \), this meaning that the ratio between the major and the minor axis attains its minimum value. This problem was studied by J. Steiner. One can also prove that each ellipse circumscribed to \( A \) has a pair of conjugate diameters that have the asymptotic directions of \( A \); see [2, Problem 45].

\(^9\)These equations are easily derived from the equation of the generic circumscribed conic, which can be written as \( \lambda \Psi + \mu \Theta = 0 \).

\(^{10}\)For a different definition, not involving conics, see §9.
Notice that, whatever choice one makes among the definitions above, there exist classical methods which produce by straight-edge and compass the asymptotic directions of a quadrangle, starting from its vertices\(^{11}\).

### 7. Oblique reflections

We are now ready to introduce oblique reflections. We recall this notion by introducing the following

**Definition.** Given an ordered pair \((r, s)\) of non parallel lines, an \((r, s)\)-reflection is the plane transformation \(\phi : P \to P'\), such that \(P - P'\) is parallel to \(r\) and the midpoint \(\frac{1}{2}(P + P')\) lies on \(s\).

An \((r, s)\)-reflection is an involutory affine transformation. Among the well-known properties of affinities, we shall use the fact that they map lines into lines, midpoints into midpoints, conics into conics. Like in a standard reflection (a particular case, when \(r, s\) are orthogonal) the line \(s\) is the locus of fixed points, the other fixed lines being parallel to \(r\). Replacing \(r\) with a parallel line \(r'\) does not affect \(\phi\); replacing \(s\) with a parallel line \(s'\) only affects \(P'\) by the translation \(s \to s'\) parallel to \(r\). Interchanging \(r\) with \(s\) amounts to letting \(P'\) undergo a half turn around the intersection of \(r\) and \(s\). An \(rs\)-reflection \(\phi\) preserves many features of quadrangles; for example, if \(\phi(A_1) = B_1\), then the diagonal triangle of \(B = B_1B_2B_3B_4\) is the \(\phi\)-image of the diagonal triangle of \(A = A_1A_2A_3A_4\). Other corresponding elements are the bi-median lines, the centroid, the medial conic, the circumscribed parabolas. As for analytic representations, if, for example, \(r : y = rx + p, s : y = sx + q\), then \(\phi\) is the bilinear mapping

\[
[x, y] \to \frac{1}{(r - s)}[(r + s)x - 2y, 2rsx - (r + s)y] + [x_0, y_0]
\]

where \([x_0, y_0]\) is the image of \([0, 0]\). The transformation matrix of \(\phi\) has determinant

\[
\frac{1}{(r - s)^2}(- (r + s)^2 + 4rs) = -1.
\]

Therefore all oriented areas undergo a change of sign.

**Theorem 6.** Let \(A\) be a complete quadrangle. Let \(\phi\) be an \((r, s)\)-reflection, where \(r, s\) are parallel to the asymptotic directions of \(A\). Let \(\chi\) be an (orthogonal) reflection in a line \(p\) parallel to a principal direction of \(A\). Define a mapping \(\psi\) as follows:

\[
\psi = \begin{cases} 
\phi, & \text{if } A \text{ is convex,} \\
\phi\chi, & \text{if } A \text{ is concave.}
\end{cases}
\]

Then \(A\) and \(B = \psi(A)\) are semi-homothetic, semi-isometric quadrangles.

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\(^{11}\)For example, a celebrated page of Newton describes how to construct the axes of a parabola if four of its points are given.
Proof. Notice that $B$ is uniquely defined by $A$, modulo translations and midturns: in fact, a translation parallel to $r$ takes place when $s$ is translated; a midturn takes place if the principal directions or the lines $r$ and $s$ are interchanged.

Along the proof we can assume, without loss of generality, that both $s$ and $p$ pass through $H_A = [0,0]$, so that $\psi(H_A) = H_A$.

First case: $A$ is convex ($s_4 > 0$). Then the lines $r, s$ have equations, say $r : y = -\frac{x}{\sqrt{s_4}}$ and $s : y = \frac{x}{\sqrt{s_4}}$. The $rs$-reflection maps the point $P[x, y]$ into $\phi(P) = [y\sqrt{s_4}, x\sqrt{s_4}]$, so that the vertex $A_i = [a_i, \frac{1}{a_i}]$ is mapped into $B_i = \phi(A_i) = [\sqrt{s_4} a_i, \frac{a_i}{\sqrt{s_4}}]$. Substituting in $xy = 1$ proves that $B_i$ lies on $\Psi_A$. Since a quadrangle has a unique circumscribed rectangular hyperbola, we have $\Psi_B = \Psi_B(B) = \Psi_A$. In particular, $A$ and $B$ have the same principal directions and $H_B = \phi(H_A) = H_{\phi(A)} = H_A$. Now consider the sides $B_i - B_j = [\sqrt{s_4} (\frac{1}{a_i} - \frac{1}{a_j}), \frac{a_i - a_j}{\sqrt{s_4}}]$ and $A_h - A_k = [a_h - a_k, \frac{1}{a_h} - \frac{1}{a_k}]$. If we take into account that $\frac{\sqrt{s_4}}{a_i a_j} = \frac{a_i a_j}{\sqrt{s_4}}$, we find that these vectors are parallel, their ratio being $\frac{B_i - B_j}{A_h - A_k} = \frac{-\sqrt{s_4} (a_i - a_j)}{(a_h - a_k) a_i a_j}$.

This proves that $A$ and $B$ are semi-homothetic (Figure 6). Moreover, the following scalar products turn out to be the same $(A_i - A_j). (A_h - A_k) = (a_i - a_j)(a_h - a_k)(1 + \frac{1}{s_4}) = (B_i - B_j). (B_h - B_k)$. Thus $A$ and $B$ are also semi-isometric: $|A_i A_j| A_h A_k| = |B_i B_j| B_h B_k|$, as we wanted. Since the matrix for $\phi = \psi$ has determinant 1, the oriented areas of the corresponding complementary triangles and quadrilaterals, as expected, undergo a sign change.

Second case: $A$ is concave ($s_4 < 0$). The argument is similar: let $r : y = -\frac{x}{\sqrt{-s_4}}$ and $s : y = \frac{x}{\sqrt{-s_4}}$. Then the reflection $\chi$, for example in the $x$-axes, takes $[x, y]$ into $[x, -y]$ and $\psi : [x, y] \rightarrow [y, \sqrt{-s_4}, \frac{x}{\sqrt{-s_4}}]$. Thus $B_i = \psi(A_i) = [-\sqrt{-s_4} a_i, \frac{a_i}{\sqrt{-s_4}}]$. If we take into account that $\frac{\sqrt{-s_4}}{a_i a_j} = \frac{a_i a_j}{\sqrt{-s_4}}$, we find that the vectors $B_i - B_j, A_h - A_k$ are parallel. For their ratio we find the equality $\frac{B_i - B_j}{A_h - A_k} = \frac{-\sqrt{-s_4} (a_i - a_j)}{(a_h - a_k) a_i a_j} = \frac{A_i - A_j}{B_h - B_k}$. The matrix for $\psi$ has now determinant 1 so that the oriented areas are conserved.

Incidentally, the foregoing argument also shows that in the semi-homothety of Theorem 2 in §5 the sign of the scalar $\mu = \frac{B_i - B_j}{A_h - A_k}, A_i - A_j$ is respectively 1 or -1 for convex and concave quadrangles. This proves that semisimilarities preserve convexity.

Since the mapping of Theorem 6 is involutory and the principal references for $A$ and $B$ have been shown to be the same, we have substantially proved that

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12The only exceptions will be considered in §12
Figure 6. An oblique reflection producing semi-homothetic semi-isometric quadrangles. Pairs of corresponding sides, central lines, central conics etc meet on line $s$.

**Theorem 7.** Two semi-homothetic quadrangles have the same asymptotic directions.

This statement will be confirmed in the next section.

**8. Behaviour of central conics**

We want now to examine how the central conics $\Psi, \Gamma, \Theta$ (see §4) of two semi-similar quadrangles are related to each other. We claim that, modulo similarities, these conics are either identical ellipses or conjugate hyperbolas.\(^{13}\)

The problem can obviously be reduced to a pair of semi-homothetic, semi-isometric quadrangles.

**Theorem 8.** Let $A$ and $B$ be semi-homothetic, semi-isometric quadrangles.

1. Assume $\Psi_A$ and $\Psi_B$ have the same center: $H_A = H_B$. Then either $\Psi_A = \Psi_B$ ($A$ convex) or $\Psi_A$ and $\Psi_B$ are conjugate ($A$ concave).
2. Assume $\Gamma_A$ and $\Gamma_B$ have the same center: $G_A = G_B$. Then either $\Gamma_A = \Gamma_B$ ($A$ concave) or $\Gamma_A$ and $\Gamma_B$ are conjugate ($A$ convex). In the latter case, the two circumscribed parabolas $\Pi_A \pm \Pi_B$ are either equal or symmetric with respect to $G$.

\(^{13}\)Two hyperbolas are said to be conjugate if their equations, in a convenient orthogonal frame, can be written as $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$. Conjugate hyperbolas have the same asymptotes and their foci form a square.
(3) Assume $\Theta_A$ and $\Theta_B$ have the same center: $J_A = J_B$. Then either $\Theta_A = \Theta_B$ (A convex) or $\Theta_A$ and $\Theta_B$ are conjugate (A concave).

**Proof.** Without loss of generality, we can assume that $\mathcal{A}$ and $\mathcal{B}$ are linked by a mapping $\psi$ as in Theorem 6. According to the various statements (1), (2), (3), it will be convenient to choose $\psi$ in such a way that a specific point $F$ is fixed. We shall denote by $\psi_F$ this particular mapping: $\psi_F(F) = F$. For example, in the proof of Theorem 6 we had $\psi = \psi_H$. To obtain $\psi_F$ from $\psi_H$ one may just apply an additional translation $H \to F$.

(1) First assume that $\mathcal{A}$ is convex: $s_4 > 0$. While proving Theorem 6 we have already noticed that the point $B_i = \psi_H(A_i) = \psi_H([a_i, \frac{1}{a_i}]) = \left[\frac{\sqrt{s_4}}{a_i}, \frac{a_i}{\sqrt{s_4}}\right]$ lies on $xy = 1$, hence $H_A = H_B, \Psi_A = \Psi_B$. Now assume $\mathcal{A}$ concave: $s_4 < 0$. A *principal* reflection, say in the $x$-axes, takes $[a_i, \frac{1}{a_i}]$ into $[a_i, -\frac{1}{a_i}]$. Then $B_i = \psi_H(A_i) = \psi_H([a_i, -\frac{1}{a_i}]) = [\frac{\sqrt{s_4}}{a_i}, \frac{a_i}{\sqrt{s_4}}]$, clearly a point of the hyperbola $xy = -1$, the conjugate of $\Psi_A$, as we wanted.

(2) Since affinities preserve midpoints and conics, for any choice of $\psi$ we have $\Gamma_B = \psi(\Gamma_A)$, and $G_B = \psi(G_A)$. The assumption $G_B = G_A$ suggests the choice $\psi = \psi_G$ in Theorem 6. Assume $\mathcal{A}$ is convex: $s_4 > 0$. Then $B_i = \psi_G(A_i)$ is obtained by applying the translation $\psi_H(G_A) \to G_A$ to the point $\psi_H([a_i, \frac{1}{a_i}]) = \left[\frac{\sqrt{s_4}}{a_i}, \frac{a_i}{\sqrt{s_4}}\right]$. Here $G_A = \left[\frac{1}{4}, \frac{a_i^2}{s_{34}}, s_3, s_1\right]$. Therefore

$$B_i = \frac{\sqrt{s_4}}{a_i} + \frac{s_1}{4} - \frac{s_3}{4\sqrt{s_4}}, \frac{a_i}{\sqrt{s_4}} + \frac{s_3}{4s_4} - \frac{s_1}{4\sqrt{s_4}}.$$ 

Straightforward calculations prove that the midpoints $\frac{1}{2}(B_i + B_j)$ satisfy the equation

$$(x - x_G)^2 - s_4(y - y_G)^2 = -\frac{1}{16s_4}(s_3^2 + s_4s_7^2),$$

which is the conjugate hyperbola of $\Gamma_A$, as we wanted.

As for the circumscribed parabolas, we already know, again by the general properties of affinities, that $\psi(\Pi_{+A})$ and $\psi(\Pi_{-A})$ will be parabolas circumscribed to $\mathcal{B}$. More precisely, one can verify that the above points $B_i = \psi_H(A_i) + G_A - \psi_H(G_A)$ satisfy the equation for

$$\Pi_{+A} : \quad (x - y\sqrt{s_4})^2 - s_1x - s_3y + s_2 + 2\sqrt{s_4} = 0.$$ 

A midturn around $G_A$ maps $B_i$ into $2G_A - B_i$. The new points are $-\psi_H(A_i) + G_A + \psi_H(G_A)$ and they are checked to satisfy the equation

$$(x + y\sqrt{s_4})^2 - s_1x - s_3y + s_2 - 2\sqrt{s_4} = 0.$$ 

Therefore $\Pi_{-B} = \Pi_{-A}, \Pi_{+B} = (\Pi_{+A})^G$.

Now assume $\mathcal{A}$ concave: $s_4 < 0$. As before, $\psi_H(A_i) = \left[-\frac{\sqrt{s_4}}{a_i}, \frac{a_i}{\sqrt{s_4}}\right]$. The mapping $\psi_G$ is again obtained by applying the translation $\psi_H(G_A) \to G_A$, but here $\psi_H(G_A) = \left[\frac{1}{4}, \frac{a_i^2}{s_{34}}, s_3, s_1\right]$. Therefore,

$$B_i = \psi_G(A_i) = \left[-\frac{\sqrt{s_4}}{a_i} + \frac{s_1}{4} - \frac{s_3}{4\sqrt{s_4}}, \frac{a_i}{\sqrt{s_4}} + \frac{s_3}{4s_4} - \frac{s_1}{4\sqrt{s_4}}\right].$$
By direct calculation, one checks that the midpoints \( \frac{1}{2}(B_i + B_j) \) lie on
\[
\Gamma_A : (x - x_G)^2 - s_4(y - y_G)^2 = \frac{1}{16}(s_3s_1^2 - s_3s_2^2).
\]
Thus the medial ellipses are the same: \( \Gamma_B = \psi_G(\Gamma_A) = \Gamma_A \), as we wanted.

(3) The proof is as above, except that we want \( \psi = \psi_J \) and the translation is \( \psi_H(J_A) \to J_A \). When \( \mathcal{A} \) is convex, one finds that the points
\[
B_i = \left( \frac{\sqrt{s_4}}{a_i} + \frac{s_1}{2} - \frac{s_3}{2\sqrt{s_4}}, \frac{a_i}{\sqrt{s_4}} + \frac{s_3}{2s_4} - \frac{s_1}{2\sqrt{s_4}} \right)
\]
lie on
\[
\Theta_A : (x - x_J)^2 + s_4(y - y_J)^2 = \frac{1}{4s_4}(s_3^2 + s_4s_1^2) - s_2.
\]
Hence \( \Theta_B = \psi_J(\Theta_A) = \Theta_A \).

When \( \mathcal{A} \) is concave, similar arguments lead to the points
\[
B_i = \psi_G(A_i) = \left( -\frac{\sqrt{s_4}}{a_i} + \frac{s_1}{2} - \frac{s_3}{2\sqrt{s_4}}, \frac{a_i}{\sqrt{s_4}} + \frac{s_3}{2s_4} + \frac{s_1}{2\sqrt{s_4}} \right)
\]
which satisfy the equation
\[
(x - x_J)^2 + s_4(y - y_J)^2 = -\frac{1}{4s_4}(s_3^2 + s_4s_1^2) + s_2,
\]
the conjugate hyperbola of \( \Theta_A \). This completes the proof. \( \square \)

**Proof of Theorem 7.** By applying proper homotheties, we can reduce the proof to the case that \( \mathcal{A} \) and \( \mathcal{B} \) are semi-isometric; by further translations, we can even assume that \( \psi : \mathcal{A} \to \mathcal{B} \) as in Theorem 6 and the conics centers are the same. Then, according to Theorem 8, the circumscribed conics \( \Gamma \) and \( \Theta \) are either equal or conjugate. In any case \( \mathcal{A} \) and \( \mathcal{B} \) have the same asymptotic directions.

## 9. A special case: cyclic quadrangles

A cyclic (or circumscriptible) quadrangle \( \mathcal{A} \) is convex and corresponds to \( s_4 = 1 \). In this case \( \Theta_A \) is the circumcircle of equation \( x^2 + y^2 - s_1x - s_3y + s_2 = 0 \). The center of \( \Theta_A \) is \( J = J_A \) and its radius is \( \rho = \frac{1}{2}\sqrt{s_1^2 + s_3^2 - 4s_2} \). Incidentally, since \( \rho^2 = |JH|^2 - s_2 \), we have discovered for \( s_2 \) a geometric interpretation, namely the power of \( H \) with respect to the circumcircle \( \Theta \). The medial conic \( \Gamma \) is the rectangular hyperbola:
\[
x^2 - y^2 - \frac{1}{2}s_1x + \frac{1}{2}s_3y = 0
\]
and the circumscribed parabolas are
\[
(x \pm y)^2 - s_1x - s_3y + s_2 = \pm 2.
\]
Therefore the lines \( r \) and \( s \) defining \( \psi (= \phi) \) in Theorem 6 are perpendicular with slope \( \pm 1 \) and \( \psi \) is just the orthogonal reflection in the line \( s \). For the asymptotic directions of cyclic quadrangles we have a simple geometric interpretation at finite, not involving conics: they merely bisect the angle formed by any pair of opposite
Semi-similar complete quadrangles

sides of $A$. In fact, the reflection $\psi$ maps the line $A_iA_j$ into the line $B_iB_j$, which is parallel to $A_hA_k$, because of the semi-homothety. We may also think of the asymptotic directions as the mean values of the directions of the radii $JA_i$, because $\psi_J$ maps the perpendicular bisectors of $A_iA_j$ into the perpendicular bisector of $A_hA_k$, namely a bisector of $\angle A_iJA_j$ into a bisector of $\angle A_hJA_k$.

For cyclic quadrangles, Theorem 1 states that the oriented angles formed by the sides of the diagonal triangle of $A$ are just the opposite (as a result of the reflection $\psi$) of those formed by the bimedians. Using the symbols of Theorem 1 this can be written as $\angle N_{14,23}GN_{12,34} = \angle N_{14,23}G, GN_{12,34} = -\angle A_{12,34}A_{13,24}, A_{13,24}A_{14,23}$. Since a triangle and its medial are obviously homothetic, we have $\angle N_{14,23}GN_{12,34} = \angle N_{14,23}N_{13,24}N_{12,34}$. Hence the four points $N_{ij,hk}$ and $G$ lie on a circle. This suggests the following statement, that we have been unable to find in the literature:

**Corollary 9.** A quadrangle is cyclic if and only if its centroid lies on the nine-point circle of its diagonal triangle.

![Figure 7. The centroid of a cyclic quadrangle lies on the nine-point circle of its diagonal triangle](image)

**Proof.** (Figure 7) It is well-known (see [2, Problem 46], or [1, §17.5.4]) that a conic circumscribed to a triangle $D$ is a rectangular hyperbola if and only if its center lies on the ninepoint circle of $D$. Let $A$ be a quadrangle whose centroid $G$ lies on the nine-point circle of its diagonal triangle $D$. Since the medial conic
Γ_A is circumscribed to D and its center is the centroid G_A, we know that Γ_A is a rectangular hyperbola. Then, by previous theorems, we have s_4 = 1, Θ_A is a circle and A is cyclic. The converse argument is similar. 

10. Another special case: trapezoids

Trapezoids form another popular family of convex quadrangles, corresponding to the case s_4 = \( \frac{s_2}{s_1} \). In fact, two opposite sides, say A_1A_2, A_3A_4 are parallel if and only if \( \frac{a_1 - a_2}{a_1 - a_2} = \frac{a_3 - a_4}{a_3 - a_4} \), hence \( a_1a_2 = a_3a_4 \); and a straightforward calculation gives \( (a_1a_2 - a_3a_4)(a_1a_3 - a_2a_4)(a_1a_4 - a_2a_3) = s_4s_1^2 - s_2^2 \).

For trapezoids the medial conic \( \Gamma \) degenerates into two lines: \( (s_1x + s_3y - \frac{1}{2}s_1^2)(s_1x - s_3y) = 0 \) which are bimedians for A; their slopes are simply \( \pm \frac{s_1}{s_3} \): the asymptotic direction \( \frac{s_1}{s_3} \) is shared by the parallel sides of A; the other is the direction of the line \( s_1x - s_3y = 0 \), on which one finds H, G, J, plus the two diagonal points at finite A_{ih,jk}, A_{ik,jh}. \( \Pi_+ \) also degenerates into the pair of parallel opposite sides. If two of the differences \( a_i a_j - a_h a_k \) vanish, then A is a parallelogram, \( H = J \) is its center and the asymptotic directions are parallel to the sides. For a cyclic trapezoid we have the additional condition \( s_2^2 = s_3^2 \) and \( HG \) is a symmetry line for A. As for the oblique reflection \( \psi \) of Theorem 6, if, for example, the parallel sides are A_1A_2 and A_3A_4, then the lines r and s are the bimedians A_{12}A_{34}, A_{14}A_{23} and the oblique reflection interchanges two pairs of vertices: \( \psi : A_1 \to A_2, A_2 \to A_1, A_3 \to A_4, A_4 \to A_3 \). Thus A and B = \( \psi(A) \) are the same quadrangle, but \( \psi \) is not the identity!

The fact that for both cyclic quadrangles and trapezoids semi-similarities leave the quadrangle shape invariant may perhaps explain why the relation of semi-similarity, to our knowledge, has not been studied.

11. Orthogonal quadrangles

We still have to consider the family of quadrangles which have a pair of orthogonal opposite sides, because in this case the foregoing analytical geometry does not work. We call these quadrangles orthogonal. We shall first assume that the other pairs of sides are not perpendicular, leaving still out the subfamily of the so-called orthocentric quadrangles, which will be considered as very last. For orthogonal (non orthocentric) quadrangles, the statements of the Theorems of §8.3 to 10 remain exactly the same, but a principal reference cannot be defined as before and different analytic proofs must be provided. First notice that for this family the hyperbola

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14 Going back to the semi-similar quadrangles A and o(A) = O_1O_2O_3O_4 which we mentioned in the introduction and appear in Johnson’s textbook [3], it follows from our previous arguments that \( A_i \to O_i \) is induced by an affine transformation which can be thought of as the product of four factors: a rotation of a straight angle, an oblique reflection, a homothety (the three of them fixing H) and a final translation \( H \to J \). It can be proved that \( A_i \to O_i \) is also induced, modulo an isometry, by a circle inversion centered at the so-called isoptic point of A, see [6, 4].
\( \Psi \) degenerates into a pair of orthogonal lines. We can represent these lines by the equation \( xy = 0 \) (replacing \( xy = 1 \)) and use them as \( xy \)-axes of a new principal reference (the unit length is arbitrary). Within this frame we can assume without loss of generality \( A_1 = [x_1, 0], A_2 = [0, y_2], A_3 = [x_3, 0], A_4 = [0, y_4] \). Notice that the product \( x_1y_2x_3y_4 \) cannot vanish, as we have excluded quadrangles with three collinear vertices. The role of the elementary symmetric polynomials can be played here by other polynomials, as \( s_x = x_1 + x_3, s_y = y_2 + y_4, p_x = x_1x_3, p_y = y_2y_4 \). We have \( H = [0, 0], G = \frac{1}{4}[s_x, s_y], J = \frac{1}{2}[s_x, s_y] \). One of the diagonal points is \( H \); the remaining two are 
\[
\frac{1}{x_1y_4 - x_3y_2} [x_1x_3(y_2 - y_4), y_2y_4(x_1 - x_3)]
\]
and 
\[
\frac{1}{x_1y_2 - x_3y_4} [-x_1x_3(y_2 - y_4), y_2y_4(x_1 - x_3)].
\]
This shows that the \( xy \)-axes bisect an angle of the diagonal triangle. The fraction \( \frac{p_x}{p_y} \) (or the product \( p_xp_y \)) plays the role of \( s_A \). More precisely: convexity and concavity are represented by \( p_xp_y > 0 \) or \( p_xp_y < 0 \) respectively (\( p_xp_y = 0 \) has been already excluded); \( A \) is cyclic if and only if \( p_x = p_y \); \( A \) is a non-cyclic trapezoid when \( p_xp_y = p_y^2s_2 \). Similar conditions can be established for \( s_x, s_y, p_x, p_y \) to characterize the various families of quadrangles (skites, diamonds, squares). The equations for the central conics are
\[
\Gamma: \quad p_yx^2 - p_xy^2 - \frac{1}{2}p_ys_xx + \frac{1}{2}p_xp_yy = 0,
\]
and
\[
\Theta: \quad p_yx^2 + p_xy^2 - p_yxxy - p_xyy = 0.
\]
The asymptotic directions have slope \( \pm \sqrt{\frac{p_y}{p_x}} \) and \( \pm \sqrt{\frac{p_y}{p_x}} \) for the convex or concave case, respectively; the corresponding affinity of Theorem 6 is \([x, y] \rightarrow [y \sqrt{\frac{p_x}{p_y}}, x \sqrt{\frac{p_x}{p_y}}]\) for convex \( A \) etc. Not surprisingly, all statements and proofs of the foregoing theorems remain substantially the same and do not deserve special attention.

12. An extreme case: orthocentric quadrangles

If two pairs of opposite sides of \( A \) are orthogonal, then the same holds for the third pair. Such a concave quadrangle is called orthocentric, as each vertex \( A_i \) is the orthocenter of the complementary triangle \( A_jA_hA_k \). For these quadrangles all the circumscribed conics are rectangular hyperbolas, so that \( \Psi, H, J, \Theta \) are not defined. On the other hand, the medial conic \( \Gamma \) is defined, being merely the common nine-point circle for all the complementary triangles \( A_jA_hA_k \). The asymptotic directions of \( A \) cannot be defined, but any pair of orthogonal directions can be used for defining the affinity of \( \S\), and semi-similar orthocentric quadrangles turn out to be just directly similar. As an example, the elementary construction we gave in the introduction, when applied to an orthocentric quadrangle \( A \), modulo homotheties, just rotates \( A \) by a straight angle. We may also notice that some statements
regarding orthocentric quadrangles can be obtained from the general case, as limits for $s_4$ tending to the value $-1$.

References


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Inversions in an Ellipse

José L. Ramírez

Abstract. In this paper we study the inversion in an ellipse which generalizes the classical inversion with respect to a circle and some properties. We also study the inversive images of lines, ellipses and other curves. Finally, we generalize the Pappus chain theorem to ellipses.

1. Introduction

In this paper we study inversions in an ellipse, which was introduced in [2], and some related properties to the distance of inverse points, cross ratio, harmonic conjugates and the images of various curves. This notion generalizes the classical inversion, which has a lot of properties and applications, see [1, 3, 4].

Definition. Let $E$ be an ellipse centered at a point $O$ with foci $F_1$ and $F_2$ in $\mathbb{R}^2$, the inversion in $E$ is the mapping $\psi : \mathbb{R}^2 \setminus \{O\} \rightarrow \mathbb{R}^2 \setminus \{O\}$ defined by $\psi(P) = P'$, where $P'$ lies on the ray $OP$ and $OP \cdot OP' = OQ^2$, where $Q$ is the intersection of the ray $OP$ with the ellipse.

The point $P'$ is said to be the inverse of $P$ in the ellipse $E$. We call $E$ the ellipse of inversion, $O$ the center of inversion, and the number $w := OQ$ the radius of inversion (see Figure 1). Unlike the classical case, the radius of inversion is not constant. Clearly, $\psi$ is an involution, i.e., $\psi(\psi(P)) = P$ for every $P \neq O$. The fixed points are the points on the ellipse $E$. Indeed, $P$ is in the exterior of $E$ if and only if $P'$ is in the interior of $E$. By introducing a point at infinity $O_\infty$ as the inversive image of $O$, we regard $\psi$ as an involution on the extended inversive plane $\mathbb{R}_\infty^2$.

Figure 1
2. Basic properties

**Proposition 1.** The inverse of $P$ in an ellipse $E$ is the intersection of the line $OP$ and the polar of $P$ with respect to $E$. More precisely, if $E$ is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the inverse of the point $(u, v)$ in the ellipse is the point

$$\left( \frac{a^2 b^2 u}{b^2 u^2 + a^2 v^2}, \frac{a^2 b^2 v}{b^2 u^2 + a^2 v^2} \right).$$

**Proof.** If $P = (u, v)$, the ray $\overrightarrow{OP}$ intersects $E$ at $Q = (tu, tv)$ for $t > 0$ satisfying $t^2 \left( \frac{u^2}{a^2} + \frac{v^2}{b^2} \right) = 1$. Now, the polar of $P$ is the line $\frac{ux}{a^2} + \frac{vy}{b^2} = 1$. This intersects the line $OP$ (with equation $vx - uy = 0$) at the point $(u', v') = (ku, kv)$ for $k$ satisfying $k \left( \frac{u^2}{a^2} + \frac{v^2}{b^2} \right) = 1$. Comparison gives $k = t^2$. Hence $OP \cdot OP' = OQ^2$, and $(u', v')$ is the inverse of $P$ in $E$. Explicitly, $u' = \frac{a^2 b^2 u}{b^2 u^2 + a^2 v^2}$ and $v' = \frac{a^2 b^2 v}{b^2 u^2 + a^2 v^2}$. \hfill \Box

**Theorem 2.** Let $P$ and $T$ be distinct points with inversion radii $w$ and $u$ with respect to $E$. If $P'$ and $T'$ are the inverses of $P$ and $T$ in $E$,

$$P'T' = \begin{cases} \frac{w^2 TP}{OP \cdot OT}, & \text{if } O, P, T \text{ are collinear,} \\ \frac{\sqrt{(u^2 - w^2)(w^2 OT^2 - u^2 OP^2) + w^2 u^2 PT^2}}{OP \cdot OT}, & \text{otherwise.} \end{cases}$$

**Proof.** If $O, P, T$ are collinear, the line containing them also contains $Q, P'$ and $T'$. Clearly,

$$P'T' = OT' - OP' = \frac{OQ^2}{OT} - \frac{OP^2}{OP} = \frac{w^2 (OP - OT)}{OP \cdot OT} = \frac{w^2 \cdot TP}{OP \cdot OT}.$$

Now suppose $O, P, T$ are not collinear. Then neither are $O, P', T'$ (see Figure 3). Let $\alpha$ be the measure of angle $P'OT'$. By the law of cosines, we have, in triangle $POT$,

$$\cos \alpha = \frac{OP^2 + OT^2 - PT^2}{2 \cdot OP \cdot OT}. $$
Also, in triangle $P'OT$,

\[
P'^2T^2 = OP'^2 + OT'^2 - 2 \cdot OP' \cdot OT' \cdot \cos \alpha
\]

\[
= \left( \frac{w^2}{OP} \right)^2 + \left( \frac{u^2}{OT} \right)^2 - 2 \cdot \frac{w^2}{OP} \cdot \frac{u^2}{OT} \cdot \frac{OP^2 + OT^2 - PT^2}{2 \cdot OP \cdot OT}
\]

\[
= \frac{w^4 \cdot OT^2 + u^4 \cdot OP^2 - w^2 u^2 (OP^2 + OT^2 - PT^2)}{OP^2 \cdot OT^2}
\]

\[
= \frac{(w^2 - u^2)(w^2 \cdot OT^2 - u^2 \cdot OP^2) + w^2 u^2 \cdot PT^2}{OP^2 \cdot OT^2}.
\]

From this the result follows. \qed


3. Cross ratios and harmonic conjugates

Let $A, B, C$ and $D$ be four distinct points on a line $\ell$. We define the cross ratio

\[(AB, CD) := \frac{AC \cdot BD}{AD \cdot BC},\]

where $AB$ denotes the signed distance from $A$ to $B$. We say that $C$, $D$ divide $A$, $B$ harmonically if the cross ratio $(AB, CD) = -1$. In this case we say that $C$ and $D$ are harmonic conjugates with respect to $A$ and $B$. The cross ratio is an invariant under inversion in a circle whose center is not any of the four points $A, B, C, D$ (see [1]). However, the inversion in an ellipse does not preserve the cross ratio. Nevertheless, in the case of harmonic conjugates, we have the following theorem.

**Theorem 3.** Let $E$ be an ellipse with center $O$, and $Q_1Q_2$ a diameter of $E$. Two points on the line $Q_1Q_2$ are harmonic conjugates with respect to $Q_1$ and $Q_2$ if and only if they are inverse to each other with respect to $E$.

**Proof.** Let $P$ and $P'$ be two points on a diameter $Q_1Q_2$. Since

\[
Q_1P \cdot Q_2P' = (Q_1O + OP) \cdot (Q_2O + OP')
\]

\[
= (w + OP)(-w + OP')
\]

\[
= -w^2 - w(OP - OP') + OP \cdot OP',
\]

\[
Q_1P' \cdot Q_2P = -w^2 + w(OP - OP') + OP \cdot OP',
\]

...
the points \(P\) and \(P'\) are harmonic conjugates with respect to \(Q_1\) and \(Q_2\) if and only if \(OP \cdot OP' = w^2\), i.e., \(P\) and \(P'\) are inverse with respect to \(E\).

\[\square\]

4. Images of curves under an inversion in an ellipse

**Theorem 4.** Consider the inversion \(\psi\) in an ellipse \(E\) with center \(O\).

(a) Every line containing \(O\) is invariant under the inversion.

(b) The image of a line not containing \(O\) is an ellipse containing \(O\) and homothetic to \(E\).

![Figure 4](image)

**Proof.**

(a) This is clear from definition.

(b) Consider a line \(\ell\) not containing \(O\), with equation \(px + qy + r = 0\) with \(r \neq 0\). \((x, y)\) is the inversive image of a point on \(\ell\), then the image of \((x, y)\) lies on \(\ell\). In other words,

\[
p \cdot \frac{a^2b^2x}{b^2x^2 + a^2y^2} + q \cdot \frac{a^2b^2y}{b^2x^2 + a^2y^2} + r = 0.
\]

\[
a^2b^2(px + qy) + r(b^2x^2 + a^2y^2) = 0. \quad (1)
\]

This is clearly an ellipse containing \(O(0, 0)\). Indeed, by rearranging its equation as

\[
\left(x + \frac{a^2p}{2r}\right)^2 + \frac{y + \frac{b^2q}{2r}}{b^2} = \frac{a^2p^2 + b^2q^2}{4r^2},
\]

we note that this is the ellipse with center \((-\frac{a^2p}{2r}, -\frac{b^2q}{2r})\), and homothetic to \(E\) with ratio \(\frac{2r}{\sqrt{a^2p^2 + b^2q^2}}\).

\[\square\]

**Corollary 5.** Let \(\ell_1\) and \(\ell_2\) be perpendicular lines intersecting at a point \(P\).

(a) If \(P = O\), then \(\psi(\ell_1)\) and \(\psi(\ell_2)\) are perpendicular lines.

(b) If \(\ell_1\) does not contain \(O\) but \(\ell_2\) does, then \(\psi(\ell_1)\) is an ellipse through \(O\) orthogonal to \(\psi(\ell_2) = \ell_2\) at \(O\).

(c) If none of the lines contains \(O\), then \(\psi(\ell_1)\) and \(\psi(\ell_2)\) are ellipses containing \(P'\) and \(O\), and are orthogonal at \(O\).
Proof. (a) The lines $\ell_1$ and $\ell_2$ are invariant.

(b) Let $\ell_1$ be the line $px + qy + r = 0$ (with $r \neq 0$). Its image in $E$ is the ellipse given by (1). The tangent at $O$ is the line whose equation is obtained by suppressing the $x^2$ and $y^2$ terms, and replacing $x$ and $y$ by $\frac{1}{2}x$ and $\frac{1}{2}y$. This results in the line $\frac{1}{2}a^2b^2(px + qy) = 0$, or simply $px + qy = 0$, parallel to $\ell_1$ and orthogonal to $\ell_2$ at $O$.

(c) Let $\ell_1$ and $\ell_2$ be the orthogonal lines $p(x - h) + q(y - k) = 0$ and $q(x - h) - p(y - k) = 0$ intersecting at $P = (h, k) \neq O$. Their inverse images in $E$ are ellipses intersecting at $O$ and $P'$. By (b) above, the tangents at $O$ are the orthogonal lines $px + qy = 0$ and $qx - py = 0$. \hfill \Box

Remark. In (c), the images are not necessarily orthogonal at $P'$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{Figure 5}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.png}
\caption{Figure 6.}
\end{figure}
Corollary 6. (a) If \( P \neq O \), the inverse images of the pencil of lines through \( P \) are coaxial ellipses through \( O \) and \( P' \) (see Figure 6).

(b) The inverse images of a system of straight lines parallel to \( \ell_0 \) through \( O \) are ellipses homothetic to \( \mathcal{E} \) tangent to \( \ell_0 \) at \( O \) (see Figure 7).

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure7.png}
\caption{Figure 7.}
\end{figure}

Theorem 7. Let \( \mathcal{E} \) be the ellipse of inversion with center \( O \), and \( \mathcal{E}' \) an ellipse homothetic to \( \mathcal{E} \). The image of \( \mathcal{E}' \) is

(a) an ellipse homothetic to \( \mathcal{E} \) if \( \mathcal{E}' \) does not pass through \( O \),

(b) a line if \( \mathcal{E}' \) passes through \( O \).

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figures89.png}
\caption{Figure 8 and Figure 9}
\end{figure}

Proof. An ellipse \( \mathcal{E}' \) homothetic to \( \mathcal{E} \) has equation

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + px + qy + r = 0.
\]

The ellipse \( \mathcal{E}' \) passes through \( O \) if and only if \( r = 0 \).

(a) If \( \mathcal{E} \) does not pass through \( O \), then \( r \neq 0 \). The inversive image consists of points \( P(x, y) \) for which \( P' \) lies on the ellipse, i.e.,

\[
\left( \frac{\frac{a^2 b^2 x}{b^2 x^2 + a^2 y^2}}{a^2} \right)^2 + \left( \frac{\frac{a^2 b^2 y}{b^2 x^2 + a^2 y^2}}{b^2} \right)^2 + p \left( \frac{a^2 b^2 x}{b^2 x^2 + a^2 y^2} \right) + q \left( \frac{a^2 b^2 y}{b^2 x^2 + a^2 y^2} \right) + r = 0.
\]
Simplifying, we obtain
\[(b^2 x^2 + a^2 y^2) \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{p}{r} x + \frac{q}{r} y + \frac{1}{r} \right) = 0.\]

Since \(b^2 x^2 + a^2 y^2 \neq 0\), we must have
\[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{p}{r} x + \frac{q}{r} y + \frac{1}{r} = 0.\]

This is an ellipse homothetic to \(E\) (see Figure 8).

(b) If \(E'\) passes through \(O\), then \(r = 0\). Equation (2) reduces to \(px + qy + 1 = 0\) (see Figure 9).

**Corollary 8.** Let \(E'\) be an ellipse with center \(O'\) homothetic to \(E\) with center \(O\). If \(E'\) is invariant under inversion in \(E\), and \(P\) is a common point of the ellipses, then \(O'P\) and \(OP\) are tangent to \(E\) and \(E'\) respectively.

![Figure 10](image-url)

**Proof.** Comparing the equations of \(E'\) and its image under inversion in \(E\) in the proof of Theorem 7 above, we conclude that the ellipse \(E'\) is invariant if and only if its equation is of the form
\[(E') : \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + px + qy + 1 = 0.\]

Note that the center \(O'\) of \(E'\) has coordinates \(\left(-\frac{pa^2}{2}, -\frac{qb^2}{2}\right)\).

Let \(P = (x_0, y_0)\) be a common point of the two ellipses. Clearly,
\[
\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1, \quad (3)
\]
\[
px_0 + qy_0 + 2 = 0. \quad (4)
\]

The tangents to \(E\) and \(E'\) at \((x_0, y_0)\) are the lines
\[(t) : \quad \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} - 1 = 0,\]
and

\[(t') : \frac{x_0x}{a^2} + \frac{y_0y}{b^2} + \frac{1}{2}p(x + x_0) + \frac{1}{2}q(y + y_0) + 1 = 0.\]

Substitution of \((x,y)\) by the coordinates \(O'\) into \((t)\) and \((0,0)\) into \((t')\) lead to \(\mp \left(\frac{px_0}{a^2} + \frac{qy_0}{b^2} + 1\right)\) respectively. By (4), this is zero in both cases. This shows that \(O'P\) is tangent to \(E\) and \(OP\) is tangent to \(E'\). \(\square\)

**Theorem 9.** Given an ellipse \(E\) with center \(O\), the image of a conic \(C\) not homothetic to \(E\) is

(i) a cubic curve if \(C\) passes through \(O\),

(ii) a quartic curve if \(C\) does not pass through \(O\).

In Figures 11, 12, 13 below, we show the inversive images of a circle, a parabola, and a hyperbola in an ellipse.

Note that the inversion in an ellipse is not conformal.
5. Pappus chain of ellipses

The classical inversion has a lot of applications, such as the Pappus chain theorem, Feuerbach’s Theorem, Steiner Porism, the problem of Apollonius, among others [1, 3, 4]. We conclude this note with a generalization of the Pappus chain theorem to ellipses.

**Theorem 10.** Let $E$ be a semiellipse with principal diameter $AB$, and $E'$, $E_0$ semiellipses on the same side of $AB$ with principal diameters $AC$ and $CB$ respectively, both homothetic to $E$ (see Figure 14). Let $E_1$, $E_2$, ..., be a sequence of ellipses tangent to $E$ and $E'$ such that $E_n$ is tangent to $E_{n-1}$ and $E_{n+1}$ for all $n \geq 1$. If $r_n$ is the semi-minor axis of $E_n$ and $h_n$ the distance of the center of $E_n$ from $AB$, then $h_n = 2nr_n$.

**Figure 14.**

**Proof.** Let $\psi_i$ be the inversion in the ellipse $E_i$. (In Figure 14 we select $i = 2$).

By Theorem 7, $\psi_i(E)$ and $\psi_i(E_0)$ are lines perpendicular to $AB$ and tangent to the ellipse $E_i$. Hence, the ellipses $\psi_i(E_1)$, $\psi_i(E_2)$, ... will be inverted to tangent ellipses to parallel lines to $\psi_i(E)$ and $\psi_i(E_0)$. Hence, $h_i = 2ir_i$. □

**References**


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On a Geometric Locus in Taxicab Geometry

Bryan Brzycki

Abstract. In axiomatic geometry, the taxicab model of geometry is important as an example of a geometry where the SAS Postulate does not hold. Some properties that hold true in Euclidean geometry are not true in taxicab geometry. For this reason, it is important to understand what happens with various classes of geometric loci in taxicab geometry. In the present study, we focus on a geometric locus question inspired by a problem originally posed by Titeica in the Euclidean context; our study presents the solution to this question in the taxicab plane.

1. Introduction

The taxicab geometry is particularly important in foundations of geometry because it provides an example of a geometry where the Side Angle Side Postulate does not hold (see e.g. [8]). In the recent decades, several investigations have focused on various properties of taxicab geometry, some of them inspired from questions studied in advanced Euclidean geometry (see e.g. [2, 4, 6]). An introduction in the fundamental properties of taxicab geometry is [3].

A well-known reference in advanced Euclidean geometry is Titeica’s problem book [7]. For the historical context in which the problem book [7] was written and on Titeica’s research, including his doctoral dissertation written under Gaston Darboux’s direction, see [1]. The problem book [7] is cited by many authors and motivated many contemporary problems. In the present work, we will focus on one particular question, namely, Problem 143. We will ask the question not in the Euclidean context, but in the context of taxicab geometry.

The taxicab distance between two points \((x_1, y_1)\) and \((x_2, y_2)\) in the Cartesian plane is defined (see e.g. [8], p. 39) by

\[
\rho((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|.
\]

A direct verification shows that \(\rho\) is a metric. The Cartesian plane endowed with the metric induced by the distance \(\rho\) yields the taxicab geometry.
2. Right triangle leg ratio

Many relationships in Euclidean geometry do not hold in taxicab geometry. A well-known example is that SAS congruence fails in the taxicab plane; another is that the area of a triangle cannot simply be expressed in the classic $\frac{1}{2}bh$ (see [2]). Nonetheless, a handful of relationships do remain valid in the taxicab plane. For example, we present the following proposition.

Proposition 1. The ratio between the two legs of a right triangle in the taxicab plane is equal to the ratio between the same two legs in the Euclidean plane.

Proof. Let $a$ and $b$ denote the legs of the right triangle. We denote the taxicab lengths of $a$ and $b$ by $a_T$ and $b_T$, and we denote the Euclidean lengths of $a$ and $b$ by $a_E$ and $b_E$. If $a$ and $b$ are parallel to the coordinate axes, then $a_T = a_E$ and $b_T = b_E$, so clearly $\frac{a_T}{b_T} = \frac{a_E}{b_E}$. Otherwise, let $a$ have nonzero slope $m$; this means that $b$ has slope $-\frac{1}{m}$. We have the relations

$$a_T = \frac{1 + |m|}{\sqrt{1 + m^2}} \cdot a_E,$$

$$b_T = \frac{1 + |-\frac{1}{m}|}{\sqrt{1 + \left(-\frac{1}{m}\right)^2}} \cdot b_E = \frac{1 + |m|}{\sqrt{1 + m^2}} \cdot b_E$$

(see [2]). Dividing these two expressions yields $\frac{a_T}{b_T} = \frac{a_E}{b_E}$. □

3. A novel locus

The problem book [7] is cited by many authors and has motivated many contemporary problems. For example, V. Pambuccian’s work [5] incorporates the axiomatic analysis of a problem found originally in Șteica’s problem book. Pursuing a similar idea, we now examine what happens to Problem 143, which was originally stated in [7] in the Euclidean context, if we study it in taxicab geometry. Thus, we ask the following:

Question. Consider a circle with center $O$ and radius $r$ in the taxicab plane. Point $A$ is located within the circle. Find the locus of midpoints of all chords of the circle that pass through $A$.

In Euclidean geometry, the locus is well-known. It is simply a circle with diameter $OA$. On the other hand, when we consider this same locus problem in the context of taxicab geometry, we quickly see that the locus is not so simple. Figure 1 shows an example of such a locus:

Theorem 2. In general, the locus of midpoints of chords that pass through a point $A$ consists of two straight line segments and two hyperbolic sections, at least one of which contains $A$. 
3.1. Set-Up for proof. Without loss of generality, we place $O$ at the origin, and let $A = (x_A, y_A)$ such that $x_A \geq 0$ and $y_A \leq -x_A$. We can make that last assumption due to the symmetry of the taxicab plane; reflecting the circle across the axes and the lines $y = \pm x$ essentially preserves the shape of the locus. In other words, given any point $A$ within the circle, we can reflect that point and the specified locus about the axes and the lines $y = \pm x$ until the image of $A$ satisfies $x_A \geq 0$ and $y_A \leq -x_A$.

With the notations specified above, let the vertices of the circle be labeled $X$, $Y$, $Z$, and $W$, where $X$ lies on the $x$-axis and the vertices are labeled counterclockwise. Particularly, $X = (r, 0)$, $Y = (0, r)$, $Z = (-r, 0)$, and $W = (0, -r)$. Furthermore, let $XA$, $YA$, $ZA$, and $WA$ intersect the circle again at $X'$, $Y'$, $Z'$, and $W'$, respectively. Then we claim that the locus consists of four parts:

1. The locus of midpoints of the chords between $XX'$ and $W'W$ is a straight line from the midpoint of $XX'$ to the midpoint of $W'W$ along the line $y = -x$.

2. The locus of midpoints of the chords between $YY'$ and $ZZ'$ is a straight line from the midpoint of $YY'$ to the midpoint of $ZZ'$ along the line $y = x$.

3. The locus of midpoints of the chords between $W'W$ and $YY'$ is a hyperbolic section from the midpoint of $W'W$ to the midpoint of $YY'$ that is centered at $(\frac{x_A}{2}, \frac{y_A}{2})$.

4. The locus of midpoints of the chords between $ZZ'$ and $X'X$ is a hyperbolic section from the midpoint of $ZZ'$ to the midpoint of $X'X$ that is centered at $(\frac{x_A}{2}, \frac{y_A-x}{2})$. Point $A$ lies on this hyperbolic section.
3.2. Proof of Theorem 2.

(1) and (2): Consider any chord passing through $A$ between $XX'$ and $W'W$. Since the endpoints of this chord lie on parallel lines, the midpoint of the chord must be halfway between these lines, also forming a parallel line with the two original ones. Particularly, the line that the midpoints fall on is $y = x$, from the midpoint of $XX'$ to the midpoint of $W'W$. By the same reasoning, the locus of midpoints of any chord passing through $A$ between $YY'$ and $Z'Z$ is on $y = x$, from the midpoint of $YY'$ to the midpoint of $Z'Z$.

(3) and (4): We prove (4); then (3) follows by a similar argument.

The equations of lines $ZW$ and $WX$ are $y = -x - r$ and $y = x - r$, respectively. Consider the endpoint $C$ on $ZW$ of a chord passing through $A$ with coordinates $(x_C, -x_C - r)$. The point $D$ at which $CA$ intersects $WX$ is uniquely determined, so we can calculate $D$ using the slope of $CA$ and hence the equations of lines $CA$ and $WX$:

$$D = \left( \frac{x_A x_C + r x_C + y_A x_C}{y_A - x_A + 2x_C + r}, \frac{x_A x_C + y_A x_C - r y_A + r x_A - r x_C - r^2}{y_A - x_A + 2x_C + r} \right).$$

The midpoint $M$ of chord $CD$ is simply the average of the points $C$ and $D$:

$$M = \left( \frac{x_C + \frac{x_C(x_A - x_C)}{y_A - x_A + 2x_C + r}}{y_A - x_A + 2x_C + r}, \frac{x_C(x_A - x_C)}{y_A - x_A + 2x_C + r} \right).$$

To find the locus of midpoints with the above expression, we wish to relate the $x$ and $y$ coordinates. We set

$$x = x_C + \frac{x_C(x_A - x_C)}{y_A - x_A + 2x_C + r} = \frac{x_C y_A + x_C^2 - r x_C}{y_A - x_A + 2x_C + r},$$

$$y = \frac{x_C(x_A - x_C)}{y_A - x_A + 2x_C + r} - r = \frac{x_C x_A - x_C^2 - 2r x_C - r y_A + r x_A - r^2}{y_A - x_A + 2x_C + r}.$$

These equations yield

$$x + y = \frac{x_C y_A + x_C x_A - r y_A + r x_A - r^2}{y_A - x_A + 2x_C + r},$$

$$x - y = x_C + r.$$

Multiplying through by the denominator and substituting $x_C = x - y - r$ yields

$$(x + y)(y_A - x_A + 2(x - y - r) + r)$$

$$= (x - y - r)y_A + (x - y - r)x_A - r(x - y - r) - r y_A + r x_A - r^2$$

$$\implies x^2 - y^2 - x x_A + y y_A - y r + y_A r = 0$$

$$\implies \left( x - \frac{x_A}{2} \right)^2 - \left( y - \frac{y_A - r}{2} \right)^2 = \frac{x_A^2}{4} - \frac{(y_A + r)^2}{4}.$$

This is precisely the form of a hyperbola, with center $(\frac{x_A}{2}, \frac{y_A - r}{2})$, as desired. Clearly, the point $A = (x_A, y_A)$ lies on this hyperbolic section since

$$\left( x_A - \frac{x_A}{2} \right)^2 - \left( y_A - \frac{y_A - r}{2} \right)^2 = \frac{x_A^2}{4} - \frac{(y_A + r)^2}{4}.$$
A similar argument proves (3) as well. This completes the proof of Theorem 2.

Remarks. (1) In some cases, such as when $A$ lies on the axes or $y = \pm x$ lines, some of these segments or sections may be degenerate. For example, if $A$ lies on either of the coordinate axes, the locus consists of two straight line segments and one hyperbolic section. If $A$ lies on $y = x$ or $y = -x$, the locus consists of one straight line segment and two hyperbolic sections. Clearly, if $A$ is at the origin, the locus is simply a point.

(2) In advanced Euclidean geometry, we work within the axiomatic context given by the postulates of Euclidean geometry, which itself can be viewed in many axiomatic contexts (see [8]). Our present study points out how much a geometric locus can change in an axiomatic framework where the SAS Postulate does not hold any longer.

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A Simple Proof of Gibert’s Generalization of the Lester Circle Theorem

Dao Thanh Oai

Abstract. We give a simple proof of Gibert’s generalization of the Lester circle theorem.

The famous Lester circle theorem states that for a triangle, the two Fermat points, the nine point center and the circumcenter lie on a circle, the Lester circle of the triangle. Here is Gibert’s generalization of the Lester circle theorem, given in [2] and [4, Theorem 6]: Every circle whose diameter is a chord of the Kiepert hyperbola perpendicular to the Euler line passes through the Fermat points. In this note we show that this follows from a property of rectangular hyperbolas.

Lemma 1. Let $F_+$ and $F_-$ be two antipodal points on a rectangular hyperbola. For every point $H$ on the hyperbola, the tangent to the circle $(F_+F_-H)$ at $H$ is parallel to the tangents of the hyperbola at $F_+$ and $F_-$. 

Proof. In a Cartesian coordinate system, let the rectangular hyperbola be represented by $xy = a$, and $F_+ \left( x_0, \frac{a}{x_0} \right)$ and $F_- \left( -x_0, \frac{-a}{x_0} \right)$ two antipodal points. The slope of the tangents at $F_\pm$ is $-\frac{a}{x_0^2}$. Let $H \left( x_H, \frac{a}{x_H} \right)$ be a point on the hyperbola. Consider the circle through $F_\pm$ and $H$. Writing its equation in the form

$$x^2 + y^2 + Ax + By + C = 0,$$

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and substituting the coordinates of $F_\pm$ and $H$ above, we obtain
\[ x_0^2 + y_0^2 + Ax_0 + By_0 + C = 0, \]
\[ x_0^2 + y_0^2 - Ax_0 - By_0 + C = 0, \]
\[ x_H^2 + y_H^2 + Ax_H + By_H + C = 0. \]
Solving these equations we have
\[ A = -x_H + y_H \cdot \frac{a}{x_0^2}, \quad B = -\frac{Ax_0^2}{a}, \quad C = -(x_0^2 + y_0^2). \]
The tangent of the circle at $H$ is the line
\[ 2x_H x + 2y_H y + A(x + x_H) + B(y + y_H) + 2C = 0. \]
It has slope
\[ -\frac{2x_H + A}{2y_H + B} = -\frac{x_H + y_H \cdot \frac{a}{x_0^2}}{y_H + x_H \cdot \frac{x_0^2}{a}} = -\frac{x_H + \frac{a}{x_H} \cdot \frac{x_0^2}{a}}{a + x_H \cdot \frac{x_0^2}{a}} = -\frac{a}{x_0^2}. \]
This tangent is parallel to the tangents of the hyperbola at $F_\pm$. \[ \square \]

**Figure 2**

**Theorem 2** ([1]). Let $H$ and $G$ lie on one branch of a rectangular hyperbola, and
(i) $F_+$ and $F_-$ antipodal points on the hyperbola the tangents at which are parallel
to the line $HG$,
(ii) $K_+$ and $K_-$ two points on the hyperbola the tangents at which intersect at a
point $E$ on the line $HG$.
If the line $K_+K_-$ intersects $HG$ at $D$, and the perpendicular bisector of $DE$
intersects the hyperbola at $G_+$ and $G_-$, then the six points $F_+$, $F_-$, $D$, $E$, $G_+$,
$G_-$ lie on a circle.
Proof. By Lemma 1, the circle \((F_+ F_- H)\) is tangent to \(HG\) at \(H\). Similarly, the circle \((F_+ F_- G)\) is tangent to the same line \(HG\) at \(G\).

Let \(M\) be the intersection of \(F_+ F_-\) and \(HG\). It lies on the radical axis of the circles \((F_+ F_- H)\) and \((F_+ F_- G)\), and satisfies \(MG^2 = MF_+ \cdot MF_- = MH^2\). Therefore, \(M\) is the midpoint of \(HG\).

Since the tangents of the hyperbola at \(K_+\) and \(K_-\) intersect at \(E\), the line \(K_+ K_-\) is the polar of \(E\). If it intersects the line \(HG\) at \(D\), then \((G, D; H, E)\) is a harmonic range. Since \(M\) is the midpoint of \(HG\), by a famous property of harmonic range, we have \(MG^2 = MD \cdot ME\). Therefore, \(MF_+ \cdot MF_- = MD \cdot ME\), and the four points \(F_+, F_-, D, E\) lie on a circle.

Now let the circle \((F_+ F_- DE)\) intersect the rectangular hyperbola at two points \(G_+\) and \(G_-\). By Lemma 1, the tangents of the circle at \(G_+\), \(G_-\) are parallel to those of the hyperbola at \(F_+\) and \(F_-\), and therefore also to \(HG\). It follows that \(G_+ G_-\) is a diameter of the circle perpendicular to \(HG\), and \(G_+, G_-\) lie on the perpendicular bisector of the chord \(DE\) of the circle. The proof of the theorem is complete.

\[\square\]

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A Note on the Fermat-Torricelli Point of a Class of Polygons

Cristinel Mortici

Abstract. The aim of this note is to prove a result related to the Fermat-Torricelli point for a class of polygons.

The French mathematician Pierre Fermat (1601-1665) proposed at the end of his book *Treatise on Minima and Maxima* the search for a point $T$ in the plane of a triangle $\Delta ABC$ for which the sum $TA + TB + TC$ of the distances from $T$ to the vertices is minimum. As the problem was first proved by the Italian scientist Evangelista Torricelli (1608-1647), the point $T$ is sometimes called the Fermat-Torricelli point. The geometric construction of the Fermat-Torricelli point can be found in many textbooks, the most well known being that which uses the equilateral triangles constructed on the sides of the given triangle. If all angles of the given triangles are smaller than or equal to $2\pi/3$, then

$$\angle ATB = \angle BTC = \angle CTA = \frac{2\pi}{3}.$$

The problem has been studied by Fejes Tóth [2], Kazarinoff [3], and other specialists in geometric inequalities.

A history of Fermat’s problem can be find in Boltyanski et al. [1].

We propose here the following new result that can be considered as an extension of Fermat’s problem for a particular class of polygons. Remarks on this form can be found in [4]. Moreover, the proof provided is quite elementary.

Let there be given in plane $n + 1$ points $T, A_1, A_2, \ldots, A_n$. We say that the figure consisting of the union of the segments $TA_1, TA_2, \ldots, TA_n$ is a *star* if $A_1A_2\ldots A_n$ is a $n$-sided polygon and

$$\angle A_1TA_2 = \angle A_2TA_3 = \cdots = \angle A_nTA_1 = \frac{2\pi}{n}.$$ 

Let us denote such a star by $[T; A_1, A_2, \ldots, A_n]$. Now we are in the position to give our result.

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Theorem 1. Let $[T; A_1, A_2, \ldots, A_n]$ be a star. Then for every point $M$, we have

$$TA_1 + TA_2 + \cdots + TA_n \leq MA_1 + MA_2 + \cdots + MA_n.$$ 

Proof. Let us consider the complex plane with the origin $T$ and the positive real axis $TA_1$. Let $r_k = TA_k$, and assume that $r_k \omega^{k-1}$ is the complex number associated with the point $A_k$, for every $1 \leq k \leq n$, where

$$\omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}.$$ 

For any point $M$ associated with a complex number $z$, we have

\begin{align*}
MA_1 + MA_2 + \cdots + MA_n & = |z - r_1| + |z - r_2\omega| + \cdots + |z - r_n\omega^{n-1}| \\
& = |z - r_1| + \left| \frac{z}{\omega} - r_2 \right| + \cdots + \left| \frac{z}{\omega^{n-1}} - r_n \right| \\
& \geq \left| (z - r_1) + \left( \frac{z}{\omega} - r_2 \right) + \cdots + \left( \frac{z}{\omega^{n-1}} - r_n \right) \right| \\
& = \left| z \left( 1 + \frac{1}{\omega} + \cdots + \frac{1}{\omega^{n-1}} \right) - (r_1 + r_2 + \cdots + r_n) \right| \\
& = | - (r_1 + r_2 + \cdots + r_n)| \\
& = r_1 + r_2 + \cdots + r_n \\
& = TA_1 + TA_2 + \cdots + TA_n,
\end{align*}

and we are done. \qed

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Properties of Equidiagonal Quadrilaterals

Martin Josefsson

Abstract. We prove eight necessary and sufficient conditions for a convex quadrilateral to have congruent diagonals, and one dual connection between equidiagonal and orthodiagonal quadrilaterals. Quadrilaterals with both congruent and perpendicular diagonals are also discussed, including a proposal for what they may be called and how to calculate their area in several ways. Finally we derive a cubic equation for calculating the lengths of the congruent diagonals.

1. Introduction

One class of quadrilaterals that have received little interest in the geometrical literature are the equidiagonal quadrilaterals. They are defined to be quadrilaterals with congruent diagonals. Three well known special cases of them are the isosceles trapezoid, the rectangle and the square, but there are other as well. Furthermore, there exists many equidiagonal quadrilaterals that besides congruent diagonals have no special properties. Take any convex quadrilateral $ABCD$ and move the vertex $D$ along the line $BD$ into a position $D'$ such that $AC = BD'$. Then $ABCD'$ is an equidiagonal quadrilateral (see Figure 1).

![Figure 1. An equidiagonal quadrilateral $ABCD'$](image)

Before we begin to study equidiagonal quadrilaterals, let us define our notations. In a convex quadrilateral $ABCD$, the sides are labeled $a = AB$, $b = BC$, $c = CD$ and $d = DA$, and the diagonals are $p = AC$ and $q = BD$. We use $\theta$ for the angle between the diagonals. The line segments connecting the midpoints of opposite sides of a quadrilateral are called the bimedians and are denoted $m$ and $n$, where $m$ connects the midpoints of the sides $a$ and $c$. 

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2. Characterizations of equidiagonal quadrilaterals

Of the seven characterizations for equidiagonal quadrilaterals that we will prove in this section, three have already appeared in our previous papers [11] and [12]. We include them here anyway for the sake of completeness. One of them is proved in a new way.

It is well known that the midpoints of the sides in any quadrilateral are the vertices of a parallelogram, called Varignon’s parallelogram. The diagonals in this parallelogram are the bimedians of the original quadrilateral and the sides in the Varignon parallelogram are half as long as the diagonal in the original quadrilateral that they are parallel to. When studying equidiagonal quadrilaterals, properties of the Varignon parallelogram proves to be useful.

![Figure 2. The Varignon parallelogram](image)

Using the parallelogram law in the Varignon parallelogram yields (see Figure 2)

\[ m^2 + n^2 = 2\left( \left( \frac{p}{2} \right)^2 + \left( \frac{q}{2} \right)^2 \right) \]

which is equivalent to

\[ p^2 + q^2 = 2(m^2 + n^2). \] (1)

This equality is valid in all convex quadrilaterals.

For the product of the diagonals we have a necessary and sufficient condition of equidiagonal quadrilaterals in terms of the bimedians.

**Proposition 1.** The product of the diagonals \( p \) and \( q \) in a convex quadrilateral with bimedians \( m \) and \( n \) satisfies

\[ pq \leq m^2 + n^2 \]

where equality holds if and only if it is an equidiagonal quadrilateral.

**Proof.** By adding and subtracting \( 2pq \) to the left hand side of (1), we get

\[ 2pq \leq (p - q)^2 + 2pq = 2(m^2 + n^2). \]

The inequality follows, with equality if and only if \( p = q \). \( \square \)

The first part in the following theorem was proved by us as Theorem 7 (ii) in [11], but we repeat the short argument here.
**Theorem 2.** A convex quadrilateral is equidiagonal if and only if
(i) the bimedians are perpendicular, or
(ii) the midpoints of its sides are the vertices of a rhombus.

**Proof.** (i) It is well known that a quadrilateral has perpendicular diagonals if and only if the sum of the squares of two opposite sides is equal to the sum of the squares of the other two sides (see Theorem 1 in [11]). Hence we get
\[ p = q \Leftrightarrow \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2 = \left(\frac{q}{2}\right)^2 + \left(\frac{q}{2}\right)^2 \Leftrightarrow m \perp n \]
since opposite sides in a parallelogram are congruent.

(ii) A parallelogram is a rhombus if and only if its diagonals are perpendicular. Since the diagonals in the Varignon parallelogram are the bimedians of the original quadrilateral (see Figure 2), (ii) is equivalent to (i). □

The next characterization is about the area of the quadrilateral. To prove it in a new way compared to what we did in [12, p.19], we need the following area formula for convex quadrilaterals. We cannot find a reference for this formula, but it is similar to one we derived in [10].

**Theorem 3.** A convex quadrilateral with diagonals \( p, q \) and bimedians \( m, n \) has the area
\[ K = \sqrt{m^2n^2 - \left(\frac{p^2 - q^2}{4}\right)^2}. \]

**Proof.** Rewriting (1), we have in all convex quadrilaterals
\[ (m^2 - n^2)^2 + 4m^2n^2 = \left(\frac{p^2 + q^2}{2}\right)^2. \]

Theorem 7 in [10] states that a convex quadrilateral has the area
\[ K = \frac{1}{2}\sqrt{p^2q^2 - (m^2 - n^2)^2}. \]

Inserting (2) yields for the area
\[ 4K^2 = \frac{4p^2q^2}{4} + 4m^2n^2 - \left(\frac{p^2 + q^2}{2}\right)^2 = 4m^2n^2 - \left(\frac{p^2 - q^2}{2}\right)^2 \]
and the formula follows. □

**Corollary 4.** The area of a convex quadrilateral is equal to the product of the bimedians if and only if it is an equidiagonal quadrilateral.

**Proof.** In Theorem 3, we have that \( p = q \) if and only if \( K = mn \). □

A direct consequence is another area formula, that also appeared in [12, p.19].

**Corollary 5.** A convex quadrilateral with consecutive sides \( a, b, c, d \) is equidiagonal if and only if it has the area
\[ K = \frac{1}{4}\sqrt{(2(a^2 + c^2) - 4v^2)(2(b^2 + d^2) - 4v^2)} \]
where \( v \) is the distance between the midpoints of the diagonals.
Proof. The length of the bimedians in a convex quadrilateral are
\[ m = \frac{1}{2} \sqrt{2(b^2 + d^2) - 4v^2} \quad \text{and} \quad n = \frac{1}{2} \sqrt{2(a^2 + c^2) - 4v^2} \] (3)
according to [10, p.162]. Using these expressions in Corollary 4 directly yields this formula. □

The next characterization is perhaps not so elegant in itself, but it will be used to derive a more symmetric one later on.

**Proposition 6.** A convex quadrilateral \(ABCD\) with consecutive sides \(a, b, c, d\) is equidiagonal if and only if
\[ ab \cos B + cd \cos D = ad \cos A + bc \cos C. \]

**Proof.** The quadrilateral is equidiagonal if and only if \(2p^2 = 2q^2\), which, according to the law of cosines, is equivalent to (see Figure 2)
\[ a^2 + b^2 - 2ab \cos B + c^2 + d^2 - 2cd \cos D = a^2 + d^2 - 2ad \cos A + b^2 + c^2 - 2bc \cos C. \]
Eliminating common terms and factors on both sides, this is equivalent to the equation in the proposition. □

This lemma, which can be thought of as a law of sines for quadrilaterals and is very similar to the previous proposition, will be used in the next proof.

**Lemma 7.** In a convex quadrilateral \(ABCD\) with consecutive sides \(a, b, c, d\),
\[ ab \sin B + cd \sin D = ad \sin A + bc \sin C. \]

**Proof.** By dividing the quadrilateral into two triangles using a diagonal, which can be done in two different ways, we have for its area that (see Figure 2)
\[ K = \frac{1}{2}ab \sin B + \frac{1}{2}cd \sin D = \frac{1}{2}ad \sin A + \frac{1}{2}bc \sin C. \]
The equation in the lemma follows at once by doubling both sides of the second equality. □

Now we come to our main characterization of equidiagonal quadrilaterals.

**Theorem 8.** A convex quadrilateral \(ABCD\) with consecutive sides \(a, b, c, d\) is equidiagonal if and only if
\[(a^2 - c^2)(b^2 - d^2) = 2abcd(\cos (A - C) - \cos (B - D)).\]

**Proof.** Squaring both sides of the equation in Lemma 7 yields
\[ a^2 b^2 \sin^2 B + c^2 d^2 \sin^2 D + 2abcd \sin B \sin D \]
\[ = a^2 d^2 \sin^2 A + b^2 c^2 \sin^2 C + 2abcd \sin A \sin C \] (4)
which is true in all convex quadrilaterals. Squaring the equation in Proposition 6, we have that a convex quadrilateral is equidiagonal if and only if
\[ a^2 b^2 \cos^2 B + c^2 d^2 \cos^2 D + 2abcd \cos B \cos D \]
\[ = a^2 d^2 \cos^2 A + b^2 c^2 \cos^2 C + 2abcd \cos A \cos C. \] (5)
By adding equations (4) and (5) and applying the identity \( \sin^2 \phi + \cos^2 \phi = 1 \) four times, we get the following equality that is equivalent to the one in Proposition 6 (due to the property that \( x = y \) if and only if \( x + z = y + z \) for any \( z \))

\[
\begin{align*}
    a^2 b^2 + c^2 d^2 + 2abcd (\sin B \sin D + \cos B \cos D) &= a^2 d^2 + b^2 c^2 + 2abcd (\sin A \sin C + \cos A \cos C). \\
\end{align*}
\]

Using the subtraction formula for cosine, this is equivalent to

\[
\begin{align*}
    a^2 b^2 - a^2 d^2 - b^2 c^2 + c^2 d^2 &= 2abcd \cos (A - C) - 2abcd \cos (B - D) \\
\end{align*}
\]

which is factored into the equation in the theorem. \( \square \)

**Corollary 9.** Two opposite sides of an equidiagonal quadrilateral are congruent if and only if it is an isosceles trapezoid.

**Proof.** Applying the trigonometric formula \( \cos \phi - \cos \psi = -2 \sin \frac{\phi + \psi}{2} \sin \frac{\phi - \psi}{2} \) and the sum of angles in a quadrilateral, we have that the equation in Theorem 8 is equivalent to

\[
(a + c)(a - c)(b + d)(b - d) = -4abcd \sin (A + B) \sin (A + D). \\
\]

Hence \( a = c \) or \( b = d \) is equivalent to \( A + B = \pi \) or \( A + D = \pi \), which are well known characterizations of a trapezoid (see [13, p.24]). \( \square \)

### 3. A new duality regarding congruent and perpendicular diagonals

Theorem 7 in [11] can be reformulated to say that a convex quadrilateral is equidiagonal if and only if its Varignon parallelogram is orthodiagonal, and the quadrilateral is orthodiagonal if and only if its Varignon parallelogram is equidiagonal. Thus it gives a sort of dual connection between a quadrilateral and its Varignon parallelogram. Here we shall prove another duality between a quadrilateral and one quadrilateral associated with it. First let us remind the reader that if squares are erected outwards on the sides of a quadrilateral, then their centers are the vertices of a quadrilateral that is both equidiagonal and orthodiagonal.\(^1\) This result is called van Aubel’s theorem. It can be proved using elementary triangle geometry (see the animated proof at [7]) or basic properties of complex numbers as in [2, pp.62–64].

What happens if we exchange the squares for equilateral triangles? Problem 5 on the shortlist for the International Mathematical Olympiad in 1992 asked for a proof that the two line segments connecting opposite centroids of those triangles are perpendicular if the quadrilateral has congruent diagonals [6, p.269]. That problem covered only a quarter of the following theorem, since the converse statement as well as a dual one and its converse are also true. Essentially the same proof of part (i) was given at [15]. We have found no reference to neither the proof nor the statement of part (ii).

---

\(^1\)An orthodiagonal quadrilateral is a quadrilateral with perpendicular diagonals.
Theorem 10. Suppose equilateral triangles are erected outwards on the sides of a convex quadrilateral $ABCD$. Then the following characterizations hold:

(i) $ABCD$ is an equidiagonal quadrilateral if and only if the triangle centroids are the vertices of an orthodiagonal quadrilateral.

(ii) $ABCD$ is an orthodiagonal quadrilateral if and only if the triangle centroids are the vertices of an equidiagonal quadrilateral.

Proof. (i) Let the triangle centroids be $G_1, G_2, G_3$ and $G_4$. In an equilateral triangle with side $x$, the distance from the centroid to a vertex (equal to the circumradius $R$) is $R = \frac{x}{\sqrt{3}}$. Applying the law of cosines in triangle $G_1A G_4$ yields (see Figure 3)

\[
(G_1G_4)^2 = \left(\frac{a}{\sqrt{3}}\right)^2 + \left(\frac{d}{\sqrt{3}}\right)^2 - \frac{a}{\sqrt{3}} \cdot \frac{d}{\sqrt{3}} \cos \left(\frac{A + \pi}{3}\right) = \frac{a^2}{3} + \frac{d^2}{3} - \frac{ad}{3} \left(\frac{1}{2} \cos A - \frac{\sqrt{3}}{2} \sin A\right).
\]

In the same way we have

\[
(G_2G_3)^2 = \frac{b^2}{3} + \frac{c^2}{3} - \frac{bc}{3} \left(\frac{1}{2} \cos C - \frac{\sqrt{3}}{2} \sin C\right),
\]

\[
(G_1G_2)^2 = \frac{a^2}{3} + \frac{b^2}{3} - \frac{ab}{3} \left(\frac{1}{2} \cos B - \frac{\sqrt{3}}{2} \sin B\right),
\]

\[
(G_3G_4)^2 = \frac{c^2}{3} + \frac{d^2}{3} - \frac{cd}{3} \left(\frac{1}{2} \cos D - \frac{\sqrt{3}}{2} \sin D\right).
\]

Thus, simplifying and collecting similar terms yields that

\[
(G_1G_2)^2 + (G_3G_4)^2 - (G_2G_3)^2 - (G_1G_4)^2 = \frac{1}{6} (ab \cos A + bc \cos C - ab \cos B - cd \cos D) + \frac{\sqrt{3}}{6} (ab \sin B + cd \sin D - ad \sin A - bc \sin C).
\]

The last parenthesis is equal to zero in all convex quadrilaterals (Lemma 7). Hence we have

\[
(G_1G_2)^2 + (G_3G_4)^2 = (G_2G_3)^2 + (G_1G_4)^2
\]

\[
\iff ab \cos B + cd \cos D = ad \cos A + bc \cos C,
\]

where the first equality is a well known characterization for $G_1 G_3 \perp G_2 G_4$ (see Theorem 1 in [11]) and the second equality is true if and only if $ABCD$ is equidiagonal according to Proposition 6.

(ii) This statement is trickier to prove with trigonometry, so instead we will use complex numbers. Let the vertices $A, B, C, D$ of a convex quadrilateral be represented by the complex numbers $z_1, z_2, z_3$ and $z_4$ respectively. Also, let the centroids $G_1, G_2, G_3$ and $G_4$ of the equilateral triangles be represented by the
Properties of equidiagonal quadrilaterals

complex numbers $g_1, g_2, g_3$ and $g_4$ respectively. The latter are related to the former according to

$$g_1 = \frac{z_1 - z_2}{\sqrt{3}} e^{i\pi/6}, \quad g_2 = \frac{z_2 - z_3}{\sqrt{3}} e^{i\pi/6}, \quad g_3 = \frac{z_3 - z_4}{\sqrt{3}} e^{i\pi/6}, \quad g_4 = \frac{z_4 - z_1}{\sqrt{3}} e^{i\pi/6}.$$  

The proof will be in two parts.

$(\Rightarrow)$ If $ABCD$ is orthodiagonal, then $z_3 - z_1 = i\mathcal{R}(z_4 - z_2)$ for some real number $\mathcal{R} \neq 0$. Using the expressions for the centroids, we get

$$\frac{|g_3 - g_1|}{|g_4 - g_2|} = \frac{|z_3 - z_4 - (z_1 - z_2)|}{|z_4 - z_1 - (z_2 - z_3)|} = \frac{|(z_3 - z_1) - (z_4 - z_2)|}{|(z_3 - z_1) + (z_4 - z_2)|} = \frac{i\mathcal{R} - 1}{i\mathcal{R} + 1} = \frac{\sqrt{1 + \mathcal{R}^2}}{\sqrt{1 + \mathcal{R}^2}} = 1$$

where the exponential functions and the $\sqrt{3}$ were canceled out in the first equality. This proves that the line segments connecting opposite centroids are congruent, so $G_1G_2G_3G_4$ is an equidiagonal quadrilateral.

$(\Leftarrow)$ If $G_1G_2G_3G_4$ is equidiagonal, then according to the rewrite in the first part,

$$|(z_3 - z_1) - (z_4 - z_2)| = |(z_3 - z_1) + (z_4 - z_2)|.$$  

We shall prove that this implies that $z_3 - z_1$ and $z_4 - z_2$ are perpendicular. Let us define the two new complex numbers $w_1$ and $w_2$ according to $w_1 = z_3 - z_1$ and $w_2 = z_4 - z_2$. Thus we are to prove that if $|w_1 - w_2| = |w_1 + w_2|$, then $w_1$ and $w_2$
are perpendicular. This is quite obvious from a geometrical perspective considering the vector nature of complex numbers, but we give an algebraic proof anyway. To this end we use the polar form. Thus we have \( w_1 = r_1(c_1 + i s_1) \) and \( w_2 = r_2(c_2 + i s_2) \). We square the two equal absolute values and rewrite \( |w_1 - w_2|^2 = |w_1 + w_2|^2 \) to get
\[
(r_1 c_1 - r_2 c_2)^2 + (r_1 s_1 - r_2 s_2)^2 = (r_1 c_1 + r_2 c_2)^2 + (r_1 s_1 + r_2 s_2)^2.
\]
Expanding these expressions and canceling equal terms, this is equivalent to
\[
4r_1r_2(c_1 c_2 + s_1 s_2) = 0 \iff \cos(c_1 - c_2) = 0.
\]
The last equation has the valid solutions \( c_1 - c_2 = \pm \frac{\pi}{2} \), which proves that the angle between \( w_1 \) and \( w_2 \) is a right angle. Hence \( ABCD \) is orthodiagonal. \( \square \)

Other generalizations of van Aubel’s theorem concerning rectangles, rhombi and parallelograms can be found in [5] and [17].

4. Quadrilaterals that are both equidiagonal and orthodiagonal

Consider Table 1, where three well known properties of the diagonals in seven of the most basic quadrilaterals are shown. The answer “no” refers to the general case for each quadrilateral. One thing is obvious, there is something missing here. No quadrilateral with just the two properties of perpendicular and congruent diagonals is included. This is because no name seems to have been given to this class of quadrilaterals.\(^2\)

<table>
<thead>
<tr>
<th>Quadrilateral</th>
<th>Bisecting diagonals</th>
<th>Perpendicular diagonals</th>
<th>Congruent diagonals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trapezoid</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Isosceles trapezoid</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Kite</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Parallelogram</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Rhombus</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Rectangle</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Square</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Table 1. Diagonal properties in basic quadrilaterals

Before we proceed, we quote in Table 2 in a somewhat expanded form a theorem we proved in [11, p.19]. The four properties on each line in this table are equivalent. The Varignon parallelogram properties follows directly from the fact

\(^2\)In [14, p.50] Gerry Leversha claims that such a quadrilateral is sometimes called a pseudo-square. We can however not find any other reference for that use of the name (neither on the web nor in any geometry books or papers we know of). Instead a Google search indicates that a pseudo-square is a squares with four cut off vertices.
that a parallelogram is a rhombus if and only if its diagonals are perpendicular, and it is a rectangle if and only if its diagonals are congruent [4, p.53].

<table>
<thead>
<tr>
<th>Original quadrilateral</th>
<th>Diagonal property</th>
<th>Bimedian property</th>
<th>Varignon parallelogram</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equidiagonal</td>
<td>( p = q )</td>
<td>( m \perp n )</td>
<td>Rhombus</td>
</tr>
<tr>
<td>Orthodiagonal</td>
<td>( p \perp q )</td>
<td>( m = n )</td>
<td>Rectangle</td>
</tr>
</tbody>
</table>

Table 2. Special cases of the Varignon parallelogram

The bimedians of a convex quadrilateral are the diagonals of its Varignon parallelogram, so the original quadrilateral has congruent and perpendicular diagonals if and only if the Varignon parallelogram has perpendicular and congruent diagonals (see Table 2). For such quadrilaterals, the Varignon parallelogram is a square, and this is a characterization of those quadrilaterals with congruent and perpendicular diagonals since a parallelogram is a square if and only if it is both a rhombus and a rectangle. Thus we have the following two necessary and sufficient conditions.

**Theorem 11.** A convex quadrilateral has congruent and perpendicular diagonals if and only if
(i) the bimedians are perpendicular and congruent, or
(ii) the midpoints of its sides are the vertices of a square.

So what shall we call these quadrilaterals? They are both equidiagonal and orthodiagonal, but trying to combine the two words yields no good name. The individual words describe the defining properties of these quadrilaterals. With that and Theorem 11 in mind, we propose that a quadrilateral with congruent and perpendicular diagonals is called a midsquare quadrilateral (see Figure 4).

![Figure 4. A midsquare quadrilateral and its Varignon square](image)

Three special cases of midsquare quadrilaterals are orthodiagonal isosceles trapezoids, equidiagonal kites and squares.
**Proposition 12.** A midsquare quadrilateral is a square if and only if its diagonals bisect each other.

**Proof.** If the diagonals of a midsquare quadrilateral bisect each other, it is obvious that it is a square since the diagonals divide it into four congruent right triangles with equal legs.

Conversely it is a well known property that in a square, the diagonals bisect each other. □

After having given a name for this neglected type of quadrilateral, we now consider its area. The first formula in the following proposition has been known at least since 1962 according to [3, p.132].

**Proposition 13.** A convex quadrilateral with diagonals \( p \) and \( q \) and bimedians \( m \) and \( n \) is a midsquare quadrilateral if and only if its area is given by

\[
K = \frac{1}{4}(p^2 + q^2) \quad \text{or} \quad K = \frac{1}{2}(m^2 + n^2).
\]

**Proof.** Using the identity \((p-q)^2 = p^2 + q^2 - 2pq\), the area of a convex quadrilateral satisfies (see [8])

\[
K = \frac{1}{2}pq \sin \theta = \frac{1}{4}(p^2 + q^2 - (p-q)^2) \sin \theta \leq \frac{1}{4}(p^2 + q^2)
\]

where equality holds if and only if \( p = q \) and \( p \perp q \).

The second formula follows at once from the first by using equality (1). □

Since the two diagonals and the two bimedians are individually congruent in a midsquare quadrilateral, its area can be calculated with the four simple formulas

\[
K = \frac{1}{2}p^2 = \frac{1}{2}q^2 = m^2 = n^2.
\]

(6)

The next proposition gives more area formulas for midsquare quadrilaterals.

**Proposition 14.** A convex quadrilateral with consecutive sides \( a, b, c, d \) is a midsquare quadrilateral if and only if its area is given by

\[
K = \frac{1}{4}\left(2(a^2 + c^2) - 4v^2\right) = \frac{1}{4}\left(2(b^2 + d^2) - 4v^2\right)
\]

where \( v \) is the distance between the midpoints of the diagonals.

**Proof.** A convex quadrilateral has congruent diagonals if and only if its area is the product of the bimedians according to Corollary 4. Since the diagonals are perpendicular if and only if the bimedians are congruent (Table 2), the two area formulas follows at once using (3). □

**Corollary 15.** A convex quadrilateral with consecutive sides \( a, b, c, d \) is a square if and only if its area is

\[
K = \frac{1}{2}(a^2 + c^2) = \frac{1}{2}(b^2 + d^2).
\]

**Proof.** These formulas are a direct consequence of the last proposition since a convex quadrilateral is a square if and only if it is a midsquare quadrilateral with bisecting diagonals \((v = 0)\) according to Proposition 12. □
Now we come to an interesting question. Can we calculate the area of a mid-square quadrilateral knowing only its four sides? The answer is yes. The origin for the next theorem is a solved problem we found in the pleasant book [9, pp.179–180]. There Heilbron states that this area is given by

\[ K = \frac{1}{4} \left( a^2 + c^2 + \sqrt{2(a^2c^2 + b^2d^2)} \right). \]

He starts his derivation thoroughly, but at the end, when he obtains a quadratic equation, he merely claims that solving it will provide the formula he was supposed to derive. When we started to analyze the solutions to this equation in more detail, we began to smell a rat, and eventually realized that Heilbrons formula is in fact incorrect. We will motivate this after our proof of the correct formula.

**Theorem 16.** A midsquare quadrilateral with consecutive sides \(a, b, c, d\) has the area

\[ K = \frac{1}{4} \left( a^2 + c^2 + \sqrt{4(a^2c^2 + b^2d^2) - (a^2 + c^2)^2} \right). \]

**Proof.** We use notations on the sides and the diagonal parts as in Figure 5, where \(w + x = y + z = p\) since the diagonals are congruent. The area is given by \(K = \frac{1}{2}p^2\), so we need to express a diagonal \(p\) in terms of the sides. Using the Pythagorean theorem, we get

\[ a^2 - b^2 = w^2 - x^2 = (w + x)(w - x) = p(2w - p) \]

and similar \(b^2 - c^2 = p(2y - p)\). Thus we have

\[ a^2 - b^2 + p^2 = 2pw \quad \text{and} \quad b^2 - c^2 + p^2 = 2py. \]

Squaring and adding these yields

\[ (a^2 - b^2 + p^2)^2 + (b^2 - c^2 + p^2)^2 = 4p^2(w^2 + y^2) = 4p^2a^2 \]

where we used the Pythagorean theorem again in the last equality. Expanding and simplifying results in a quadratic equation in \(p^2\):

\[ 2p^4 - 2(a^2 + c^2)p^2 + (a^2 - b^2)^2 + (b^2 - c^2)^2 = 0. \]
This has the solutions
\[ p^2 = \frac{a^2 + c^2 \pm \sqrt{-a^4 - c^4 + 2a^2c^2 - 4b^4 + 4a^2b^2 + 4b^2c^2}}{2}. \]

The radicand can be simplified to
\[-a^4 - c^4 + 2a^2c^2 + 4b^2d^2 = 4(a^2c^2 + b^2d^2) - (a^2 + c^2)^2 \]
where we used \( a^2 - b^2 + c^2 = d^2 \) (see Theorem 1 in [11]). Thus
\[ p^2 = \frac{1}{2} \left( a^2 + c^2 \pm \sqrt{4(a^2c^2 + b^2d^2) - (a^2 + c^2)^2} \right). \]  
(7)

To decide the correct sign we study the special case when the quadrilateral is a square. Using \( a = b = c = d \) in (7) yields
\[ p^2 = \frac{1}{2}(2a^2 \pm 2a^2) \]
where we see that the solution with the negative sign is obviously false. The area formula now follows when inserting (7) into \( K = \frac{1}{2}p^2 \).

Note that it is easy to get formulas for the lengths of the diagonals and the bimedians in a midsquare quadrilateral in terms of the sides. We simply have to combine (6) and Theorem 16.

Remark. Let us comment on the formula suggested by Heilbron. It gives the correct area for a square, so we need to do a more thorough investigation. If his formula were correct, it would mean that \( 2(a^2c^2 + b^2d^2) = (a^2 + c^2)^2 \). But then his formula could be simplified to \( K = \frac{1}{2}(a^2 + c^2) \). According to Corollary 15, this is a characterization for a square. Hence his formula must be incorrect, since the quadrilateral has perpendicular and congruent diagonals, but need not to be a square. Another way to dispute it is by considering a right kite with \( a = d \) and \( c = b \). It has the area \( K = ac \), but Heilbron’s formula gives \( K = \frac{1}{4}(a + c)^2 \). Equating these expressions yields \( (a - c)^2 = 0 \) which again imply the quadrilateral must be a square, which it is not.

5. When are certain quadrilaterals equidiagonal?

So far we have several ways of determining when a convex quadrilateral is equidiagonal. An isosceles trapezoid, a rectangle and a square are always equidiagonal, but how can we know when the diagonals are congruent in other basic quadrilaterals, such as a parallelogram or a cyclic quadrilateral?

Theorem 17. The following characterizations hold:
(i) A parallelogram is equidiagonal if and only if it is a rectangle.
(ii) A rhombus is equidiagonal if and only if it is a square.
(iii) A trapezoid is equidiagonal if and only if it is an isosceles trapezoid.
(iv) A cyclic quadrilateral is equidiagonal if and only if it is an isosceles trapezoid.

Proof. (i) In a parallelogram \( ABCD \) with the two different side lengths \( a \) and \( b \), the law of cosines yields that
\[ p^2 = q^2 \iff a^2 + b^2 - 2ab \cos B = a^2 + b^2 - 2ab \cos A \iff A = B. \]
Two adjacent angles in a parallelogram are equal if and only if it is a rectangle.

(ii) The first part of the proof is the same as in (i) except that \( a = b \), which does not effect the outcome. Two adjacent angles in a rhombus are equal if and only if it is a square.

(iii) The lengths of the diagonals in a trapezoid with consecutive sides \( a, b, c, d \) are given by (see [13, p.31])

\[
p = \sqrt{\frac{ac(a - c) + ad^2 - cb^2}{a - c}} \quad \text{and} \quad q = \sqrt{\frac{ac(a - c) + ab^2 - cd^2}{a - c}}
\]

where \( a \parallel c \) and \( a \neq c \). Thus we get

\[
p^2 = q^2 \iff ad^2 - cb^2 = ab^2 - cd^2 \iff (a - c)(d^2 - b^2) = 0.
\]

Since \( a \neq c \), the only valid solution is \( b = d \), so we have an isosceles trapezoid.

(iv) In a cyclic quadrilateral we can apply Ptolemy’s second theorem, according to which (see [1, p.65])

\[
\frac{p}{q} = \frac{ad + bc}{ab + cd}
\]

Hence

\[
p = q \iff ab + cd = ad + bc \iff (a - c)(b - d) = 0
\]

where the last equality has the two possible solutions \( a = c \) and \( b = d \). Any cyclic quadrilateral with a pair of opposite congruent sides is an isosceles trapezoid. One way of realizing this is by connecting the vertices to the circumcenter and thus conclude that this cyclic quadrilateral has a line of symmetry (see Figure 6).

Conversely it is well known that an isosceles trapezoid has congruent diagonals.

\[\square\]

![Figure 6. This is an isosceles trapezoid](image)

In the previous section we concluded that an orthodiagonal quadrilateral is also equidiagonal if and only if the midpoints of the sides are the vertices of a square. There don’t seem to be any similar easy ways of determining when a kite or a tangential quadrilateral are equidiagonal.
6. The diagonal length in equidiagonal quadrilaterals

We conclude this paper by discussing how the equal length of the diagonals in a general equidiagonal quadrilateral can be calculated given only the four sides, and also how this is related to finding the area of the quadrilateral. Thus this will lead up to a generalization of Theorem 16.

There is a formula relating the four sides and the two diagonals of a convex quadrilateral, sometimes known as Euler’s four point relation. It is quite rare to find this relation in geometry books and even rarer to find a proof of it that does not involve determinants, so we start by deriving it here. For this purpose we need the following trigonometric formula.

Lemma 18. For any two angles \( \alpha \) and \( \beta \) we have the identity
\[
\cos^2 \alpha + \cos^2 \beta + \cos^2(\alpha + \beta) - 2 \cos \alpha \cos \beta \cos(\alpha + \beta) = 1.
\]

Proof. The addition formula for cosines can be rewritten in the form
\[
\cos \alpha \cos \beta - \cos(\alpha + \beta) = \sin \alpha \sin \beta.
\]
Squaring both sides, we have
\[
(\cos \alpha \cos \beta - \cos(\alpha + \beta))^2 = (1 - \cos^2 \alpha)(1 - \cos^2 \beta).
\]
Now the identity follows after expansion and simplification. \( \square \)

The following relation has been derived independently by several mathematicians. It cannot be factored, but there are several ways to collect the terms. The version we present with only four terms is definitely one of the most compact, and except for some basic algebra we only use the law of cosines in the short proof.

Theorem 19 (Euler’s four point relation). In all convex quadrilaterals with consecutive sides \( a, b, c, d \) and diagonals \( p, q \), it holds that
\[
p^2q^2(a^2 + b^2 + c^2 + d^2 - p^2 - q^2) - (a^2 - b^2 + c^2 - d^2)(a^2c^2 - b^2d^2) - p^2(a^2 - d^2)(b^2 - c^2) + q^2(a^2 - b^2)(c^2 - d^2) = 0.
\]

Proof. Let \( \alpha = \angle BAC \) and \( \beta = \angle DAC \) in quadrilateral \( ABCD \). The law of cosines applied in triangles \( BAC, DAC \) and \( ABD \) yields respectively (see Figure 7)
\[
\cos \alpha = \frac{a^2 + p^2 - b^2}{2ap}, \quad \cos \beta = \frac{d^2 + p^2 - c^2}{2dp}, \quad \cos(\alpha + \beta) = \frac{a^2 + d^2 - q^2}{2ad}.
\]
Inserting these into the identity in Lemma 18 and multiplying both sides of the equation by the least common multiple \( 4a^2d^2p^2 \), we get after simplification
\[
da^2(a^2 + p^2 - b^2)^2 + a^2(d^2 + p^2 - c^2)^2 + p^2(a^2 + d^2 - q^2)^2 - (a^2 + p^2 - b^2)(d^2 + p^2 - c^2)(a^2 + d^2 - q^2) = 4a^2d^2p^2.
\]
Now expanding these expressions and collecting similar terms results in Euler’s four point relation. \( \square \)

\( ^3 \)The history of this six variable polynomial dates back to the 15th century, and it is closely related to the volume of a tetrahedron. The Italian painter Piero della Francesca was also interested in
Returning to the initial goal of calculating the length of the equal diagonals, we set $p = q$ in Theorem 19. This results in the following cubic equation in $p^2$:
\[
2p^6 - (a^2 + b^2 + c^2 + d^2)p^4 + ((a^2 + c^2)(b^2 + d^2) - 2(a^2c^2 + b^2d^2))p^2
+ (a^2 - b^2 + c^2 - d^2)(a^2c^2 - b^2d^2) = 0.
\]

Cubic equations have been solved for five centuries and there are several different solution methods known. However they all have one thing in common as anyone who has used one of them has noticed: the expressions for the roots they produce are very complicated and in most of the times completely useless. In fact, solving a cubic equation with coefficients like the one above with a computer algebra system can produce several pages of output formulas. Should it be necessary in a practical situation, a numerical solution (on a calculator or computer) is almost always preferable.

After having solved the cubic equation numerically, the area of the equidiagonal quadrilateral (with $p = q$) is given by the formula of Staudt (see [16, p.35])
\[
K = \frac{1}{4} \sqrt{4p^4 - (a^2 - b^2 + c^2 - d^2)^2}.
\]

So it may come as a little disappointment that we did not get a nice formula for the diagonals and the area like in the case when the diagonals are also perpendicular. There are however lots of cubic equations arising when solving problems in the geometry of triangles and quadrilaterals, so this is quite a common occurrence. On geometry and derived a formula for the volume $V$ of a tetrahedron expressed in terms of its six edges. The formula states that the left hand side of the equation in Theorem 19 is equal to $144V^2$. The formula was rediscovered in the 16th century by the Italian mathematician Niccolò Fontana Tartaglia, who was also involved in the first solution of the cubic equation. In the 18th century the famous Swiss mathematician Leonhard Euler solved the same problem. The invention of determinants made it possible for the 19th century British mathematician Arthur Cayley to express the tetrahedron volume in a very compact form using the so called Cayley-Menger determinant. We do not know which one of these gentlemen was the first to conclude that setting the tetrahedron volume equal to zero would result in an interesting identity for quadrilaterals. A trigonometric derivation that did not involve the tetrahedron has surely been known at least since the 19th century when several mathematicians made thorough trigonometric studies of the geometry of quadrilaterals.
the other hand, we can now get a second derivation of Theorem 16. If the diagonals are both congruent and perpendicular, the constant term of the cubic equation vanishes (since \(a^2 + c^2 = b^2 + d^2\)), so after simplifying the equation and dividing it by the positive number \(p^2\) we get

\[
2p^4 - 2(a^2 + c^2)p^2 + (a^2 + c^2)^2 - 2(a^2c^2 + b^2d^2) = 0.
\]

This directly yields the solution

\[
p^2 = \frac{1}{2} \left( a^2 + c^2 + \sqrt{4(a^2c^2 + b^2d^2) - (a^2 + c^2)^2} \right)
\]

which we recognize from (7).

References


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The Miquel Points, Pseudocircumcenter, and Euler-Poncelet Point of a Complete Quadrilateral

Michal Rolínek and Le Anh Dung

Abstract. We prove over 40 similarities in the configuration of a complete quadrilateral and the Miquel points. Then we introduce a generalized circumcenter and prove a theorem on the Euler-Poncelet point.

1. Introduction

In this paper we will study, using purely synthetical methods, the configuration concerning complete quadrilateral $ABCD$ denoting $P = AC \cap BD$, $Q = AB \cap CD$, and $R = AD \cap BC$. We begin with revealing a large number of similarities within the configuration concerning the associated Miquel points. Then we proceed to introduce a quadrilateral center which generalizes the circumcenter of a cyclic quadrilateral, and finally we will prove a result regarding the Euler-Poncelet Point.

The Euler-Poncelet point $X$ is the common point of the nine-point circles of triangles $ABC$, $BCD$, $CDA$, $DAB$. Also, it is known to lie on the pedal circles of $A$ with respect to triangle $BCD$, and the cyclic variants. More on the Euler-Poncelet point can be found in [5].

Our main result is the following:

Theorem 1. The Euler-Poncelet point $X$ lies on the circumcircle of the triangle $PQR$.

It is of particular interest that our result is a strong generalization of the celebrated result by Emelyanov and Emelyanova (see [2]).

Theorem 2 (Emelyanov, Emelyanova). Let $ABC$ be a triangle with incenter $I$. Let $AI \cap BC = P$, $BI \cap CA = Q$, $CI \cap AB = R$, then the Feuerbach point $F_e$ lies on the circumcircle of triangle $PQR$.

Proof. If we take the complete quadrilateral to be $ABCI$, then the Euler-Poncelet point lies on both the incircle (pedal of $I$ with respect to triangle $ABC$) and the nine-point circle of $ABC$, thus it coincides with their point of contact, i.e., the Feuerbach point. Theorem 1 now implies the result. □
2. Similarities on the Miquel points

In this section we define Miquel points (which were previously studied in [3]) as spiral similarity centers and uncover a surprising amount of similarities within the configuration.

The following two results on spiral similarities are well-known. Their proofs can be found for example in [4].

**Proposition 3.** Let \( A, B, A', B' \) be points in plane such that no three of them are collinear. Assume that the lines \( AB \) and \( A'B' \) intersect at \( P \). Then there exists a unique spiral similarity that sends \( A \) to \( A' \) and \( B \) to \( B' \). The center of this spiral similarity is the second intersection of the circles \((AA'P)\) and \((BB'P)\).

![Figure 1](image1.png)

**Proposition 4.** Let \( S(S,k,\varphi) \) be the spiral similarity that maps \( A \) to \( A' \) and \( B \) to \( B' \).
(a) \( \triangle SAB \sim \triangle SA'B' \).
(b) \( \triangle SAA' \sim \triangle SBB' \).
(c) There is a spiral similarity \( S'(S,k',\varphi') \) that maps \( A \) to \( B \) and \( A' \) to \( B' \) for suitable choice of \( k' \) and \( \varphi' \).

![Figure 2](image2.png)

From these propositions we immediately deduce that there is a unique point which is the center of two spiral similarities.

With this notion we can define three different Miquel points associated to a complete quadrilateral.
Definition. We define the Miquel points $M_p$, $M_q$, and $M_r$ as the following spiral similarity centers (segments are considered directed):

<table>
<thead>
<tr>
<th>Point</th>
<th>Center taking and (at the same time)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_p$</td>
<td>$AB \mapsto DC$ $AD \mapsto BC$</td>
</tr>
<tr>
<td>$M_q$</td>
<td>$AC \mapsto DB$ $AD \mapsto CB$</td>
</tr>
<tr>
<td>$M_r$</td>
<td>$AC \mapsto BD$ $AB \mapsto CD$</td>
</tr>
</tbody>
</table>

As expected, Proposition 3 gives us many circles passing through the Miquel points. Also let us use the notation $M_{XY}$ for the midpoint of the segment $XY$.

Proposition 5. The following sets of points are concyclic:
(a) $(M_p RAB), (M_p RDC), (M_p QAD), (M_p QBC),$
(b) $(M_q RAC), (M_q RDB), (M_q PAD), (M_q PCD),$
(c) $(M_r QAC), (M_r QBD), (M_r PAB), (M_r PCD),$
(d) $(M_p M_q RABD M_{AB} M_{CD}), (M_q M_r PM_{AC} M_{DB}), (M_r M_p QM_{AB} M_{CD}).$

Proof. Parts (a), (b), and (c) follow from the definition of the Miquel points and Proposition 3 as each of them is a center of two different spiral similarities.

For part (d) observe that from $\triangle M_p AD \sim \triangle M_p BC$ it follows that $\triangle M_p M_{AD} A \sim \triangle M_p M_{BC} B$ (directly) and hence $M_p$ is the spiral similarity center which takes $M_{AD}$ to $A$ and $M_{BC}$ to $B$. Proposition 3 now implies that $M_p$ lies on the circumcircle of $\triangle R M_{AD} M_{BC}$. Point $M_q$ lies on the same circle for analogous reasons. The other circles are established in the same way. $\square$

Proposition 6. The following sets of triangles are directly similar:
(a) $\triangle M_p AB \sim \triangle M_p DC \sim \triangle M_B A M_{BD} M_{AC} \sim \triangle M_C D M_{AC} M_{DB},$
$\triangle M_p AD \sim \triangle M_p BC \sim \triangle M_D A M_{DB} M_{AC} \sim \triangle M_C B M_{AC} M_{BD};$
(b) $\triangle M_q AC \sim \triangle M_q CB \sim \triangle M_D A M_{DC} M_{AB} \sim \triangle M_B C M_{AB} M_{CD},$
$\triangle M_q AC \sim \triangle M_q DB \sim \triangle M_C A M_{CD} M_{AB} \sim \triangle M_B D M_{AB} M_{DC};$
(c) $\triangle M_r AC \sim \triangle M_r BD \sim \triangle M_C A M_{CB} M_{AD} \sim \triangle M_D C M_{AD} M_{CB},$
$\triangle M_r AC \sim \triangle M_r BD \sim \triangle M_C A M_{CB} M_{AD} \sim \triangle M_D B M_{AD} M_{BC}.$
Proof. Again, from symmetry of the complete quadrilateral it suffices to prove only one part and this time we choose (thinking of a nice diagram) to prove part (c). In the first chain of similarities the first and the third one are immediate from the definition of the Miquel point and the fact that the points $M_{BA}, M_{AD}, M_{DC}, M_{CB}$ form a Varignon parallelogram, respectively.

Note that the lines $M_{AB}M_{BC}$ and $M_{AB}M_{AD}$ are midlines in triangles $ABC$ and $ABD$, respectively. Angle-chasing (with the use of Proposition 5(c)) now gives

$$\angle(M_{BC}M_{AB}, M_{AD}M_{AB}) = \angle(AC, BD) = \angle(CP, PD) = \angle(CM_{r}, M_{r}D)$$

and after the ratio calculation (using $\triangle M_{p}AC \sim \triangle M_{p}BD$)

$$\frac{M_{BC}M_{AB}}{M_{AD}M_{AB}} = \frac{AC}{BD} = \frac{CM_{r}}{DM_{r}},$$

we have the desired similarity by SAS. The second part is proved likewise. \qed

Lemma 7. If $A'B'C'D'$ is the image of $ABCD$ in the inversion with respect to $M_{p}$ ($M_{q}$, $M_{r}$, respectively), then quadrilateral $ABCD$ is indirectly similar to $C'D'A'B'$ ($B'A'D'C'$, $D'C'B'A'$, respectively).

Proof. From the definition of the Miquel point and inversion, respectively, we have $\triangle M_{p}BC \sim \triangle M_{p}AD \sim \triangle M_{p}D'A'$, where the first similarity is direct and the second one indirect. Similarly, we get $\triangle M_{p}DC \sim \triangle M_{p}AB \sim \triangle M_{p}B'A'$ (again first directly and then indirectly). But this implies indirect similarity of the quadrilaterals $M_{p}DCB$ and $M_{p}B'A'D'$, from which we obtain indirect similarity of $\triangle DCB$ and $\triangle B'A'D'$. In the same vein, we find that $\triangle ABD \sim \triangle C'D'B'$ (indirectly) and the desired similarity of quadrilaterals now follows. The proof for $M_{q}$ and $M_{r}$ goes along the same lines. \qed
Proposition 8.
(a) $M_p$ is the center of spiral similarity which sends $M_r R$ to $QM_q$. 
(b) $M_q$ is the center of spiral similarity which sends $M_p P$ to $RM_r$. 
(c) $M_r$ is the center of spiral similarity which sends $M_q Q$ to $PM_p$. 

Proof. From symmetry it is enough to prove part (a). Consider inversion with respect to point $M_p$ and use standard notation for images.

Further, according to Lemma 7 $ABCD$ is indirectly similar to $C'D'A'B'$. In this similarity $Q$ (intersection of $AB$ and $CD$) corresponds to $R'$ (intersection of $C'D'$ and $A'B'$). Analogously $R$ corresponds to $Q'$. Also, $M_r$ (second intersection of $(QBD)$ and $(QAC)$) corresponds to $M'_q$ (second intersection of $(R'D'B')$ and $(R'C'A')$) and $M_p$ (second intersection of $(RAB)$ and $(RCD)$) corresponds to $M_p$ (second intersection of $(Q'C'D')$ and $(Q'A'B')$).

From these observations we can deduce that $\triangle M_p M_r R \sim \triangle M_p M'_q Q'$ (indirectly). From inversion we also have $\triangle M_p M'_q Q' \sim \triangle M_p M_q Q$ (indirectly). Altogether we have the direct similarity of $\triangle M_p M_r R$ and $\triangle M_p M_q Q$, which implies the result. □

3. The pseudocircumcenter

It is well-known (see e.g. [6]) that in the case of a cyclic quadrilateral $ABCD$ inscribed in a circle centered at $O$ the triangle $PQR$ has $M_p$, $M_q$, and $M_r$ as its feet of altitudes and point $O$ as its orthocenter. Also, the Euler-Poncelet point $X$ is symmetric to $O$ with respect to the centroid $G$ of $ABCD$.

We introduce a point $O$ associated to a (not necessarily cyclic) complete quadrilateral which inherits most of these properties.

Theorem 9. The circles $(PM_qM_r)$, $(QM_rM_p)$, and $(RM_pM_q)$ meet in one point.
Proof. Define $O$ as the second intersection of $(QM_r, M_p)$ and $(RM_p, M_q)$. Then
\[ \angle(M_q O, OM_r) = \angle(M_q O, OM_p) + \angle(M_p O, OM_r) \]
\[ = \angle(M_q R, RM_p) + \angle(M_p Q, QM_r). \]

Now using the circles from Proposition 5 we obtain
\[ \angle(M_q R, RM_p) = \angle(M_q R, RA) + \angle(AR, RM_p) = \angle(M_q C, CA) + \angle(AB, BM_p) \]
and likewise
\[ \angle(M_p Q, QM_r) = \angle(M_p Q, QC) + \angle(CQ, QM_r) = \angle(M_p B, BC) + \angle(CA, AM_r), \]
\[ \angle(M_r P, PM_q) = \angle(M_r P, PB) + \angle(BP, PM_q) = \angle(M_r A, AB) + \angle(BC, CM_q) \]

After careful inspection, the three right-hand sides add up to 0, hence
\[ \angle(M_q O, OM_r) - \angle(M_q P, PM_r) = 0 \]
and the conclusion follows. \hfill \square

We call this point $O$ the pseudocircumcenter of $ABCD$.

**Theorem 10.** The lines $PM_p$, $RM_r$, $QM_q$ meet in $O$.

Proof. It suffices to prove $O$, $R$, and $M_r$ are collinear. With the use of Proposition 8, $M_q$ is the center of spiral similarity which takes $M_r$ to $R$ and $P$ to $M_p$, therefore $\triangle M_q M_r P \sim \triangle M_q R M_p$ (directly) and so $\angle(M_r P, PM_q) = \angle(R M_p, M_p M_q)$. We can now conclude after using the circles $(M_q M_r PO)$ and $(M_p M_q RO)$ as follows:
\[ \angle(M_r O, OM_q) = \angle(M_r P, PM_q) = \angle(R M_p, M_p M_q) = \angle(RO, OM_q) \]
which immediately implies collinearity of $O$, $R$, $M_r$. \hfill \square
Theorem 11.
(a) The circles \((M_{AB}M_{AC}M_{AD})\), \((M_{BA}M_{BC}M_{BD})\), \((M_{CA}M_{CB}M_{CD})\), and \((M_{DA}M_{DB}M_{DC})\) all pass through \(O\).
(b) \(O\) and \(X\) are symmetric with respect to \(G\).

**Proof.** From symmetry, it suffices to prove part (a) for one of the mentioned circles, for example \((M_{AB}M_{AC}M_{AD})\). Using midlines in triangles \(ACD\) and \(ABC\), we obtain

\[\angle(M_{AD}M_{AC}, M_{AC}M_{AB}) = \angle(CD, CB).\]

Further, point \(O\) lies on \((RM_pM_{AD})\) and \((QM_pM_{AB})\) (consult Proposition 5(d) and Theorem 9), hence (using also the basic circles from Proposition 5(c))

\[\angle(M_{AD}O, OM_{AB}) = \angle(M_{AD}O, OM_p) + \angle(M_pO, OM_{AB})\]
\[= \angle(DR, RM_p) + \angle(M_pQ, QB)\]
\[= \angle(DC, CM_p) + \angle(M_pC, CB) = \angle(DC, CB).\]

Hence \(O\) indeed lies on \((M_{AB}M_{AC}M_{AD})\).

For part (b) apply the symmetry with respect to \(G\). It is well-known that \(M_{AB}, M_{AC}, M_{AD}, M_{BC}, M_{BD}, M_{CD}\) are sent to \(M_{CD}, M_{BD}, M_{BC}, M_{AD}, M_{AC}, M_{AB}\), respectively. Therefore the circles \((M_{AB}M_{AC}M_{AD})\), \((M_{BA}M_{BC}M_{BD})\), \((M_{CA}M_{CB}M_{CD})\), \((M_{DA}M_{DB}M_{DC})\) are sent to the nine-point circles of the triangles \(BCD, ACD, ABD, ABC\), respectively and these circles have \(X\) as their common point. Hence \(X\) is the image of \(O\). \(\square\)
4. Proof of Theorem 1

**Lemma 12.** Reflect a triangle $ABC$ about a point $P$ in its plane to triangle $A'B'C'$. Then the circles $(AB'C')$, $(BC'A')$, $(CA'B')$ are concurrent on $(ABC)$.

![Figure 8](image_url)

**Proof.** Intersect $(ABC)$ and $(AB'C')$ at $X$. Then angle-chase using the circles and parallel lines

$$
\angle(BX, XC') = \angle(BX,XA) + \angle(AX,XC')
$$
$$
= \angle(BC, CA) + \angle(AB', B'C')
$$
$$
= \angle(BC, A'C') + \angle(A'B, BC)
$$
$$
= \angle(BA', A'C'),
$$
which proves that $X$ lies on $(BC'A')$. Analogously, we prove it lies on $(CA'B')$. \[\square\]

**Theorem 1.** Point $X$ lies on $(PQR)$.

**Proof.** Let us denote by $P'$, $Q'$, $R'$ the reflections of $P$, $Q$, $R$, respectively, about the centroid $G$. From Theorem 11, it suffices to prove that $O$ lies on the circle $(P'Q'R')$. We will prove that $O$ lies on the circles $(P'QR)$, $(Q'RP)$, $(R'PQ)$ and then the result will follow from Lemma 12 applied on triangle $P'Q'R'$. In fact, we only need (by symmetry) to prove the circle $(P'QRO)$.

The line $M_{AB}M_{CD}$ is the Newton-Gauss line of $ABCD$ so it passes through the midpoint of $PR$. As it also passes through $G$, it is the midline in $\triangle RPP'$. Similarly, we prove that $M_{AD}M_{BC}$ is the midline in $\triangle QPP'$. It follows that

$$
\angle(RP', P'O) = \angle(M_{AB}M_{CD}, M_{AD}M_{BC}).
$$

At the same time using Theorems 9 and 10 we obtain

$$
\angle(RO, OQ) = \angle(M'rO, OQ) = \angle(M'rM_{CD}, M_{CD}Q) = \angle(M'rM_{CD}, CD).
$$
But in Proposition 6 we proved the direct similarity
$$\triangle M_r CD \sim \triangle M_{BA} M_{BC} M_{AD}.$$ 
As the angles $\angle (M_r M_{CD}, CD)$ and $\angle (M_{AB} M_{CD}, M_{AD} M_{BC})$ correspond in this similarity (angles by medians), they are equal. It follows that
$$\angle (RP', P'Q) = \angle (RO, OQ),$$
which concludes the entire proof.

References


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A Note on Reflections

Emmanuel Antonio José García

Abstract. We prove some simple results associated with the triangle formed by the reflections of a point in the midpoints of the sides of a given triangle.

Let \(ABC\) be a given triangle with midpoints \(M_a, M_b, M_c\) of the sides \(BC, CA, AB\) respectively. Consider a point \(P\) and its reflections \(X, Y, Z\) in \(M_a, M_b, M_c\) respectively.

**Proposition 1.** Triangle \(XYZ\) is oppositely congruent to \(ABC\) at the complement (inferior) of \(P\).

**Proof.** Let \(G\) be the centroid of triangle \(ABC\). Consider triangle \(APX\). It has the segment \(AM_a\) as a median, and so has centroid \(G\). If \(Q\) is the midpoint of \(AX\), then \(PQ\) is another median of triangle \(APX\). Therefore, \(G\) divides \(PQ\) in the ratio \(PG : GQ = 2 : 1\), and \(Q\) is the complement of \(P\).

Similarly, the same point \(Q\) is the midpoint of \(BY\) and \(CZ\). It follows that \(XYZ\) is oppositely congruent to \(ABC\) at \(Q\).

\(\square\)

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Let $X^*, Y^*, Z^*$ be the reflections of $X, Y, Z$ in the sidelines $BC, CA, AB$ respectively.

**Proposition 2.** The circle through $X^*, Y^*, Z^*$ has center $O$ and contains the $P$ and its reflection in $O$.

**Theorem 3.** If the Euler lines of $PBC$, $PCA$, $PAB$ are concurrent at $S$, then the Euler lines of $AZY$, $BXZ$, $CYX$ are concurrent at the superior (anticomplement) of $S$.

**Proof.** Triangle $AZY$ is a translation of triangle $PBC$.

\[
Y - C = (C + A - P) - C = A - P = (A + B - P) - B = Z - B.
\]

Clearly the two Euler lines of the two triangles are parallel. Since the centroid of $AZY$ is the superior of the centroid of $PBC$, every point on the Euler line of $AZY$ is the superior of a point on the Euler line of $PBC$.

The same is true for the pairs $BXZ$, $PCA$ and $CYX$, $PAB$. It follows that if the Euler lines of $PBC$, $PCA$, $PAB$ are concurrent at $S$, then those of $AZY$, $BXZ$, $CYX$ are concurrent at the superior (anticomplement) of $S$. □

**Remark.** It is well known that for $P = I$, the incenter, the Euler lines of $IBC$, $ICA$, $IAB$ are concurrent at the Schiffler point $X(21)$ on the Euler line of $ABC$ (see [4]). It follows that the Euler lines of $AYZ$, $BZX$, $CXY$ are concurrent at the superior (anticomplement) of $X(21)$ (see Figure 3). This is the triangle center $X(2475)$ of [5], also on the Euler line of $ABC$. 

![Figure 2](image-url)
On the other hand, Yiu [8] has noted that, for $P = I$, the Euler lines of $XBC$, $YCA$, $ZAB$ are concurrent at the cevian quotient $Q/I$, where $Q$ is the Spieker center, the inferior (complement) of $I$.

**Lemma 4.** For $P = I$, the incenter, the line $AX$ intersects the Euler line of triangle $XBC$ on the side $BC$.

**Proof.** Let $I_a$ be the center of the excircle tangent to $BC$ at $T'_a$. Denote by $r$ and $r_a$ the inradius and radius of the $A$-excircle. We shall make use of the formulas $r = \frac{\Delta}{s}$ and $r_a = \frac{\Delta}{s-a}$, where $\Delta$ and $s$ are the area and semiperimeter of triangle $ABC$.

Since $IX$ and $BC$ have a common midpoint $M_a$, $IBXC$ is a parallelogram. Therefore, $BX$ and $CX$ are perpendicular to $CI_a$ and $BI_a$ respectively. From
this we note that $X$ is the orthocenter of triangle $I_aBC$. Consequently, $I_a$ is the orthocenter of triangle $XBC$.

Let $G_a$ be the point dividing $XM_a$ in the ratio $XG_a : G_aM_a = 2 : 1$. This is the centroid of triangle $XBC$, and $I_aG_a$ is the Euler line.

Extend $I_aG_a$ to intersect $AT'_a$ at $G'_a$. Since the line $AT'_a$ contains the antipode of $T_a$ on the incircle, it is parallel to $IX$. Therefore, $AG'_a : G'_aT'_a = TG_a : G_aX = 2 : 1$.

Let $AI_a$ intersect $BC$ at $K$. Consider triangle $AT'_aI_a$ with $X$ on $T'_aI_a$, $K$ on $I_aA$, and $G'_a$ on $AT'_a$. We have

$$\frac{AG'_a}{G'_aT'_a} \cdot \frac{T'_aX}{XI_a} \cdot \frac{I_aK}{KA} = 1.$$  

By Ceva’s theorem, $AX, I_aG'_a$ and $T'_aK$ are concurrent. This means that $AX$ and the Euler line $I_aG_a$ of triangle $XBC$ intersect on $BC$.

**Theorem 5.** For $P = I$, the incenter, the Euler lines of triangles $XBC$, $YCA$, $ZAB$ are concurrent at the cevian quotient $Q/I$, where $Q$ is the Spieker center.
Proof. The line $AX$ contains the Spieker center $Q$ as its midpoint. Denote by $X'Y'Z'$ the cevian triangle of $Q$. By Lemma 4 above, the Euler line of triangle $XBC$ is $I_aX'$. Similarly, the Euler lines of $YCA$ and $ZAB$ are the lines $I_bY'$ and $I_cZ'$. Now, $I_aI_bI_c$ is the anticevian triangle of $I$, and $XYZ$ is the cevian triangle of $S$. These lines are concurrent at the cevian quotient $Q/I$ (see [7, §8.3]). □

Remark. For $Q =$ the Spieker center, the cevian quotient $Q/I$ is the triangle center $X(191)$ of [5]. It is the reflection of $I$ in the Schiffler point $X(21)$ (see Remark after Theorem 3 above).

**Theorem 6** (Collings). The circles $(AYZ)$, $(BZX)$, $(CXY)$ are concurrent at a point on the circumcircle of $ABC$, which is the superior of the center of the rectangular hyperbola through $A$, $B$, $C$, and $P$.

Proof. Collings [1] actually shows that if $A'B'C'$ is the superior (anticomplementary) triangle of $ABC$, and $P'$ is the superior of $P$, then the circles $(AYZ)$, $(BZX)$, $(CXY)$ and $(ABC)$ intersect at the center $M'$ of the rectangular hyperbola through $A'$, $B'$, $C'$, $P'$ (see Figure 3). Since $A'$, $B'$, $C'$, $P'$ are the superiors of $A$, $B$, $C$, $P$ respectively, $M'$ is the superior of the center $M$ of the rectangular hyperbola through $A$, $B$, $C$, $P$. This is a point on the nine-point circle of $ABC$. It follows that $M'$ is a point on the circumcircle. □
For example, if \( P = I \), the incenter, the rectangular hyperbola through \( A, B, C, I \) (and \( H \)) has center the Feuerbach point \( F \). The common point of the circles \((AZY), (BXZ), (CYX)\) is the inferior of \( F \). This is the point \( X(100) \).

Peter Moses has informed us [2], among other things, that triangle \( XYZ \) is perspective to the mid-arc triangle at \( X(100) \) (see Figure 7). The vertices of the mid-arc triangle are the intersections of the angle bisectors with the circumcircle. Since the inferiors (complements) of \( X, Y, Z \) are the midpoints \( X'', Y'', Z'' \) of \( IA, IB, IC \) respectively, it is enough to prove that \( X''Y''Z'' \) is perspective with the mid-arc triangle of the medial triangle \( M_aM_bM_c \).

\[
\text{Figure 7}
\]

**Proposition 7.** Let \( X'', Y'', Z'' \) be the midpoints of \( IA, IB, IC \) respectively, and \( M'_a, M'_b, M'_c \) the midpoints of the the arcs \( M_bM_c, M_cM_a, M_aM_b \) of the nine-point circle of triangle \( ABC \) not containing \( M_a, M_b, M_c \). The triangles \( X''Y''Z'' \) and \( M'_aM'_bM'_c \) are perspective at the Feuerbach point of triangle \( ABC \).

**Proof.** The nine-point circle of triangle \( ABC \) is tangent to the incircle at the Feuerbach point \( F \). Let \( M'_a \) be the midpoint of the arc \( M_bM_c \) of the nine-point circle (not containing \( M_a \)).

\[
\angle M'_aFMc = \frac{1}{2} \angle MbFc = \frac{1}{2} \angle M_bM_aM_c = \frac{1}{2} \angle BAC. \tag{1}
\]

On the other hand, if \( X'' \) is the midpoint of \( IA \), then the circle through \( X'', M_c \) and \( T_c \) is the nine-point circle of triangle \( IAB \), and it passes through the Feuerbach point \( F \) as well (see Remark below), and

\[
\angle X''FM_b = \angle X''T_bA = \angle X''AT_b = \frac{1}{2} \angle BAC. \tag{2}
\]
It follows from (1) and (2) that $M'_a$, $X''$, and $F$ are collinear.

The same reasoning shows that $M'_b$, $Y''$, and $F$ are collinear, so are $M'_c$, $Z''$ and $F$. Therefore, the triangles $X''Y''Z''$ and $M'_aM'_bM'_c$ are perspective at the Feuerbach point. \hfill \Box

**Remark.** The nine-point circles of $IBC$, $ICA$, $IAB$, and $ABC$ are concurrent at the center of the rectangular hyperbola through $A$, $B$, $C$, $I$. This is the Feuerbach point $F$. See Theorem 6 above. A synthetic proof of this fact can be found in [6].

**References**


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Antirhombi

Nikolaos Dergiades

Abstract. First we give the definition and some properties of the non-rhombus quadrilateral that we call antirhombus [1], that is circumscribed around a circle with center the centroid of the quadrilateral. Then we try to cut a triangle $ABC$ with a line to form an antirhombus, we prove that there are three such lines forming with the sides of $ABC$ an hexagon circumscribed around the incircle of $ABC$ and then investigate their interesting configuration.

1. Circumscribed quadrilaterals with the same incenter and centroid

Let $ABCD$ be a quadrilateral circumscribed around a circle with center $O$ and radius $r$, and $x, y, z, w$ be the distances of the vertices $A, B, C, D$ from the points of tangency $T_a, T_b, T_c, T_d$ (Figure 1).

We have

\[ x = r \cot \frac{A}{2}, \quad z = r \cot \frac{C}{2}, \quad OA = \frac{r}{\sin \frac{A}{2}}, \quad OC = \frac{r}{\sin \frac{C}{2}}. \]

Hence,

\[ x + z = r \left( \cot \frac{A}{2} + \cot \frac{C}{2} \right) = r \cdot \frac{\sin \frac{A+C}{2}}{\sin \frac{A}{2} \sin \frac{C}{2}}, \]

or

\[ \frac{OA \cdot OC}{x + z} = \frac{r}{\sin \frac{A+C}{2}}. \]
Similarly, we have

\[ \frac{OB \cdot OD}{y + w} = \frac{r}{\sin \frac{B+D}{2}}. \]

Since \( \sin \frac{A+C}{2} = \sin \frac{B+D}{2} \), we have

\[ \frac{OA \cdot OC}{x + z} = \frac{OB \cdot OD}{y + w}. \]  

(1)

It is obvious that every rhombus is a circumscribed parallelogram and its center is both the incenter and centroid. If the centroid of a circumscribed parallelogram is also its incenter, then this parallelogram is a rhombus. We will investigate this double property for other quadrilaterals.

(1) It is easy to see that if the incenter of a circumscribed trapezium (Figure 2) is also the centroid, i.e., the midpoint of its median, then from the right angled triangles \( OAB \) and \( ODC \) we get \( AB = 2MO = 2ON = CD \). Hence the trapezium is isosceles with \( AB = CD = MN \).

![Figure 2](image.png)

(2) If in a circumscribed trapezium \( ABCD \) (with bases \( BC \) and \( AD \)) we have \( OA \cdot OC = OB \cdot OD \), then

\[ \frac{r}{\sin \frac{A}{2}} \cdot \frac{r}{\sin \frac{C}{2}} = \frac{r}{\sin \frac{B}{2}} \cdot \frac{r}{\sin \frac{D}{2}} \]

\[ \Rightarrow \sin \frac{A}{2} \cos \frac{D}{2} = \cos \frac{A}{2} \sin \frac{D}{2} \]

\[ \Rightarrow \tan \frac{A}{2} = \tan \frac{D}{2} \]

\[ \Rightarrow A = D. \]

Hence the trapezium is isosceles and the incenter \( O \) is also its centroid.

We will generalize this property by adopting the following definition formulated by George Baloglou [1].

**Definition.** An antirhombus is a circumscribed quadrilateral \( ABCD \) with incenter \( O \) that satisfies the condition

\[ OA \cdot OC = OB \cdot OD. \]
Theorem 1. A circumscribed quadrilateral $ABCD$ with incenter $O$ is an antirhombus if and only if $x + z = y + w$.

Proof. It is obvious from equality (1). □

Theorem 2 ([2, Theorem 13]). A circumscribed quadrilateral with no parallel sides has the same incenter and centroid if and only if it is an antirhombus.

![Figure 3](image-url)

Proof. (1) Let $ABCD$ be a circumscribed quadrilateral with centroid and incenter the point $O$, and $E$, $F$ be the intersection points of the opposite sides $AB$, $CD$ and $BC$, $AD$ (Figure 3). This point $O$ must be the midpoint of the bimedians $M_aM_c$, $M_bM_d$. Let $T_a$, $T_b$, $T_c$, $T_d$ be the tangency points with the incircle. The triangle $EM_aM_c$ is isosceles because $EO$ is a bisector and a median. The triangle $ET_aT_c$ is also isosceles. Therefore, $M_aM_c$ and $T_aT_c$ are parallel, and are both perpendicular to $EO$. If $x > y$, then $w > z$, and we have $\frac{x-y}{2} = T_aM_a = T_cM_c = \frac{w-z}{2}$. It follows that $x + z = y + w$, and $ABCD$ is an antirhombus. Similarly, $M_bM_d$ and $T_bT_d$ are parallel, and are both perpendicular to $FO$.

(2) If $ABCD$ is an antirhombus with incenter $O$, then we have $x + z = y + w$. If $x > y$, then $w > z$ and $T_aM_a = \frac{x-y}{2} = \frac{w-z}{2} = T_cM_c$. The right angled triangles $OT_aM_a$ and $OT_cM_c$ are congruent. Hence, $OM_a = OM_c$. This means that the incenter $O$ lies on the perpendicular bisector $L_1$ of the bimedian $M_aM_c$. Similarly $O$ also lies on the perpendicular bisector $L_2$ of the bimedian $M_bM_d$. But the only common point of $L_1$ and $L_2$ is the centroid of $ABCD$, the common midpoint of the bimedians. Hence the antirhombus has the same incenter and centroid. Again we have $M_aM_c \parallel T_aT_c$, $M_aM_c \perp EO$ and the same for the other bimedian $M_bM_d$. □

Corollary 3. A circumscribed quadrilateral is an antirhombus if and only if a bimedian connecting two opposite sides is perpendicular to the bisector of the angle of these sides and hence parallel to the chord of the corresponding contact points.
2. Antirhombi from a triangle

Let $ABC$ be a triangle with intouch triangle $A_1B_1C_1$. We begin with the construction of the unique line $L_a$ which cuts the sidelines of $ABC$ at the points $A_a$, $A_b$, $A_c$ and tangent to the incircle of $ABC$ at a point $A_2$ such that the quadrilateral $BCA_bA_c$ is an antirhombus.

Construction 4. Let the perpendicular to $AI$ at $I$ intersect $AB$ at $J$ and $AC$ at $K$. For the quadrilateral $BCA_bA_c$ to be an antirhombus it is sufficient from Theorem 3 that the line $JK$ (Figure 4) be a bimedian of the antirhombus. Hence the points $A_c$ and $A_b$ are the symmetrics of $B, C$ in $J, K$ respectively. The line $A_bA_c$ is the line $L_a$.

![Figure 4](image)

Similarly, we construct the lines $L_b$ and $L_c$ intersecting the sidelines at $B_a$, $B_b$, $B_c$, and $C_a$, $C_b$, $C_c$ respectively such that $CAB_cB_a$ and $ABC_aC_b$ are antirhomobi.

The following property gives easily the barycentric coordinates of the above points.

Proposition 5. The lines $B_cC_b$, $C_aA_c$, $A_bB_a$ are parallel to the sides of $ABC$ and are concurrent at the image of the centroid $G$ under the homothety $h(I, -3)$.

Proof: The parallel from $A_b$ to $AC$ meets the parallel from $A_c$ to $AB$ at a point $P$. Let $A'$ be the midpoint of $A_bA_c$ (Figure 4). $G$ is the centroid of $ABC$ so we have

$$
GA + GB + GC = 0.
$$

Since $I$ is the centroid of the quadrilateral $BCA_bA_c$,

$$
2IA_b + IA_c + IB + IC = 0,
$$

$$
2IA_c + IB + IC = 0,
$$

$$
2IA + IB + IC = 0,
$$

$$
2IP + (IG + GA) + (IG + GB) + (IG + GC) = 0.
$$
Hence $\overrightarrow{IP} = -3\overrightarrow{IG}$, which means that $P$ is the image of $G$ under the homothety $h(I, -3)$. From $P = 4I - 3G$, we obtain

$$P = (3a - b - c : 3b - c - a : 3c - a - b)$$

in homogeneous barycentric coordinates. Similarly, the lines $B_cC_b$, $C_aA_c$, and $A_bB_a$ are parallel to the sides of $ABC$ and concurrent at $P$. □

So we have an interesting special case mentioned in [5, §12.1.2] and hence in homogeneous barycentric coordinates we have

$$C_b = (u : 0 : v + w), \quad B_c = (u : v + w : 0);$$
$$A_c = (w + u : v : 0), \quad C_a = (0 : v : w + u);$$
$$B_a = (0 : u + v : w), \quad A_b = (u + v : 0 : w).$$

The equations of the lines are

$$L_a := A_bA_c : -x + \frac{w + u}{v}y + \frac{u + v}{w}z = 0,$$
$$L_b := B_cB_a : \frac{v + w}{u}x - y + \frac{u + v}{w}z = 0,$$
$$L_c := C_aC_b : \frac{v + w}{u}x + \frac{w + u}{v}y - z = 0,$$

Since the lines $B_cC_b$, $C_aA_c$, $A_bB_a$ are concurrent at $P$, from the converse of Brianchon’s theorem we conclude that there is a conic $C_1$ inscribed in the hexagon $A_bA_cB_cB_aC_aC_b$. It is known [5] that the coordinates of the lines $BC$, $CA$, $AB$, $L_a$, $L_b$, $L_c$ correspond to points on the dual conic. Since these points are the vertices of $ABC$ and the points $(-1 : \frac{w + u}{v} : \frac{w + u}{v})$, $(-1 : \frac{w + u}{v} : -1)$, that all lie on the circumconic

$$\frac{v + w}{x} + \frac{w + u}{y} + \frac{u + v}{z} = 0$$

the dual conic, which is tangent to the 6 lines is the conic with equation

$$\sum_{\text{cyclic}} (v + w)^2x^2 - 2(w + u)(u + v)yz = 0.$$ 

This conic has center

$$O_1 = (2u + v + w : u + 2v + w : u + v + 2w).$$

In our case with $P = X_{145}$ given in (2) above, we have

$$\sum_{\text{cyclic}} (s - a)^2x^2 - 2(s - b)(s - c)yz = 0,$$

which is clearly the incircle with center $I$. Hence, the incenter $I$ of $ABC$ is also the incenter of the quadrilaterals $CAB_cB_a$ and $ABC_aC_b$.

These quadrilaterals are convex if and only if for the sides $a \leq b \leq c$ of triangle $ABC$, we have $3a - b - c > 0$. 

Proposition 6. The tangency points $A_2, B_2, C_2$ of the lines $L_a, L_b, L_c$ are the vertices of a triangle perspective with $ABC$.

Proof. The point $A_2$ is the pole of $L_a$ with respect to $C_1$:

$$A_2 = (-wv \ w(w + u) \ v(u + v)) \begin{pmatrix} 0 & u + v & w + u \\ u + v & 0 & v + w \\ w + u & v + w & 0 \end{pmatrix}$$

$$= ((v + w)(w + u)(u + v) \ v^2(u + v) \ w^2(w + u)),$$

or

$$A_2 = \begin{pmatrix} v + w & v^2 & w^2 \\ v + w & u + v & u + v \end{pmatrix}.$$

Similarly, from the coordinates of the points $B_2, C_2$ we conclude the triangle $A_2B_2C_2$ is perspective with $ABC$ at the point

$$\begin{pmatrix} u^2 & v^2 & w^2 \\ v + w & w + u & u + v \end{pmatrix}.$$

□

In our case with $P = X_{145}$, the perspector is

$$Q = \begin{pmatrix} (3a - b - c)^2 & (3b - c - a)^2 & (3c - a - b)^2 \\ b + c - a & c + a - b & a + b - c \end{pmatrix}.$$

This point is not in the current edition of the ENCYCLOPEDIA OF TRIANGLE CENTERS [4]. It has (6-9-13)-search number 0.0267031360104 ···. This divides the line $IG_e$ in the ratio

$$IQ : QG_e = 4R + r : -8r.$$

The point of tangency with $BC$ is

$$A_1 = (1 \ 0 \ 0) \begin{pmatrix} 0 & u + v & w + u \\ u + v & 0 & v + w \\ w + u & v + w & 0 \end{pmatrix} = (0 \ u + v \ w + u).$$

Proposition 7. The hexagon $A_bA_cB_cB_aC_aC_b$ is inscribed in a conic.

Proof. Since

$$\frac{BB_a}{B_aC} \cdot \frac{BC_a}{C_aC} \cdot \frac{CC_b}{C_bA} \cdot \frac{CA_b}{A_bA} \cdot \frac{AA_c}{A_cB} \cdot \frac{AB_c}{B_cB} = \frac{w}{u + v} \cdot \frac{w + u}{v} \cdot \frac{u + v}{v + w} \cdot \frac{v + w}{w + u} \cdot \frac{v}{u} = 1,$$

from Carnot’s theorem we conclude that the hexagon $A_bA_cB_cB_aC_aC_b$ is inscribed in a conic $\mathcal{C}_2$ that has [5, p.141] equation

$$\sum_{\text{cyclic}} v(w + u)x^2 - u(vw + (w + u)(u + v))yz = 0.$$
The center of this conic is the point
\[ O_2 = \left( u(2vw+u(v+w-u)) : v(2wu+v(w+u-v)) : w(2uv+w(u+v-w)) \right), \]
which is the midpoint of \( P \) and \( G/P \). □

**Proposition 8.** The points \( A_a, B_b, C_c \) lie on the trilinear polar of \( X_{5435} \).

**Proof.** The lines \( L_a, L_b, L_c \) meet the sides \( BC, CA, AB \) at the points \( A_a, B_b, C_c \) respectively. These are collinear on the Pascal line of the hexagon \( A_bA_cB_cB_aC_aC_b \).
The line \( L_a \) meets \( BC \) at the point \( A_a = \left( 0 : -\frac{v}{u+w} : \frac{w}{u+v} \right) \), which is the intersection of \( BC \) with the trilinear polar of the point \( Q = \left( \frac{u}{v+w} : \frac{v}{w+u} : \frac{w}{u+v} \right) \). For \( P = X_{145} \), this is the point \( X_{5435} = \left( \frac{3a-b-c}{b+c-a} : \frac{3b-c-a}{c+a-b} : \frac{3c-a-b}{a+b-c} \right) \); see [3]. The same holds for \( B_b \) and \( C_c \). □

**Proposition 9.** The Brianchon points of the quadrilaterals \( A_bA_cBC, B_cB_aCA, C_aC_bAB \) are the vertices of a triangle \( P_1P_2P_3 \) which is perspective with \( ABC \) at \( X_{5435} \).

**Proof.** The Brianchon point \( P_1 \) of the circumscribed quadrilateral \( A_bA_cBC \) is the common point of the diagonals \( BA_b, CA_c \) and the contact chords \( A_1A_2, B_1C_1 \) (Figure 5). Hence
\[ P_1 = \left( (w+u)(u+v) : v(u+v) : w(w+u) \right) \]
and the line \( AP_1 \) passes through \( X_{5435} \). Similarly, the lines \( BP_2 \) and \( CP_3 \) also pass through the same point. □
Proposition 10. The triangle $P_1P_2P_3$ is perspective with the contacts triangle $A_1B_1C_1$ at $P$.

Proof. The points $P, A_1, A_2$ are collinear because
\[
\begin{vmatrix}
  u & v & w \\
  0 & u + v & w + u \\
  v + w & \frac{v^2}{w+u} & \frac{w^2}{w+u}
\end{vmatrix} = u(w^2 - v^2) + (v + w)(w(u + v) - w(u + v)) = 0.
\]
Hence the line $A_1P_1$ passes through $P$, and the same holds for the lines $B_1P_2$ and $C_1P_3$.

Since the line $A_1A_2$ is the polar of the point $A_a$ and passes through $P$, we conclude that the line $L$ of the points $A_a, B_b, C_c$ (which is the tripolar of $X_{5435}$) is the polar of $P$ relative to the incircle of $ABC$. □

Proposition 11. The lines $L_a, L_b, L_c$ bound a triangle perspective with $ABC$ at $P$.

Proof. It is easy to see that the lines $L_b$ and $L_c$ meet at (Figure 6) the point $A_3 = (-\frac{u^2}{v+w} : v : w)$. Similarly $L_c$ and $L_a$ meet at $B_3 = (u : -\frac{v^2}{w+u} : w)$, and $L_a$ and $L_b$ meet at $C_3 = (u : v : -\frac{w^2}{u+v})$. From these coordinates it is clear that $A_3B_3C_3$ and $ABC$ are perspective at $P = (u : v : w)$. □

Proposition 12. The points $A, B, C, A_3, B_3, C_3$ lie on a conic $C_3$, and the centers of the conics $C_1, C_2, C_3$ are collinear.

Figure 6
Proof. It is easy to see that the conic $\mathcal{C}_3$ in question is

$$\frac{u^2}{x} + \frac{v^2}{y} + \frac{w^2}{z} = 0$$

with center

$$O_3 = (u(v^2 + w^2 - u^2) : v^2(w^2 + u^2 - v^2) : w^2(u^2 + v^2 - w^2)).$$

The centers $O_1, O_2, O_3$ are all on the line

$$\sum_{\text{cyclic}} (v - w)(u(v + w) - (v^2 + vw + w^2))x = 0.$$

□

References


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Asymptotic Directions of Pivotal Isocubics

Bernard Gibert

Abstract. Given the pivotal isocubic $K = pK(\Omega, P)$, we seek all the other isocubics $K_1 = pK(\Omega_1, P_1)$ with pole $\Omega_1$ and pivot $P_1$ which have the same points at infinity, i.e., the same asymptotic directions. We also examine the connection with the Simson lines concurring at a certain given point.

1. Generalities

Recall that the pivotal isocubic $K = pK(\Omega, P)$ is the locus of point $M$ such that $P, M$ and the $\Omega$–isoconjugate $M^*$ of $M$ are collinear. With a pole $\Omega(p : q : r)$ and a pivot $P(u : v : w)$, its barycentric equation is

$$\sum_{\text{cyclic}} ux(ry^2 - qz^2) = 0 \iff \sum_{\text{cyclic}} pyz(wy - vz) = 0$$ (1)

We denote by $\mathcal{F}_K$ the family of all pivotal isocubics having the same points at infinity as $K$.

Theorem 1 (Pole theorem). Given $K = pK(\Omega, P)$, a pivotal isocubic $K_1 = pK(\Omega_1, P_1)$ has the same points at infinity as $K$ if its pole $\Omega_1$ lies on the cubic $K_\Omega$ with equation

$$\sum_{\text{cyclic}} ux(y + z - x)(ry - qz) = 0$$ (2)

$$\iff \sum_{\text{cyclic}} pyz(v(x - y + z) - w(x + y - z)) = 0.$$ (3)

Theorem 2 (Pivot theorem). Given $K = pK(\Omega, P)$, a pivotal isocubic $K_1 = pK(\Omega_1, P_1)$ has the same points at infinity as $K$ if its pivot $P_1$ lies on the cubic $K_P$ with equation

$$\sum_{\text{cyclic}} u(y + z)(ry(x + z) - qz(x + y)) = 0$$ (4)

$$\iff \sum_{\text{cyclic}} p(x + y)(x + z)(wy - vz) = 0.$$ (5)
1.1. Properties of $K_\Omega$. $K_\Omega$ is the pseudo-isocubic $psK(\Omega \times cP, G, \Omega)$ where $cP$ is the complement of $P$ and $X_\Omega = \Omega \times cP$ is the barycentric product of $\Omega$ and $cP$. See [3].

$K_\Omega$ is a circum-cubic passing through $\Omega$ and the midpoints $M_a, M_b, M_c$ of $BC$, $CA$, $AB$.

The tangents at $A, B, C$ concur at $X_\Omega$. This point lies on $K_\Omega$ when $G, \Omega, P$ are collinear in which case $K_\Omega$ is a $pK$ (see §2).

The equation (2) shows that, for a given $\Omega$, all $K_\Omega$ form a net of circum-cubics which is generated by three decomposed cubics, one of them being the union of the line $BC$, the line through the midpoints of $AB$ and $AC$, the line $A\Omega$, the other two similarly. Generally, this net contains only one circular cubic and only one equilateral cubic.

1.2. Properties of $K_P$. $K_P$ is the anticomplement of the pseudo-isocubic $K'_P = psK(\Omega \times cP, G, cP)$. See [3].

$K_P$ is a circum-cubic passing through $P$ and the vertices $G_a, G_b, G_c$ of the antimedial triangle. Moreover, it has the same points at infinity as $K$ hence, $K_P$ is a circular cubic (an equilateral cubic) if and only if $K$ is itself a circular cubic (an equilateral cubic).

The tangents at $G_a, G_b, G_c$ concur at a point which is the anticomplement of $X_\Omega$. This point lies on $K_P$ when $G, \Omega, P$ are collinear in which case $K_\Omega$ is also a $pK$ (see §2).

$K_P$ is itself a pseudo-isocubic if and only if its tangents at $A, B, C$ concur which is realized when $G, \Omega, P$ are collinear as above but also when $P$ lies on the circum-conic with center $\Omega$ or equivalently when $\Omega$ lies on the bicevian conic $C(G, P)$ with center $ccP$, the complement of $cP$. In this latter case, the pseudo-pivot of $K_P$ lies on the Steiner ellipse and the pseudo-pole lies on $psK(ta\Omega, t(G/\Omega), G)$.

The equation (5) shows that, for a given $P$, all $K_P$ form another net of circum-cubics which is generated by three decomposed cubics, one of them being the union of the parallels at $C, B$ to $AB$, $AC$ respectively and the line $AP$, the other two similarly.

1.3. A special case. With $\Omega = P^2$ (barycentric square), $K_\Omega$ (resp. $K_P$) is the locus of poles (resp. pivots) of all $pK$ having asymptotes parallel to the cevian lines of $P$.

1.4. Examples. We show several examples with known cubics for which $G, \Omega, P$ are not collinear.

1.4.1. $K$ is the McCay cubic. With $\Omega = K$ and $P = O$, $K$ is the McCay cubic $K_{003}$. We know that it has three real asymptotes perpendicular to the sidelines of the Morley triangle.

$K_\Omega$ is $K_{307} = psK(X_{51}, X_2, X_6)$ with equation:

$$\sum_{\text{cyclic}} a^2 S_A x(y + z - x)(c^2 y - b^2 z) = 0$$  \hspace{1cm} (6)
It passes through $K, X_{53}, X_{216}, X_{1249}$. The tangents at $A, B, C$ concur at $X_{51}$, centroid of the orthic triangle. See Figure 1.

![Figure 1. $K\Omega$ with $\Omega = K$ and $P = O$](image)

Each point on the curve is the pole of a $pK$ having asymptotes parallel to those of the McCay cubic. $pK(X_{53}, X_4) = K_{049}$ is the McCay cubic of the orthic triangle, $pK(X_{216}, X_{20}) = K_{096}$ contains $X_3, X_5, X_{20}$ and the tangential $E_{367}$ of $H$ in the Darboux cubic.

$K_P$ is denoted by $K_{O}^{++} = K_{080}$: it is a central cubic with center $O$, having three real asymptotes perpendicular to the sidelines of the Morley triangle and concurring at $O$. Each point on the curve is the pivot of a $pK$ having asymptotes parallel to those of the McCay cubic.

Its equation is:

$$
\sum_{\text{cyclic}} a^2 (x + y)(x + z)(c^2 S_C y - b^2 S_B z) = 0 \quad (7)
$$

$K_{O}^{++}$ is a circum-cubic passing through:

- $O, H, L, X_{1670}, X_{1671}$.
- the vertices $G_a, G_b, G_c$ of the antimedial triangle,
- the reflections $H_A, H_B, H_C$ of $H$ in $A, B, C$,
- the reflections $A_O, B_O, C_O$ of $A, B, C$ in $O$,
- the points $U_a, U_b, U_c$ which also lie on the Neuberg cubic and on the circle with center $L$ and radius $2R$. These points are the images under $h(H, 2)$ of the points $V_a, V_b, V_c$, intersections of the Napoleon cubic with the circumcircle. See Figure 2.

1.4.2. $K$ is the orthocubic. With $\Omega = K$ and $P = H$, $K$ is the orthocubic $K_{006}$. $K_{\Omega}$ is $K_{260}$, a nodal cubic with node $K$ and passes through $X_{69}, X_{206}, X_{219},$
The tangents at $A$, $B$, $C$ concur at $X_{184}$. Its equation is:

$$\sum_{\text{cyclic}} a^4 S_A (y - z)yz - (b^2 - c^2)(c^2 - a^2)(a^2 - b^2) xyz = 0 \quad (8)$$

An easy construction of $K_\Omega$ with $\Omega = K$ and $P = H$ is the following: the trilinear polar $q$ of any point $Q$ on the Euler line meets the lines $KM_a$, $KM_b$, $KM_c$ at $Q_a$, $Q_b$, $Q_c$. The triangles
Asymptotic directions of pivotal isocubics

ABC and $Q_aQ_bQ_c$ are perspective at $M$ and the locus of $M$ is $K_\Omega$. Furthermore, $q$ envelopes the inscribed parabola with focus $X_{112}$ and directrix the line $HK$. See Figure 3.

$K_P$ is $K617$, the anticomplement of $K009$. It is also a nodal cubic with node $H$ and it passes through $X_{20}$, $X_{68}$, $X_{254}$, $X_{315}$, $X_{2996}$. The construction seen above for $K_\Omega$ is easily adapted for $K_P$ : it is enough to replace the Euler line by the line $X_2X_{216}$ and $K$ by $H$. Similarly we obtain another inscribed parabola with focus $X_{107}$ and directrix the line $X_4X_{51}$. See Figure 4. The equation of $K_P$ is :

$$\sum_{\text{cyclic}} a^2 (x+y)(x+z)(y/S_C - z/S_B) = 0 \quad (9)$$

![Figure 4. $K_P$ with $\Omega = K$ and $P = H$](image)

In the two cubics, the tangents at the nodes are parallel to the asymptotes of the Jerabek hyperbola.

From all this, we see that each point $Q$ on the Euler line gives a pole $\omega$ on $K_\Omega$ and a corresponding pivot $\pi$ on $K_P$ such that the cubic $pK(\omega, \pi)$ has its asymptotes parallel to those of the orthocubic. For example, with $Q = G$, $Q = O$ and $Q = H$, we find $pK(X_{69}, X_{315})$, $pK(X_{577}, X_{20})$ and $pK(X_{2165}, X_{68})$ respectively.

1.4.3. A selection of cubics $K_\Omega$, $K_P$ and $K'_P$. Table 1 below shows a small selection of these cubics, specially those connected with the usual and well known cubics $K$ in triangle geometry. When a cubic is not listed in [2], a list of centers is provided.

Some of these examples are detailed below.
2. Pivotal $K_\Omega$ and $K_P$

2.1. Theorem and corollaries.

**Theorem 3.** $K_\Omega$ and $K_P$ are pivotal isocubics if and only if

$$(q - r)u + (r - p)v + (p - q)w = 0$$

i.e. if and only if $G, \Omega, P$ are collinear.

**Corollary 4.** $K_\Omega$ has pivot $G$ and pole $\omega$, the barycentric product of $\Omega$ and the complement of $P$. This point is the common tangential of $A, B, C$.

This pole lies on the line through $\Omega$, the barycentric square of $\Omega$, the complement of the isotomic conjugate of $\Omega$.

$K_\Omega$ contains the following points:
- $A, B, C, G$
- $M_a, M_b, M_c$
- $\Omega, \omega, cP$ (complement of $P$), $P^*$ ($\Omega$–isoconjugate of $P$)
- the isotomic conjugate of the anticomplement of $\omega$ and its complement

**Corollary 5.** $K_P$ is an isotomic pivotal isocubic with pivot the anticomplement of $\omega$. This point is the common tangential of $G_a, G_b, G_c$.

This pivot lies on the line through the anticomplement of $\Omega$, the isotomic conjugate of $\Omega$, the anticomplement of the barycentric square of $\Omega$.

$K_P$ contains the following points:
- $A, B, C, G$
- $G_a, G_b, G_c$
- $P$, the anticomplements of $\Omega, \omega, P^*$
- the isotomic conjugate of the anticomplement of $\omega$

**Corollary 6.** $K_\Omega$ is the complement of $K_P$. The two cubics are tangent at $G$ to the line $G\omega$.

**Corollary 7.** $K_\Omega, K_P$ have the same points at infinity which are those of $K$.

From all this, we notice that $K_\Omega$ and $K_P$ have nine common points: $A, B, C, G$ (double), the three points at infinity of $K$, the isotomic conjugate of the anticomplement of $\omega$. 

### Table 1. $\mathcal{K}$ and the related cubics $K_\Omega, K_P, K_P'$

<table>
<thead>
<tr>
<th>$\mathcal{K}$</th>
<th>$\mathcal{K}_\Omega$</th>
<th>$\mathcal{K}_P$</th>
<th>$\mathcal{K}_P'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>K001</td>
<td>$X_6, X_{1249}, X_{1989}, X_{1990}, X_{3163}$</td>
<td>K449</td>
<td>K446</td>
</tr>
<tr>
<td>K002</td>
<td>K002</td>
<td>K007</td>
<td>K002</td>
</tr>
<tr>
<td>K003</td>
<td>K307</td>
<td>K080</td>
<td>K026</td>
</tr>
<tr>
<td>K004</td>
<td>$X_6, X_{393}, X_{1249}$</td>
<td>$X_4, X_{2963}$</td>
<td>K376</td>
</tr>
<tr>
<td>K005</td>
<td>$X_6, X_{233}, X_{1249}, X_{2963}$</td>
<td>$X_4, X_5, X_{20}, X_{2888}$</td>
<td>K569</td>
</tr>
<tr>
<td>K006</td>
<td>K260</td>
<td>K617</td>
<td>K009</td>
</tr>
<tr>
<td>K034</td>
<td>K345</td>
<td>K034</td>
<td>K345</td>
</tr>
</tbody>
</table>
When we choose $\Omega$ or $P$ at $G$, we obtain:

**Corollary 8.** (1) Any isotomic $pK$ is the locus of pivots of all $pK$ having the same asymptotic directions as itself.

(2) Any $pK$ with pivot $G$ is the locus of poles of all $pK$ having the same asymptotic directions as itself.

See the three examples below.

A line $\ell_G$ through $G$ meets $K_\Omega$ at two points $\Omega_1$ and $\Omega_2$ which are $\omega$–isoconjugate and collinear with $G$.

Denote by $P_1$, $P_2$ the anticomplements of $\Omega_2$, $\Omega_1$ respectively (reverse order). These points lie on $K_P$.

**Theorem 9.** The two pivotal isocubics $K_1 = pK(\Omega_1, P_1)$ and $K_2 = pK(\Omega_2, P_2)$ have the same points at infinity as $K = pK(\Omega, P)$

Construction: given $\Omega$ and $P$ collinear with $G$, let $\Gamma_\ell$ be the inscribed conic in $ABC$ which is tangent at $G$ to $\ell_G$. The trilinear pole $Q_\ell$ of $\ell_G$ lies on the Steiner circum-ellipse, and the trilinear pole of the tangent at $Q_\ell$ to the Steiner circum-ellipse is the perspector of $\Gamma_\ell$.

Now draw the two (not necessarily real) tangents to $\Gamma_\ell$ passing through $\omega$. These tangents meet $\ell_G$ at $\Omega_2$, $\Omega_1$. The construction of $P_1$, $P_2$ follows easily.

Beware $P_1$, $P_2$ are two points on $K_P$ but are not isotomic conjugates on this cubic.

2.2. Examples.

2.2.1. $K$ is the Thomson cubic. When $K$ is the Thomson cubic $K_{002}$ with $\Omega = K$ (Lemoine point) and $P = G$ (centroid), $K_\Omega$ is the Thomson cubic again and $K_P$ is the Lucas cubic $K_{007}$. In other words, all the corresponding cubics $K_1 = pK(\Omega_1, P_1)$ and $K_2 = pK(\Omega_2, P_2)$ have the same points at infinity which are those of the Thomson and Lucas cubics.

The following table shows several examples of corresponding points $\Omega_1$, $P_1$ and $\Omega_2$, $P_2$.

<table>
<thead>
<tr>
<th>$\Omega_1$</th>
<th>$P_1$</th>
<th>cubic or $X_i$ for $i =$</th>
<th>$\Omega_2$</th>
<th>$P_2$</th>
<th>cubic or $X_i$ for $i =$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>$X_8$</td>
<td>$K_{308}$</td>
<td>$X_1$</td>
<td>$X_8$</td>
<td>$K_{308}$</td>
</tr>
<tr>
<td>$X_2$</td>
<td>$x_{69}$</td>
<td>$K_007$</td>
<td>$X_6$</td>
<td>$x_2$</td>
<td>$K_002$</td>
</tr>
<tr>
<td>$X_3$</td>
<td>$X_20$</td>
<td>2,3,20,1032,1073,1498</td>
<td>$X_4$</td>
<td>$X_4$</td>
<td>$K_{181}$</td>
</tr>
<tr>
<td>$X_9$</td>
<td>$X_{329}$</td>
<td>2,9,188,282,329,1034,1490</td>
<td>$X_57$</td>
<td>$X_7$</td>
<td>1,2,7,57,145,174,1488,2089</td>
</tr>
<tr>
<td>$X_{223}$</td>
<td>$E_{623}$</td>
<td>$X_{282}$</td>
<td>$X_{189}$</td>
<td>2,8,9,84,189,282</td>
<td></td>
</tr>
<tr>
<td>$X_{1073}$</td>
<td>$X_{253}$</td>
<td>2,3,64,69,253,1073,3146</td>
<td>$X_{1249}$</td>
<td>$E_{624}$</td>
<td></td>
</tr>
<tr>
<td>$E_{630}$</td>
<td>?</td>
<td>$E_{668}$</td>
<td>$X_{1034}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_{382}$</td>
<td>$E_{625}$</td>
<td>$E_{553}$</td>
<td>$X_{1032}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Remark.** $E_{623}$ is the anticomplement of $X_{282}$, $E_{624}$ is the anticomplement of $X_{1073}$, $E_{382}$ is the complement of $X_{1032}$, $E_{630}$ is the complement of $X_{1034}$. 

2.2.2. \( \mathcal{K} \) is the Grebe cubic. When \( \mathcal{K} \) is the Grebe cubic \( \mathbf{K102} \) with \( \Omega = P = K \), we obtain \( \mathcal{K}_\Omega = p\mathcal{K}(X_{39}, X_2) \) and \( \mathcal{K}_P = p\mathcal{K}(X_2, X_{76}) = \mathbf{K141} \). We find six cubics with the same points at infinity:

<table>
<thead>
<tr>
<th>( \Omega_1 )</th>
<th>( P_1 )</th>
<th>cubic or ( X_i ) for ( i = )</th>
<th>( \Omega_2 )</th>
<th>( P_2 )</th>
<th>cubic or ( X_i ) for ( i = )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_3 )</td>
<td>( X_{22} )</td>
<td>2,3,22,159</td>
<td>( X_{427} )</td>
<td>( X_4 )</td>
<td>2,4,76,141,193,427,1843</td>
</tr>
<tr>
<td>( X_6 )</td>
<td>( X_6 )</td>
<td>( \mathbf{K102} )</td>
<td>( X_{141} )</td>
<td>( X_{69} )</td>
<td>2,20,69,141,427</td>
</tr>
<tr>
<td>( X_{39} )</td>
<td>( X_2 )</td>
<td>2,3,6,39,141,427</td>
<td>( X_2 )</td>
<td>( X_{76} )</td>
<td>( \mathbf{K141} )</td>
</tr>
</tbody>
</table>

2.2.3. \( \mathcal{K} \) is an isogonal circular cubic. The points at infinity of \( \mathcal{K} \) need not be real. For example, with \( \Omega = K \) and \( P = X_{524} \) (infinite point of the line \( GK \)), \( \mathcal{K} \) is now a circular cubic passing through \( G, K, X_{111} \) (Parry point), with singular focus \( X_{1296} \) (antipode of the Parry point on the circumcircle). The real asymptote is the parallel to \( GK \) at \( X_{111} \).

In this case, \( \mathcal{K}_\Omega = p\mathcal{K}(X_{187}, X_2) = \mathbf{K043} \) (Droussent medial cubic) and \( \mathcal{K}_P = p\mathcal{K}(X_2, X_{316}) = \mathbf{K008} \) (Droussent cubic). All the cubics defined in the following table are circular with a real infinite point \( X_{524} \):

<table>
<thead>
<tr>
<th>( \Omega_1 )</th>
<th>( P_1 )</th>
<th>cubic or ( X_i ) for ( i = )</th>
<th>( \Omega_2 )</th>
<th>( P_2 )</th>
<th>cubic or ( X_i ) for ( i = )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_3 )</td>
<td>( X_{858} )</td>
<td>2,3,6,524,858,895</td>
<td>( X_{468} )</td>
<td>( X_4 )</td>
<td>( \mathbf{K209} )</td>
</tr>
<tr>
<td>( X_6 )</td>
<td>( X_{524} )</td>
<td>1,2,6,111,524,2930</td>
<td>( X_{524} )</td>
<td>( X_{69} )</td>
<td>2,20,69,468,524,2373</td>
</tr>
<tr>
<td>( X_{67} )</td>
<td>( X_{67} )</td>
<td>( \mathbf{K103} )</td>
<td>( E_{406} )</td>
<td>( E_{618} )</td>
<td></td>
</tr>
<tr>
<td>( X_{111} )</td>
<td>( X_{671} )</td>
<td>( \mathbf{K273} )</td>
<td>( X_{2482} )</td>
<td>( E_{620} )</td>
<td></td>
</tr>
<tr>
<td>( X_{187} )</td>
<td>( X_2 )</td>
<td>( \mathbf{K043} )</td>
<td>( X_2 )</td>
<td>( X_{316} )</td>
<td>( \mathbf{K008} )</td>
</tr>
</tbody>
</table>

Remark. \( E_{618} \) is the anticomplement of \( X_{67} \), \( E_{620} \) is the anticomplement of \( X_{111} \), \( E_{406} \) is the midpoint of \( X_6, X_{110} \).

3. Isogonal and isotomic cubics of the pencil \( \mathcal{F}_K \)

We suppose now that \( \mathcal{K} = p\mathcal{K}(\Omega, P) \) is neither an isogonal nor an isotomic isocubic, i.e., that \( \Omega \) is distinct of \( K = X_6 \) and \( G = X_2 \).

3.1. Theorem and consequences.

**Theorem 10.** (1) \( \mathcal{F}_K \) contains one isogonal isocubic if and only if the pivot \( P \) of \( \mathcal{K} \) lies on the line \( L_g \) passing through \( H \) and the \( \Omega \)-isoconjugate of \( O \).

(2) \( \mathcal{F}_K \) contains one isotomic isocubic if and only if the pivot \( P \) of \( \mathcal{K} \) lies on the line \( L_t \) passing through \( G \) and \( \Omega \).

Remark. These two lines are always perfectly defined since \( \Omega \) is neither \( K \) nor \( G \). They coincide if and only if \( \Omega = O \) : for any \( p\mathcal{K}(O, P) \) with \( P \) on the Euler line, there is one isogonal isocubic and one isotomic isocubic having asymptotes parallel to those of \( p\mathcal{K}(O, P) \).

**Corollary 11.** \( \mathcal{F}_K \) contains one isogonal isocubic and one isotomic isocubic if and only if the pivot \( P \) of \( \mathcal{K} \) is the intersection \( Q \) of \( L_g \) and \( L_t \).

Remark. \( L_g \) and \( L_t \) are parallel if and only if \( \Omega \) lies on the rectangular hyperbola \( \mathcal{H} \) passing through \( G, O, K, X_{110} \) (focus of the Kiepert parabola) and also \( X_{154} \),
Asymptotic directions of pivotal isocubics

\[ X_{354}, X_{392}, X_{1201}, X_{2574}, X_{2575} \]. Its asymptotes are parallel to those of the Jer- 
abek hyperbola.

The intersections (other than \( X_{110} \)) with the circumcircle are three points on the 
Thomson cubic which are the vertices of the Thomson triangle. The center of \( \mathcal{H} \) is 
the midpoint of \( GX_{110} \). See Figure 5. The equation of this hyperbola is :

\[
\sum_{\text{cyclic}} (b^2 - c^2)(b^2 c^2 x^2 + a^2 S_A y z) = 0
\]

![Figure 5. The rectangular hyperbola \( \mathcal{H} \)]

3.2. Examples.

- With \( \Omega = I \) (incenter), we find \( Q = X_8 \) (Nagel point). Hence, \( pK(X_1, X_8) = \mathbf{K308} \) generates a family of isocubics containing one isogonal isocubic (the Thomson cubic \( \mathbf{K002} \)) and one isotomic isocubic (the Lucas cubic \( \mathbf{K007} \)).
- With \( \Omega = O \) and \( P = G \), \( K = \mathbf{K168} \). The isogonal cubic is \( pK(K, X_{193}) \) 
  and the isotomic cubic is \( pK(G, H) = \mathbf{K170} \). See Figure 6.

4. Circular \( \mathcal{K}_{\Omega} \) cubics

We have seen that \( \mathcal{K}_{P} \) is circular if and only if \( \mathcal{K} \) is itself circular. We have the 
following theorems for \( \mathcal{K}_{\Omega} \).

**Theorem 12.** (1) For any pole \( \Omega \) distinct of \( O \), there is one and only one circular 
\( \mathcal{K}_{\Omega} \) which passes through \( O \) and \( \Omega \).

(2) When \( \Omega = O \), there are infinitely many circular \( \mathcal{K}_{\Omega} \) forming a pencil of 
cubics passing through \( O \). In this case, \( P \) must lie on the de Longchamps axis.
For example, with $P = X_{858}$, $K_{Ω}$ is the Droussent medial cubic $K043$.

**Theorem 13.** (1) For any pivot $P$ distinct of $X_{69}$, there is one and only one circular $K_{Ω}$.

(2) When $P = X_{69}$, there are infinitely many circular $K_{Ω}$ forming a pencil of cubics passing through $O$ and $Ω$ which must lie on the line at infinity. The isogonal conjugate of $Ω$ lies on the cubic (and on the circumcircle).

The table gives a selection of such cubics.

<table>
<thead>
<tr>
<th>$Ω$</th>
<th>centers on the cubic</th>
<th>cubic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{30}$</td>
<td>$X_3, X_4, X_{30}, X_{74}, X_{133}, X_{1511}$</td>
<td>$K446$</td>
</tr>
<tr>
<td>$X_{519}$</td>
<td>$X_1, X_3, X_{106}, X_{214}, X_{519}, X_{1319}$</td>
<td></td>
</tr>
<tr>
<td>$X_{524}$</td>
<td>$X_2, X_3, X_6, X_{67}, X_{111}, X_{187}, X_{468}, X_{524}, X_{1560}, X_{2482}$</td>
<td>$K043$</td>
</tr>
<tr>
<td>$X_{527}$</td>
<td>$X_3, X_9, X_{57}, X_{527}, X_{1155}, X_{2291}$</td>
<td></td>
</tr>
<tr>
<td>$X_{532}$</td>
<td>$X_3, X_{13}, X_{16}, X_{532}, X_{618}, X_{2380}$</td>
<td></td>
</tr>
<tr>
<td>$X_{533}$</td>
<td>$X_3, X_{14}, X_{15}, X_{533}, X_{619}, X_{2381}$</td>
<td></td>
</tr>
<tr>
<td>$X_{758}$</td>
<td>$X_3, X_{10}, X_{36}, X_{65}, X_{758}, X_{759}$</td>
<td></td>
</tr>
<tr>
<td>$X_{2393}$</td>
<td>$X_3, X_{25}, X_{206}, X_{858}, X_{2373}, X_{2393}$</td>
<td></td>
</tr>
</tbody>
</table>

5. $pK$ with three real asymptotes

Let $K = pK(Ω, P)$ be the pivotal isocubic with pole $Ω(p : q : r)$ and pivot $P(u : v : w)$. Let $C_{Ω}$ be the circum-conic with perspector $Ω$ and center $O_{Ω} = p(q + r - p) : q(r + p - q) : r(p + q - r)$, the $G$—Ceva conjugate of $Ω$ i.e. the perspector of the medial triangle and the anticevian triangle of $Ω$. Let $Γ_{Ω}$ be the homothetic of $C_{Ω}$ under $h(O_{Ω}, 3)$. 
Theorem 14. $\mathcal{K}$ has three real asymptotes if and only if $P$ lies inside $^1$ a tricuspidal quartic $\mathcal{Q}_\Omega$ tritangent to $\mathcal{C}_\Omega$ and having its cusps on $\Gamma_\Omega$.

Remark. $\mathcal{Q}_\Omega$ is bitangent to the line at infinity at the two points where $\mathcal{C}_\Omega$ meets the line at infinity.

Corollary 15 (isogonal $\mathcal{K}$). From this remark, it is clear that $\mathcal{Q}_\Omega$ is bicircular if and only if $\Omega = K$. In this case, $O_\Omega = O$, $\mathcal{C}_\Omega$ is the circumcircle and $\mathcal{Q}_\Omega$ is a deltoid, the envelope of axes of inscribed parabolas. This result is already mentioned in [1].

Corollary 16 (isotomic $\mathcal{K}$). $\mathcal{Q}_\Omega$ contains $A$, $B$, $C$ if and only if $\Omega = G$. In this case, $O_\Omega = G$, $\mathcal{C}_\Omega$ is the Steiner ellipse.

$\mathcal{Q}_G$ has three cusps lying on the medians of $ABC$ and on $\Gamma_\Omega$, the homothetic of the Steiner ellipse under $h(G, 3)$. It is tangent at $A$, $B$, $C$ to the Steiner ellipse. See Figure 7.

Figure 7. The tricuspidal quartic $\mathcal{Q}_G$

The equation of $\mathcal{Q}_G$ is:

$$32 \sum_{\text{cyclic}} x (y^3 + z^3) + 61 \sum_{\text{cyclic}} y^2 z^2 + 118 \sum_{\text{cyclic}} x^2 y z = 0 \quad (10)$$

Remark. When $\Omega$ lies on the inscribed Steiner ellipse, $\mathcal{Q}_\Omega$ decomposes into the line at infinity counted twice and a parabola. In this case, one of the fixed points of the isoconjugation lies at infinity.

$^1P$ is said to be inside the quartic when it lies in the same region of the plane as $O_\Omega$.
6. Asymptotes and Simson lines

Let $K = pK(\Omega, P)$ be the pivotal isocubic with pole $\Omega(p : q : r)$ and pivot $P(u : v : w)$ and let $M$ be a point. It is known that there are three (real or not, distinct or not) Simson lines that pass through $M$ since the envelope of all Simson lines is the well known Steiner deltoid $H_3$, a tricuspidal bicircular quartic of class 3.

6.1. Construction of the Simson lines passing through $M$. Jean-Pierre Ehrmann has found a simple conic construction of these lines which is as follows. Draw the rectangular circum-hyperbola $H_M$ passing through $M$ and its image $H'_M$ under the translation that maps $H$ onto $M$. $H'_M$ meets the circumcircle of $ABC$ at four (real or not) points. One of them (always real) is the reflection of $H$ about the center of $H_M$ and this point also lies on $H_M$. The three other points (one is always real) are those whose Simson lines pass through $M$.

Recall that these three points are all real when $M$ lies inside the Steiner deltoid $H_3$.

6.2. Concurrent Simson lines and cevian lines. Three Simson lines concurring at $Q$ are parallel to the cevian lines of a certain point $M$ if and only if $M$ lies on the McCay cubic $K003$. If $M = (\alpha : \beta : \gamma)$, this point $Q$ is given by $Q = (b^2 c^2 \alpha^2 (\beta + \gamma) : \beta \alpha + \gamma : \gamma \alpha + \beta)$. Hence it is the barycentric product $tgM \times ctM$. The mapping $M \mapsto Q$ has numerous properties we shall not consider in this paper. See [5].

For example, if we take $M = X_3 = O$, we obtain a point $Q$ which is $X_{5562}$ in ETC. Its first barycentric coordinate is $a^2 S_A^2 [a^2 (b^2 + c^2) - (b^2 - c^2)^2]$ and its SEARCH number is 1.84961021841713 · · ·. This point is the intersection of many lines such as $X_2 X_{389}$, $X_3 X_{49}$, $X_4 X_{69}$, $X_5 X_{51}$, etc.

Figure 8 shows these Simson lines and the corresponding hyperbolas $H_O$ (here the Jerabek hyperbola), $H'_O$ meeting the circumcircle at $X_{74}$ and three points $Q_1$, $Q_2$, $Q_3$ whose Simson lines pass through $M$.

6.3. Theorems. In this section, we characterize the cubics $K$ for which the asymptotes are parallel to three Simson lines concurring at a certain point $M(\alpha : \beta : \gamma)$. The equation of these three Simson lines is given by

$$\sum_{cyclic} a^2 \left( \frac{\alpha(y - z) - x(\beta - \gamma)}{A_1} \right) \left( \beta(x + z) - y(\alpha + \gamma) \right) \left( \gamma(x + y) - z(\alpha + \beta) \right) = 0,$$

which is then more simply rewritten under the form:

$$\sum_{cyclic} a^2 A_1 A_2 A_3 = 0.$$

In this case, the equation of the parallels at $M$ to the asymptotes of $K$ is:

$$\sum_{cyclic} p \left( w A_2 - v A_3 \right) A_2 A_3 = 0.$$

When we express that these two equations have the same solutions when $(x : y : z)$ is a point at infinity, we obtain three conditions which are linear with respect
to all the variables namely \((p : q : r)\), \((u : v : w)\) and \((\alpha : \beta : \gamma)\). In other words, if two of the three points \(\Omega, P, M\) are chosen, the coordinates of the third point are given by a system of three linear equations with a corresponding \(3 \times 3\) matrix generally of rank 3.

Since the system has always the trivial and improper solution \((0 : 0 : 0)\), we will find at least one proper solution if and only if the determinant of each of the three matrices above is zero. This gives three conditions involving two of the three points \(\Omega, P, M\). These conditions are

\[
(\Omega P) : \sum_{\text{cyclic}} u a^2 S_A(c^2q - b^2r) = 0 \iff \sum_{\text{cyclic}} p b^2 c^2 (S_B v - S_C w) = 0, \quad (11)
\]

\[
(\Omega M) : \sum_{\text{cyclic}} \alpha q r \left( (b^2 - c^2)p - a^2(q - r) \right) = 0 \iff \sum_{\text{cyclic}} a^2 q r \left( \alpha (q - r) + p(\beta - \gamma) \right) = 0, \quad (12)
\]

\[
(PM) : \sum_{\text{cyclic}} \alpha(u + v)(u + w)(S_B v - S_C w) = 0 \iff \sum_{\text{cyclic}} a^2(u + v)(u + w) \left( (\alpha + \beta - \gamma) v - (\alpha - \beta + \gamma) w \right) = 0, \quad (13)
\]

Remarks. (1) \((\Omega P)\) is linear in \(\Omega\) and \(P\), \((\Omega M)\) and \((PM)\) are linear in \(M\).

(2) \((\Omega P)\) identically vanishes when \(\Omega = X_6\) or \(P = X_4\).
(3) \((\Omega M)\) identically vanishes when \(\Omega = X_6\) (or when \(\Omega\) is a midpoint of \(ABC\) not giving a proper cubic).

(4) \((PM)\) identically vanishes when \(P = X_4\) (or when \(P\) is a vertex of the antimedial triangle not giving a proper cubic).

(5) When \(P = X_{69}\), the conditions \((\Omega P)\) and \((PM)\) show that the points \(\Omega\) and \(M\) must lie on the line \(GK = X_2X_6\).

Each time a condition identically vanishes, the system above has one and only one solution since the rank of at least one of the matrices above is 2. This is examined in the next section.

6.4. Special cases.

6.4.1. \(\Omega = X_6\). When \(\Omega = X_6\), the cubic \(\mathcal{K}\) is a pivotal isogonal cubic and its asymptotes are parallel to the Simson lines that pass through \(M = cP\), the complement of the pivot.

For example, with \(P = X_3\) we have \(M = cP = X_5\) : the asymptotes of the McCay cubic \(K_{003}\) are the parallels at \(G\) to the Simson lines passing through \(X_5\) which are in fact the axes of the Steiner deltoid.

6.4.2. \(P = X_4\). When \(P = X_4\), the cubic \(\mathcal{K}\) is a pivotal isogonal cubic with respect to the orthic triangle. Its asymptotes are parallel to the Simson lines that pass through \(M = \Omega \times X_{69}\) (barycentric product). Conversely, if \(M\) is given, the pole \(\Omega\) is that of the isogonal conjugation that swaps the orthocenter \(H = X_4\) and \(M\).

For example, with \(M = X_5\) we have \(\Omega = X_4 \times X_5 = X_{53}\) : the corresponding cubic is the McCay orthic cubic \(K_{049}\) whose asymptotes are the parallels at \(X_{51}\) to the Simson lines passing through \(X_5\) as above.

6.4.3. \(\Omega = X_6\) and \(P = X_4\). This gives the orthocubic \(K_{006}\) whose asymptotes are parallel to the Simson lines passing through \(O = X_3\). In this case, these asymptotes are not concurrent.

6.5. Interpretation of the three conditions in the general case.

6.5.1. The linear conditions. The conditions (11), (11), (13) above represent actually six equations and four of them are linear with respect to the coordinates of at least one of the three points \(\Omega, P, M\). These four equations give the following propositions.

**Proposition 17.** For a given pole \(\Omega \neq X_6\), the asymptotes of \(\mathcal{K}\) are parallel to the Simson lines of a certain point \(M\) if and only if its pivot \(P\) lies on the line \(\mathcal{L}(\Omega, P)\) with equation

\[
\sum_{\text{cyclic}} a^2 S_A(c^2 q - b^2 r)x = 0.
\]

In this case, this point \(M\) must lie on the line \(\mathcal{L}(\Omega, M)\) with equation

\[
\sum_{\text{cyclic}} qr((b^2 - c^2)p - a^2(q - r))x = 0.
\]
Asymptotic directions of pivotal isocubics

$\mathcal{L}(\Omega, P)$ is the Steiner line of the isogonal conjugate of the infinite point of the trilinear polar of the isogonal conjugate of $\Omega$ and therefore passes through $X_4$.

If $\Gamma_\Omega$ is the circum-conic with perspector $\Omega$, the line $\mathcal{L}(\Omega, M)$ is the conjugated diameter with respect to $\Gamma_\Omega$ of the trilinear polar of the isotomic conjugate of the isogonal conjugate of $\Omega$. Obviously, this line contains the center of $\Gamma_\Omega$.

**Proposition 18.** For a given pivot $P \neq X_4$, the asymptotes of $K$ are parallel to the Simson lines of a certain point $M$ if and only if its pole $\Omega$ lies on the line $\mathcal{L}(P, \Omega)$ with equation

$$\sum_{\text{cyclic}} b^2 c^2 (S_B v - S_C w)x = 0.$$ 

In this case, this point $M$ must lie on the line $\mathcal{L}(P, M)$ with equation

$$\sum_{\text{cyclic}} (u + v)(u + w)(S_B v - S_C w)x = 0.$$ 

$\mathcal{L}(P, \Omega)$ is the trilinear polar of the isogonal conjugate of the infinite point of the trilinear polar of the barycentric quotient $X_4 \div P$ or equivalently the $X_4$-isoconjugate of $P$.

$\mathcal{L}(P, M)$ contains $cP$. It is the trilinear polar of the isoconjugate of the infinite point of the trilinear polar of the barycentric quotient $X_4 \div P$ under the isoconjugation with pole $cP$.

For example,

- $\mathcal{L}(\Omega = X_2, P)$ is the line through $X_4, X_{69}, X_{76}$, etc, and $\mathcal{L}(\Omega, P = X_2)$ is the Brocard axis,
- $\mathcal{L}(\Omega = X_2, M)$ and $\mathcal{L}(P = X_2, M)$ coincide into the line $X_2, X_6, X_{69}$, etc.

6.5.2. The other conditions. The remaining two equations are of degree 3 and lead to two cubic curves. They correspond to the choice of a given point $M$ whose isogonal conjugate of isotomic conjugate is denoted $gtM$, also the barycentric product $M \times X_6$.

**Property 1.** For a given point $M$, there is a cubic $K$ whose asymptotes are parallel to the Simson lines passing through $M$ if and only if its pole $\Omega$ lies on the cubic $K(M, \Omega)$ which is $psK(gtM, X_2, X_6) = psK(M \times X_6, X_2, X_6)$ with equation

$$\sum_{\text{cyclic}} a^2 \left[ \alpha (y - z) + x (\beta - \gamma) \right] yz = 0.$$ 

**Property 2.** For a given point $M$, there is a cubic $K$ whose asymptotes are parallel to the Simson lines passing through $M$ if and only if its pivot $P$ lies on the cubic $K(M, P)$ which is the anticomplement of $psK(gtM, X_2, X_4) = psK(M \times X_6, X_2, X_4)$. The equation of $K(M, P)$ is

$$\sum_{\text{cyclic}} a^2 \left[ (\alpha + \beta - \gamma) y - (\alpha - \beta + \gamma) z \right] (x + y)(x + z) = 0.$$
and that of its complement is very similar to the equation above namely

\[ \sum_{\text{cyclic}} a^2 \left[ y \left( y - z \right) - x \left( \beta - \gamma \right) \right] yz = 0. \]

**Example 1:** When we consider the cubics having their asymptotes parallel to the Simson lines passing through the circumcenter \( O = X_3 \), we find

- \( \mathcal{K}(M = X_3, \Omega) = \mathcal{K}(M = X_3, \Omega) = \text{ps} \mathcal{K}(X_{184}, X_2, X_6) \) passing through \( X_6, X_{69}, X_{206}, X_{219}, X_{1249}, X_{2165} \).
- \( \mathcal{K}(M = X_3, \Omega) = \text{ps} \mathcal{K}(X_{184}, X_2, X_6) \) passing through \( X_4, X_{20}, X_{68}, X_{254}, X_{315}, X_{2996} \). It is the anticomplement of \( \text{ps} \mathcal{K}(X_{184}, X_2, X_4) \) which contains \( X_3, X_4, X_32, X_56, X_{1147} \).

Among them, we have the Orthocubic \( \mathcal{K}_{006} \) as already said and also

- \( \mathcal{K}(X_{69}, X_{315}) \), \( \mathcal{K}(X_{219}, X_{3436}) \), \( \mathcal{K}(X_{577}, X_{20}) \), \( \mathcal{K}(X_{2165}, X_{68}) \), \( \mathcal{K}(X_{32} \times X_{2996}, X_{2996}) \).

All these cubics have three asymptotes parallel to the Simson lines passing through \( X_3 \) and also to the asymptotes of the Orthocubic \( \mathcal{K}_{006} \).

The most interesting probably is \( \mathcal{K}_{690} = \mathcal{K}(X_{2165}, X_{68}) \) since it contains \( X_4, X_{68}, X_{485}, X_{486}, X_{637}, X_{638} \). See Figure 9.

![Figure 9. \( \mathcal{K}_{690} = \mathcal{K}(X_{2165}, X_{68}) \)](image)

**Example 2:** When we consider the cubics having their asymptotes parallel to the Simson lines passing through the incenter \( I = X_1 \), we find

- \( \mathcal{K}(M = X_1, \Omega) = \text{ps} \mathcal{K}(X_{31}, X_2, X_6) \) passing through \( X_6, X_9, X_{19}, X_{478} \).
Asymptotic directions of pivotal isocubics

- $\mathcal{K}(M = X_1, P)$ passing through $X_4$, $X_8$. It is the anticomplement of $ps\mathcal{K}(X_3, X_2, X_1)$ which contains $X_1, X_3, X_56$.

There are two interesting related cubics namely $\mathbf{K691} = p\mathcal{K}(X_{19}, X_4)$ and $\mathbf{K692} = p\mathcal{K}(X_6, X_8)$ generating a pencil which contains the decomposed cubic which is the union of the line $X_4, X_9$ and the circum-conic passing through $X_1$ and $X_4$.

**Example 3**: The Simson lines passing through the nine point center $X_5$ form a (decomposed) stelloid and the cubics having their asymptotes parallel to these Simson lines are equilateral cubics.

- $\mathcal{K}(M = X_5, \Omega) = \mathbf{K307} = ps\mathcal{K}(X_{51}, X_2, X_6)$ passes through $X_6, X_{53}, X_{216}, X_{1249}$.
- $\mathcal{K}(M = X_5, P)$ passes through $X_3, X_4, X_{20}, X_{1670}, X_{1671}$. It is the anticomplement of $\mathbf{K026} = ps\mathcal{K}(X_{51}, X_2, X_3)$ which contains $X_3, X_4, X_5$.

The most remarkable corresponding cubics are the McCay cubic $\mathbf{K003} = p\mathcal{K}(X_6, X_3)$, the McCay orthic cubic $\mathbf{K049} = p\mathcal{K}(X_{53}, X_4)$ and $\mathbf{K096} = p\mathcal{K}(X_{216}, X_{20})$.

**References**


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The Cevian Simson Transformation

Bernard Gibert

Abstract. We study a transformation whose origin lies in the relation between concurrent Simson lines parallel to cevian lines as seen in [4].

1. Introduction

Let \( M = (u : v : w) \) be a point. In [4] we raised the following question: to find a point \( P \) such that the three Simson lines passing through \( P \) are parallel to the three cevian lines of \( M \).

The answer to this question is that \( M \) must lie on the McCay cubic \( K003 \) and, in this case, the corresponding point \( P \) is given by

\[
P = \left( \frac{u^2(v + w)}{a^2} : \frac{v^2(w + u)}{b^2} : \frac{w^2(u + v)}{c^2} \right),
\]

In this case one can find an isogonal pivotal cubic whose asymptotes are also parallel to the cevian lines of \( M \).

We note the strong connection with the cubic \( K024 \) whose equation is

\[
\sum_{\text{cyclic}} x^2(y + z) a^2 = 0.
\]

When \( M \) lies on \( K024 \), \( P \) lies on the line at infinity.

If we denote by \( gM, tM, cM, aM \) the isogonal conjugate, the isotomic conjugate, the complement, the anticomplement of \( M \) respectively then \( P = tgM \times ctM \) where \( \times \) is the barycentric product. \( \mathcal{L}(M) \) will denote the trilinear polar of \( M \).

In this paper, we extend to the whole plane the mapping CST that sends \( M \) onto \( P \) which we call the Cevian Simson Transformation.

2. Properties of CST

2.1. Singular points and consequences.

Proposition 1. CST has six singular points which are \( A, B, C \) each counted twice.
This is obvious from the coordinates of $P$. It follows that CST transforms any curve $C$ of degree $n$ into a curve $C'$ of degree $3n$ which must be reduced according to the number and the nature of the singular points on the original curve.

More precisely, let $G_aG_bG_c$ be the antimedial triangle.

1. If $C$ contains only $A$ and is not tangent to $G_bG_c$, the degree of $C'$ is $3n - 1$.
2. If $C$ contains $A$, $B$, $C$ and is not tangent at these points to a sideline of $G_aG_bG_c$, the degree of $C'$ is $3n - 3$.
3. If $C$ contains $A$, $B$, $C$ and has a double contact at these points to a sideline of $G_aG_bG_c$, the degree of $C'$ is $3n - 6$.

In particular,

4. The transform of a line is generally a cubic which must be tangent to the sidelines of ABC. See §3 below.
5. The transform of a circum-conic is generally a circum-cubic. See §4 below.
6. The transform of a circum-cubic tangent at $A$, $B$, $C$ to the sidelines of $G_aG_bG_c$ is generally a circum-cubic.

A very special case: the Steiner ellipse is tangent $A$, $B$, $C$ to the sidelines of $G_aG_bG_c$ hence its transform is a “curve” of degree 0, namely a point. This point is actually $X_{76}$, the isotomic conjugate of the Lemoine point $K = X_6$. Note that $X_{76}$ is also CST($X_2$).

Consequently, the curve $C'$ above will have a singular point at $X_{76}$ whose multiplicity is $2n$ lowered according to the singular points on $C$ as above. The nature of this singular point, i.e., the reality of the nodal tangents, will depend of the nature of the intersections of $C$ and the Steiner ellipse. If $C$ contains $X_2$, the multiplicity must be increased.

This will be developed in the following sections.

2.2. Fixed points.

**Proposition 2.** CST has one and only one fixed point which is the orthocenter $H = X_4$ of ABC.

Indeed, $M$ is a fixed point of CST if and only if $P = M \iff ctM = X_6 \iff M = X_4$. It follows that the transform $C'$ of any curve $C$ passing through $H$ also passes through $H$.

2.3. Some special CST images.

$G_a$, $G_b$, $G_c$ are transformed into $A$, $B$, $C$.

The infinite points of the sidelines of $ABC$ are transformed into the traces of the de Longchamps axis $L(X_{76})$ on these same sidelines.

The infinite points of K003 are transformed into the cusps of the Steiner deltoid $H_3$.

The infinite points of an equilateral cubic whose asymptotes are not parallel to those of K024 are transformed into the cusps of a deltoid inscribed in $ABC$.

If these asymptotes are parallel to those of K024, their infinite points are transformed into the infinite points of the sidelines of $ABC$. 

2.4. Pre-images of a point. We already know that $X_{76}$ has infinitely many pre-images which are $G = X_2$ and the points on the Steiner ellipse and also $H = X_4$, being a fixed point, has already at least one pre-image namely itself.

We consider a point $P$ different of $X_{76}$ and not lying on a sideline of $ABC$ or $G_aG_bG_c$. We wish to characterize all the points $M$ such that CST$(M) = P$.

When expressing that CST$(M) = P$ we obtain three equations representing three nodal circum-cubic curves with nodes at $A, B, C$. Their isogonal transforms are three conics each passing through one vertex of $ABC$. These conics have generally three common points hence $P$ has three pre-images $M_1, M_2, M_3$.

The nature of these points (real or not, distinct or not) depends of the position of $P$ with respect to the sidelines of $ABC$, the cevian lines of $X_{76}$ and mainly the Ehrmann-MacBeath cubic $K_{244}$ which is the locus of the cusps of all the deltoids inscribed in $ABC$ and also the CST image of the line at infinity. For more informations about $K_{244}$, see [1].

More precisely, see Figure 1,

(i) when $P$ lies inside the yellow region (excluding its “edges” mentioned above) there are three real distinct points $M_1, M_2, M_3$;

(ii) when $M$ lies outside, there is only one real point;

(iii) when $P$ lies on $K_{244}$ (but not on the other lines above), there is only one point (counted twice) and this point lies on the line at infinity. For example, when $P = X_{764}$ we obtain $X_{513}$. This will be detailed in section 3.

The net generated by the three conics above contains the circum-conic which is the isogonal transform of the line passing through $X_2$ and $gtP$ always defined since $gtP \neq X_2$. This line must contain the points $M_1, M_2, M_3$.

On the other hand, each cubic which is the union of one conic and the opposite sideline of $ABC$ must contain the isogonal conjugates of the points $M_1, M_2, M_3$. Hence the three isogonal transforms of these three cubics contain $M_1, M_2, M_3$.

These three latter cubics generate a pencil which contains several simple cubics and, in particular, the $nK_0(\Omega, \Omega)$ where $\Omega$ is the isogonal conjugate of the infinite point of the trilinear polar of $tP$, a point clearly on the circumcircle of $ABC$.

This cubic is a member of the class $\text{C1.026}$ and has always three concurring asymptotes and is tritangent at $A, B, C$ to the Steiner ellipse unless it decomposes. See Example 3 below.

The equation of $nK_0(\Omega, \Omega)$

$$\sum_{\text{cyclic}} (v - w) \frac{x^2(y + z)}{a^2} = 0$$

clearly shows that its CST image is the line $X_2, P$. See §5 for more details.

Furthermore, if the coordinates of $X_2 + \lambda gtP$ are inserted into the equation of $nK_0(\Omega, \Omega)$ then the 3rd degree polynomial in $\lambda$ has no term in $\lambda^2$. Hence the sum of the three values corresponding to the points $M_1, M_2, M_3$ is zero. It follows that the isobarycenter of $M_1, M_2, M_3$ is $X_2$.

In conclusion and generally speaking, we have
Proposition 3. (a) The pre-images $M_1$, $M_2$, $M_3$ of a point $P \neq X_{76}$ are the intersections of the line joining $X_2$ and $g_P$ with the cubic $nK_0(\Omega, \Omega)$.
(b) The CST image of this line is the line $X_2, P$.
(c) The centroid $G$ of $ABC$ is the isobarycenter of $M_1$, $M_2$, $M_3$.

Example 1. With $P = X_4$, we find the Euler line and $nK_0(X_{112}, X_{112})$. Hence the pre-images of $X_4$ are $X_4, X_{1113}, X_{1114}$.

Example 2. With $P = X_3$, we find the line through $X_2$, $X_{98}$, $X_{110}$, etc, and $nK_0(X_{112}, X_{112})$ again. One of the pre-images is $X_{110}$ and the other are real when $ABC$ is acute angled.

Example 3. The cubic $nK_0(\Omega, \Omega)$ contains $X_2$ if and only if $P$ lies on the line $X_2X_{76}$. In this case, it splits into the Steiner ellipse and the line $X_2X_{6}$.

2.5. CST images of cevian triangles. Let $P_aP_bP_c$ be the cevian triangle of $P = (p : q : r)$ and let $Q_aQ_bQ_c$ be its anticomplement.

We have $P_a = (0 : q : r)$, $Q_a = (q + r : -q + r : q - r)$. It is easy to see that $\text{CST}(P_a) = \text{CST}(Q_a) = (0 : c^2q : b^2r) = R_a$. The points $R_b, R_c$ are defined likewise and these three points are the vertices of the cevian triangle of $tgP$. Hence,

Proposition 4. CST maps the vertices of the cevian triangle of $P$ and the vertices of its anticomplement to the vertices of the same cevian triangle, that of $tgP$. 
2.6. **CST images of some common triangle centers.** Table 1 gives a selection of some CST images. A (6-9-13)-search number is given for each unlisted point in ETC.

<table>
<thead>
<tr>
<th>( M )</th>
<th>CST(( M ))</th>
<th>( M )</th>
<th>CST(( M ))</th>
<th>( M )</th>
<th>CST(( M ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_1 )</td>
<td>( X_{10} )</td>
<td>( X_2 )</td>
<td>( X_{76} )</td>
<td>( X_3 )</td>
<td>( X_{5562} )</td>
</tr>
<tr>
<td>( X_4 )</td>
<td>( X_4 )</td>
<td>( X_5 )</td>
<td>4.3423321955522807</td>
<td>( X_6 )</td>
<td>( X_{39} )</td>
</tr>
<tr>
<td>( X_7 )</td>
<td>( X_{85} )</td>
<td>( X_8 )</td>
<td>( X_{341} )</td>
<td>( X_9 )</td>
<td>5.493555510910763</td>
</tr>
<tr>
<td>( X_{10} )</td>
<td>5.329221045166122</td>
<td>( X_{11} )</td>
<td>4.196262646186122</td>
<td>( X_{12} )</td>
<td>2.698123376290196</td>
</tr>
<tr>
<td>( X_{13} )</td>
<td>0.1427165061182335</td>
<td>( X_{14} )</td>
<td>5.228738830014126</td>
<td>( X_{15} )</td>
<td>4.707520749612165</td>
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<tr>
<td>( X_{16} )</td>
<td>-15.70210201702076</td>
<td>( X_{17} )</td>
<td>2.708683938139388</td>
<td>( X_{18} )</td>
<td>12.30617330317703</td>
</tr>
</tbody>
</table>

Peter Moses has kindly provided all the pairs \( \{ M, \text{CST}(M) \} = \{ X_i, X_j \} \) in the ETC (up to \( X_{5573} \)) for these \( \{ i, j \} \). Apart from those listed in Table 1 above and excluding \( X_2 \) and all the points on the Steiner ellipse for which CST(\( M \)) is \( X_{76} \), he has found

\[
\begin{array}{cccccccc}
  i & 66 & 69 & 100 & 101 & 110 & 513 & 651 & 879 & 925 \\
  j & 2353 & 3926 & 8 & 3730 & 3 & 764 & 348 & 5489 & 847 \\
  i & 1113 & 1114 & 1379 & 1380 & 1576 & 3952 \\
  j & 4 & 4 & 3557 & 3558 & 3202 & 1089 \\
\end{array}
\]

3. **CST images of lines**

Let \( \mathcal{L} \) be the line with equation \( px + qy + rz = 0 \) and trilinear pole \( Q = (qr : rp : pq) \).

3.1. **The general case.** In general, the CST image of \( \mathcal{L} \) is a nodal cubic with node \( X_{76} \) which is tangent to the sidelines of \( ABC \) at the traces \( A_2, B_2, C_2 \) of \( \mathcal{L}(tgQ) \) and meeting these lines again at the traces \( A_1, B_1, C_1 \) of \( \mathcal{L}(tgatQ) \).

Indeed, if \( \mathcal{L} \) meets \( BC \) at \( U = (0 : r : -q) \) and \( G_nG_c \) at \( U' = (q - r : -p : p) \) then CST(\( U \)) = \( A_2 = (0 : c^2r : -b^2q) \) and CST(\( U' \)) = \( A_1 = (0 : c^2(p + q - r) : -b^2(p + q + r)) \).

Note that the CST image of the infinite point of \( \mathcal{L} \) is the point

\[
\left( \frac{(q - r)^3}{a^2} : \frac{(r - p)^3}{b^2} : \frac{(p - q)^3}{c^2} \right)
\]

on the cubic. It is also on \( K_{244} \) as seen below.

The most remarkable example is obtained when \( Q = X_2 \). Since \( \mathcal{L} \) is the line at infinity and since the two trilinear polars coincide into the de Longchamps axis (the isotomic transform of the circumcircle of \( ABC \)), we find the cubic \( K_{244} \) meeting the sidelines of \( ABC \) at three inflexion points on the curve (see Figure 2).
3.2. Special cases.

(1) If $\mathcal{L}$ contains $X_2$ and another point $M$, the cubic is the line $\mathcal{L}'$ passing through $X_{76}$ and $\text{CST}(M)$ counted three times.

More precisely, if $\mathcal{L}$ meets the Kiepert hyperbola again at $E$ then $\mathcal{L}'$ is the line $X_{76}E$.

(2) If $\mathcal{L}$ is tangent to the Steiner ellipse, the cubic is cuspidal (with cusp $X_{76}$) and the lines $\mathcal{L}(tgQ), \mathcal{L}(tgtaQ)$ envelope the circum-conic and the in-conic with same perspector $X_{76}$ respectively.

4. CST images of circum-conics

Let $\mathcal{C}(Q)$ be the circum-conic with perspector $Q = (p : q : r) \neq X_2$ (to eliminate the Steiner ellipse case) and equation $pyz + qzx + rxy = 0$.

4.1. The general case. In general, the CST image of $\mathcal{C}(Q)$ is a nodal circum-cubic with node $N_Q$ passing through $X_{76}$ which turns out to be a $psK$ as in [2]. This cubic has the following properties.

(1) Its pseudo-pivot $P_Q = \left( \frac{1}{a^2(-p + q + r)} : \cdots : \cdots \right)$ is $tgtaQ$.

(2) Its pseudo-isopivot $P_Q^* = \left( \frac{p^2}{a^2 : \cdots : \cdots} \right)$ is $tgQ^2$. 
(3) Its node $N_Q = \left( \frac{p}{a^2(-p + q + r)} : \cdots : \cdots \right)$ is $P_Q \times Q$.

(4) Its pseudo-pole $\Omega_Q = \left( \frac{p^2}{a^4(-p + q + r)} : \cdots : \cdots \right)$ is $P_Q \times P_Q^*$ or $N_Q \times tgQ$. This node is obtained when the intersections of $C(Q)$ with the line through its center and $X_2$ are transformed under CST.

(5) The isoconjugate $X_{76}^*$ of $X_{76}$ is $Q \times N_Q = \left( \frac{p^2}{a^2(-p + q + r)} : \cdots : \cdots \right)$, obviously on the cubic.

The most remarkable example is obtained when $Q = X_6$ since $C(Q)$ is the circumcircle $(O)$ of $ABC$. In this case we find the (third) Musselman cubic $K028$, a stelloid which is $psK(X_4, X_{264}, X_3)$. See details in [1] and Figure 3.

![Figure 3. The cubic K028](image)

4.2. Special cases. The CST image of $C(Q)$ is a cuspidal circum-cubic if and only if $Q$ lies on two cubics which are the complement of $K196$ (the isotomic transform of $K024$ with no remarkable center on it) and $K219$ (the complement of $K015$) containing $X_2, X_{1645}, X_{1646}, X_{1647}, X_{1648}, X_{1649}, X_{1650}$. In this latter case, the cusp lies on $K244$. 
Figure 4 shows the cubic which is the CST image of \( C(X_{1646}) \), a circum-cubic passing through \( X_{513}, X_{668}, X_{891}, X_{1015} \). The cusp is \( X_{764} = \text{CST}(X_{513}) \). Since \( X_{891} \) is a point at infinity, its image also lies on \( K_{244} \).

![Figure 4. A cuspidal cubic, CST image of \( C(X_{1646}) \)](image)

4.3. CST images of some usual circum-conics. Any \( C(Q) \) which is a rectangular hyperbola must have its perspector \( Q \) on the orthic axis, the trilinear polar of \( X_{4} \). Its CST image \( K(Q) \) is a nodal cubic passing through \( X_{76} \) and \( X_{4} \). Furthermore, its node lies on \( K_{028} \), its pseudo-pivot lies on the Steiner ellipse, its pseudo-isopivot lies on the inscribed conic with perspector \( X_{2052} \). The pseudo-pole lies on a complicated quartic.

Table 2 gives a selection of such hyperbolas.

<table>
<thead>
<tr>
<th>( Q )</th>
<th>( C(Q) )</th>
<th>( K(Q) )</th>
<th>( N_Q )</th>
<th>other centers on ( K(Q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_{523} )</td>
<td>Kiepert ( psK(X_{850} \times X_{76}, X_{670}, X_{76}) )</td>
<td>( X_{76} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( X_{647} )</td>
<td>Jerabek ( psK(X_{520}, X_{99}, X_{3}) )</td>
<td>( X_{3} )</td>
<td>( X_{39}, X_{2353} )</td>
<td></td>
</tr>
<tr>
<td>( X_{650} )</td>
<td>Feuerbach ( psK(X_{4397}, X_{668}, X_{4}) )</td>
<td>( X_{8} )</td>
<td>( X_{10}, X_{85}, X_{341} )</td>
<td></td>
</tr>
</tbody>
</table>

Remark. When \( M \) lies on the Jerabek hyperbola, the points \( X_{3}, M \) and \( \text{CST}(M) \) are collinear. This is also true when \( M \) lies on the circumcircle.
More generally, for any point \(N_Q\) on \(K028\), the points \(M, \text{CST}(M), N_Q\) are collinear if and only if \(M\) lies on two circum-conics \(\gamma_1, \gamma_2\).

\(\gamma_1\) is the isogonal conjugate of the parallel \(\delta_1\) at \(X_3\) to the line \(X_4N_Q\). \(\gamma_1\) is obviously a rectangular hyperbola.

\(\gamma_2\) is the isogonal conjugate of the perpendicular \(\delta_2\) at \(X_3\) to the line \(X_4N_Q\). The perspector of \(\gamma_2\) lies on the circum-conic passing through \(X_2, X_6\).

Note that \(\delta_2\) envelopes the Kiepert parabola and that \(\delta_1, \delta_2\) meet on the Stammler strophoid \(K038\).

The CST images of \(\gamma_1, \gamma_2\) are two nodal cubics \(psK\) with nodes \(N_1 = N_Q, N_2\) on the Kiepert hyperbola respectively.

5. CST images of some circum-cubics

5.1. CST images of the cubics \(nK_0(P, P)\). If \(P = (p : q : r)\), the cubic \(nK_0(P, P)\) has an equation of the form

\[
\sum \text{cyclic} \frac{x^2(y + z)}{p} = 0 \iff \sum \text{cyclic} \frac{a^2}{p} \times \frac{x^2(y + z)}{p} = 0,
\]

which shows that its CST image is the line \(L(tgP)\).

Recall that \(nK_0(P, P)\) is a member of the class \(CL026\). It is a cubic having three asymptotes concurring at \(X_2\).

With \(P = X_2, X_6, X_{1989}\) we find the cubics \(K016, K024, K064\) whose CST images are the de Longchamps axis, the line at infinity, the perpendicular bisector of \(\overline{OH}\) respectively.

The cubics \(nK_0(X_{112}, X_{112}), nK_0(X_{1576}, X_{1576}), nK_0(X_{32}, X_{32})\) give the Euler line, the Brocard axis, the Lemoine axis.

With \(P = gtX_{107}\) we have the cubic whose CST image is the line \(HK\). See Figure 5.

5.2. CST images of the cubics \(cK(\#P, P^2) = nK(P^2, P^2, P)\). If \(P = (p : q : r)\), the cubic \(cK(\#P, P^2)\) has an equation of the form

\[
\sum \text{cyclic} p^2 x (ry - qz)^2 = 0.
\]

It is a nodal cubic with node \(P\). Since it is tangent at \(A, B, C\) to the sidelines of the antimedial triangle, its CST image must be a cubic curve with node \(\text{CST}(P)\).

This cubic is tangent to the sidelines of \(ABC\) at their intersections with \(L(tgP^2)\) and meets these sidelines again on \(L(tgcP)\).

The most remarkable example is obtained when \(P = X_2\) since the CST image of the nodal Tucker cubic \(K015 = cK(\#X_2, X_2)\) is the cubic \(K244\). In this case, the two trilinear polars coincide as already point out above.

We conclude with a summary of interesting CST images.
Figure 5. The cubic $nK_0(P, P)$ with $P = g t X_{107}$

<table>
<thead>
<tr>
<th>$C$</th>
<th>CST($C$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lines</td>
<td>Line at infinity</td>
</tr>
<tr>
<td>Euler line</td>
<td>Line $X_4, X_50, X_76$, etc</td>
</tr>
<tr>
<td>Conics</td>
<td>Steiner ellipse</td>
</tr>
<tr>
<td>Circumcircle</td>
<td>$X_76$</td>
</tr>
<tr>
<td>Kiepert hyperbola</td>
<td>$psK(X_{850} \times X_{76}, X_{670}, X_{76})$</td>
</tr>
<tr>
<td>Jerabek hyperbola</td>
<td>$psK(X_{520}, X_{99}, X_{3})$</td>
</tr>
<tr>
<td>Feuerbach hyperbola</td>
<td>$psK(X_{4397}, X_{608}, X_{4})$</td>
</tr>
<tr>
<td>Cubics</td>
<td>$K024$</td>
</tr>
<tr>
<td></td>
<td>Line at infinity</td>
</tr>
<tr>
<td></td>
<td>$K015$</td>
</tr>
<tr>
<td></td>
<td>$K244$</td>
</tr>
<tr>
<td></td>
<td>$psK(X_{850} \times X_{76}, X_{670}, X_{76})$</td>
</tr>
<tr>
<td>Others</td>
<td>$Q066$</td>
</tr>
<tr>
<td></td>
<td>Kiepert hyperbola</td>
</tr>
</tbody>
</table>

References


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Two Pairs of Archimedean Circles in the Arbelos

Dao Thanh Oai

Abstract. We construct four circles congruent to the Archimedean twin circles in the arbelos.

Consider an arbelos formed by semicircles \((O_1), (O_2),\) and \((O)\) of radii \(a, b,\) and \(a + b.\) The famous Archimedean twin circles associated in the arbelos have equal radii \(\frac{ab}{a+b}\) (see [2, 3]).

Let \(CD\) be the dividing line of the smaller semicircles, and extend their common tangent \(PQ\) to intersect \((O)\) at \(T_a\) and \(T_b.\)

**Theorem 1.** Let \(A'\) and \(B'\) be the orthogonal projections of \(D\) on the tangents to \((O)\) at \(T_a\) and \(T_b\) respectively. The circles with diameters \(DA'\) and \(DB'\) are congruent to the Archimedean twin circles.

**Proof.** Let the tangents at \(T_a\) and \(T_b\) intersect at \(T.\) Since \(OT\) is the perpendicular bisector of \(T_aT_b,\) it intersects the semicircle \((O)\) at the midpoint \(D\) of the arc \(T_aT_b\) (see [3, §5.2.1]). Since \(O_1P, O_1M\) and \(O_2Q\) are parallel, and \(O_1P = OO_2 = a,\) \(O_2Q = O_1O = b,\)

\[
OM = \frac{a}{a+b} \cdot O_1P + \frac{b}{a+b} \cdot O_2Q = \frac{a^2 + b^2}{a + b} \quad \Rightarrow \quad DM = OD - OM = \frac{2ab}{a + b}.
\]
Now, $\angle DT_aT = \angle DT_bT_a = \angle DT_bT_b$. Therefore, $T_aD$ bisects angle $TT_aT_b$. Similarly, $T_bD$ bisects angle $TT_bT_a$, and $D$ is the incenter of triangle $TT_aT_b$. It follows that $DA' = DB' = DM$, and the circles with $DA'$ and $DB'$ are congruent to the Archimedean twin circles. □

Remark. The circle with $DM$ as diameter is the Archimedean circle $(A_3)$ in [2] (or $(W_4)$ in [1]).

**Theorem 2.** Let $A_1A_2$ and $B_1B_2$ be tangents to the smaller semicircles with $A_1, B_1$ on the line $AB$ and $A_1A_2 = a, B_1B_2 = b$. If $H$ and $K$ are the midpoints of the semicircles $(O_1)$ and $(O_2)$ respectively, and $A'' = CH \cap A_1B_2$, $B'' = CK \cap B_1A_2$, then the circles through $C$ with centers $A''$ and $B''$ are congruent to the Archimedean twin circles.

**Proof.** Clearly, $\angle A''CA_1 = \angle HCO_1 = 45^\circ$. Since $B_1B_2 = O_2B_2 = b$, $\angle B_2B_1O_2 = 45^\circ$, the lines $CA''$ and $B_1B_2$ are parallel. Also, $B_1O_2 = \sqrt{2}b$. Similarly, $A_1O_1 = \sqrt{2}a$, and $A_1B_1 = (\sqrt{2} + 1)(a + b)$. Therefore,

$$CA'' = B_1B_2 \cdot \frac{A_1C}{A_1B_1} = b \cdot \frac{(\sqrt{2} + 1)a}{(\sqrt{2} + 1)(a + b)} = \frac{ab}{a + b}.$$ 

Similarly, $CB'' = \frac{ab}{a + b}$. Therefore, the circles through $C$ with centers $A''$ and $B''$ are congruent to the Archimedean twin circles. □

**References**


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On Some Triads of Homothetic Triangles

Gotthard Weise

Abstract. To a given reference triangle $\Delta$ and three directions $\alpha, \beta, \gamma$ we construct four triads of homothetic triangles and investigate relations between their homothetic centers, centroids, midway triangles, medial triangles and areas.

1. Two triads of homothetic triangles

Given an arbitrary triangle $\Delta = ABC$ with sidelines $a, b, c$ and an ordered set $\{\alpha, \beta, \gamma\}$ of three directions in the plane of $\Delta$. Let

$$
p_1 = \alpha \beta \gamma, \quad p_2 = \gamma \alpha \beta, \quad p_3 = \beta \gamma \alpha, \quad \text{and} \quad p^1 = \gamma \beta \alpha, \quad p^2 = \beta \alpha \gamma, \quad p^3 = \alpha \gamma \beta
$$

be the even and odd permutations of these directions. Each such permutation $p = \pi_1 \pi_2 \pi_3$ defines three lines with directions $\pi_1$ at $A$, $\pi_2$ at $B$ and $\pi_3$ at $C$, which are the sidelines of two triads $T_\Delta$ and $T^\Delta$ of homothetic triangles $\Delta_i = U_iV_iW_i$ and $\Delta^i = U^iV^iW^i$ ($i = 1, 2, 3$) with angles $U, V, W$. The assignment of the indexed symbols $U, V, W$ to the vertices of these triangles is chosen so that homologous vertices have the same symbol (see Figure 1).

We shall consider some properties of these triads and relationships with other triads of triangles.

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2. Coordinate representations of geometric objects

Geometric objects are described in this paper by homogeneous barycentric coordinates with reference to the triangle \( \Delta \). \( P = (u : v : w) \) is a point, \( l = [p : q : r] \) a line. For a triangle given by its vertices we use the matrix representation (round brackets) with vertex coordinates in the rows. The same triangle can be represented in another matrix form (square brackets), where the rows mean the coordinates of the lines.

We shall regard each direction as a point on the line at infinity with the same name. Then the ordered direction triple \((\alpha, \beta, \gamma)\) has the matrix form

\[
D = \begin{pmatrix}
\alpha \\
\beta \\
\gamma
\end{pmatrix} = \begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 \\
\beta_1 & \beta_2 & \beta_3 \\
\gamma_1 & \gamma_2 & \gamma_3
\end{pmatrix}
\]

with vanishing row sums. By suitable factors in each row it is possible, that not only the row sums of \( D \) vanish, but also the column sums, and that all cofactors are equal of unity. In order to verify the propositions in the sections below, we shall use some other properties of such matrices (see [2]):

\[
\begin{align*}
\beta_1 - \gamma_2 &= \gamma_3 - \alpha_1 = \alpha_2 - \beta_3 = \alpha_1 \beta_3 \gamma_2 - \alpha_2 \beta_1 \gamma_3 =: \lambda_1, \\
\gamma_2 - \alpha_3 &= \alpha_1 - \beta_2 = \beta_3 - \gamma_1 = \alpha_3 \beta_2 \gamma_1 - \alpha_1 \beta_3 \gamma_2 =: \lambda_2, \\
\alpha_3 - \beta_1 &= \beta_2 - \gamma_3 = \gamma_1 - \alpha_2 = \alpha_2 \beta_3 \gamma_3 - \alpha_3 \beta_2 \gamma_1 =: \lambda_3; \\
\beta_3 - \gamma_2 &= \gamma_1 - \alpha_3 = \alpha_2 - \beta_1 = \alpha_2 \beta_3 \gamma_1 - \alpha_3 \beta_1 \gamma_3 =: \mu_1, \\
\gamma_2 - \alpha_1 &= \alpha_3 - \beta_2 = \beta_1 - \gamma_3 = \alpha_3 \beta_1 \gamma_2 - \alpha_1 \beta_2 \gamma_3 =: \mu_2, \\
\alpha_1 - \beta_3 &= \beta_2 - \gamma_1 = \gamma_3 - \alpha_2 = \alpha_1 \beta_2 \gamma_3 - \alpha_2 \beta_3 \gamma_1 =: \mu_3
\end{align*}
\]

with

\[
\sum_{i=1}^{3} \lambda_i = \sum_{i=1}^{3} \mu_i = 0 \quad \text{and} \quad \sum_{k=1}^{3} \mu_k^2 = 6 + \sum_{k=1}^{3} \lambda_k^2; \quad (7)
\]

\[
\begin{align*}
\alpha_2 \alpha_3 - \beta_3 \gamma_2 &= \beta_2 \beta_3 - \gamma_3 \alpha_2 = \gamma_2 \gamma_3 - \alpha_3 \beta_2 =: \xi_1, \\
\alpha_3 \alpha_1 - \beta_1 \gamma_3 &= \beta_3 \beta_1 - \gamma_1 \alpha_3 = \gamma_3 \gamma_1 - \alpha_1 \beta_3 =: \xi_2, \\
\alpha_1 \alpha_2 - \beta_2 \gamma_1 &= \beta_1 \beta_2 - \gamma_2 \alpha_1 = \gamma_1 \gamma_2 - \alpha_2 \beta_1 =: \xi_3; \\
\alpha_2 \alpha_3 - \beta_2 \gamma_3 &= \beta_2 \beta_3 - \gamma_2 \alpha_3 = \gamma_2 \gamma_3 - \alpha_2 \beta_3 =: \eta_1, \\
\alpha_3 \alpha_1 - \beta_3 \gamma_1 &= \beta_3 \beta_1 - \gamma_3 \alpha_1 = \gamma_3 \gamma_1 - \alpha_3 \beta_1 =: \eta_2, \\
\alpha_1 \alpha_2 - \beta_1 \gamma_2 &= \beta_1 \beta_2 - \gamma_1 \alpha_2 = \gamma_1 \gamma_2 - \alpha_1 \beta_2 =: \eta_3.
\end{align*}
\]

Furthermore, for each row \( i \) of \( D \),

\[
\sum_{k=1}^{3} \left( \prod_{j \neq i} d_{jk} \prod_{j \neq k} d_{ij} \right) = 1, \quad (14)
\]
and for each column \( k \),

\[
\sum_{i=1}^{3} \left( \prod_{j \neq i} d_{jk} \prod_{j \neq k} d_{ij} \right) = 1.
\] (15)

Here is an example:

\[
D = \begin{pmatrix}
1 & 2 & -3 \\
1 & 3 & -4 \\
-2 & -5 & 7
\end{pmatrix}.
\]

The 9 lines at the points \( A, B, C \) with the directions \( \alpha, \beta, \gamma \) have following coordinate representations:

<table>
<thead>
<tr>
<th></th>
<th>( A )</th>
<th>( B )</th>
<th>( C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>( \alpha_A = 0 : -\alpha_3 : \alpha_2 )</td>
<td>( \alpha_B = [\alpha_3 : 0 : -\alpha_1] )</td>
<td>( \alpha_C = [-\alpha_2 : \alpha_1 : 0] )</td>
</tr>
<tr>
<td>( \beta )</td>
<td>( \beta_A = 0 : -\beta_3 : \beta_2 )</td>
<td>( \beta_B = [\beta_3 : 0 : -\beta_1] )</td>
<td>( \beta_C = [-\beta_2 : \beta_1 : 0] )</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>( \gamma_A = 0 : -\gamma_3 : \gamma_2 )</td>
<td>( \gamma_B = [\gamma_3 : 0 : -\gamma_1] )</td>
<td>( \gamma_C = [-\gamma_2 : \gamma_1 : 0] )</td>
</tr>
</tbody>
</table>

From this it is easy to determine the matrix forms of the triangles \( \Delta_i \) and \( \Delta^j \) in normalized barycentric coordinates (row sums are equal of unity):

\[
\Delta_1 = \begin{pmatrix} \alpha_A \\ \beta_B \\ \gamma_C \end{pmatrix} \Rightarrow \begin{pmatrix} \beta_B \cap \gamma_C \\ \gamma_C \cap \alpha_A \\ \alpha_A \cap \beta_B \end{pmatrix} = \begin{pmatrix} U_1 \\ V_1 \\ W_1 \end{pmatrix} = \begin{pmatrix} \beta_1 \gamma_1 & \beta_1 \gamma_2 & \beta_3 \gamma_3 \\ \gamma_1 \alpha_2 & \gamma_2 \alpha_2 & \gamma_2 \alpha_3 \\ \alpha_3 \beta_1 & \alpha_2 \beta_2 & \alpha_3 \beta_3 \end{pmatrix},
\]

\[
\Delta_2 = \begin{pmatrix} \gamma_A \\ \alpha_B \\ \beta_C \end{pmatrix} \Rightarrow \begin{pmatrix} \beta_C \cap \gamma_A \\ \gamma_A \cap \alpha_B \\ \alpha_B \cap \beta_C \end{pmatrix} = \begin{pmatrix} U_2 \\ V_2 \\ W_2 \end{pmatrix} = \begin{pmatrix} \beta_1 \gamma_2 & \beta_2 \gamma_2 & \beta_2 \gamma_3 \\ \gamma_3 \alpha_1 & \gamma_2 \alpha_3 & \gamma_3 \alpha_3 \\ \alpha_1 \beta_2 & \alpha_2 \beta_2 & \alpha_3 \beta_1 \end{pmatrix},
\]

\[
\Delta_3 = \begin{pmatrix} \beta_A \\ \gamma_B \\ \alpha_C \end{pmatrix} \Rightarrow \begin{pmatrix} \beta_A \cap \gamma_B \\ \gamma_B \cap \alpha_C \\ \alpha_C \cap \beta_A \end{pmatrix} = \begin{pmatrix} U_3 \\ V_3 \\ W_3 \end{pmatrix} = \begin{pmatrix} \beta_3 \gamma_1 & \beta_2 \gamma_3 & \beta_3 \gamma_3 \\ \gamma_1 \alpha_1 & \gamma_1 \alpha_2 & \gamma_3 \alpha_1 \\ \alpha_1 \beta_2 & \alpha_2 \beta_2 & \alpha_2 \beta_3 \end{pmatrix},
\]

\[
\Delta^1 = \begin{pmatrix} \gamma_A \\ \beta_B \\ \alpha_C \end{pmatrix} \Rightarrow \begin{pmatrix} \beta_B \cap \gamma_A \\ \gamma_A \cap \alpha_C \\ \alpha_C \cap \beta_B \end{pmatrix} = \begin{pmatrix} U^1 \\ V^1 \\ W^1 \end{pmatrix} = \begin{pmatrix} \beta_1 \gamma_3 & \beta_3 \gamma_2 & \beta_3 \gamma_3 \\ \gamma_2 \alpha_1 & \gamma_3 \alpha_2 & \gamma_3 \alpha_2 \\ \alpha_1 \beta_1 & \alpha_2 \beta_1 & \alpha_1 \beta_3 \end{pmatrix},
\]

\[
\Delta^2 = \begin{pmatrix} \beta_A \\ \gamma_C \\ \alpha_B \end{pmatrix} \Rightarrow \begin{pmatrix} \beta_A \cap \gamma_C \\ \gamma_C \cap \alpha_B \\ \alpha_B \cap \beta_A \end{pmatrix} = \begin{pmatrix} U^2 \\ V^2 \\ W^2 \end{pmatrix} = \begin{pmatrix} \beta_2 \gamma_1 & \beta_2 \gamma_2 & \beta_2 \gamma_2 \\ \gamma_1 \alpha_1 & \gamma_2 \alpha_1 & \gamma_1 \alpha_3 \\ \alpha_3 \beta_1 & \alpha_3 \beta_2 & \alpha_3 \beta_3 \end{pmatrix},
\]

\[
\Delta^3 = \begin{pmatrix} \alpha_A \\ \gamma_B \\ \beta_C \end{pmatrix} \Rightarrow \begin{pmatrix} \beta_C \cap \gamma_B \\ \gamma_B \cap \alpha_A \\ \alpha_A \cap \beta_C \end{pmatrix} = \begin{pmatrix} U^3 \\ V^3 \\ W^3 \end{pmatrix} = \begin{pmatrix} \alpha_2 \beta_1 & \alpha_2 \beta_2 & \alpha_2 \beta_3 \\ \beta_1 \gamma_1 & \beta_2 \gamma_1 & \beta_1 \gamma_3 \\ \gamma_1 \alpha_3 & \gamma_3 \alpha_2 & \gamma_3 \alpha_3 \end{pmatrix}.
\]

In §5 we shall use the concept of the **medial map** \( m \) of a point \( P = (u : v : w) \), defined by \( mP = (v + w : w + u : u + v) \). The **medial image** \( mP \) lies on the line \( PG \). The centroid \( G \) of \( \Delta \) divides the segment between \( P \) and \( mP \) in the ratio 2 : 1. We call a triangle \( PQR \) the **medial image** of a triangle \( UVW \), if \( P = mU, Q = mV \) and \( R = mW \).

The isotomic conjugate \( P^* \) of a point \( P \) has the coordinates \((1/u : 1/v : 1/w)\).
3. Centroids and homothetic centers of the triangles \( \Delta_i \) and \( \Delta^i \)

The centroid of \( \Delta \) is \( G = (1 : 1 : 1) \), \( G_i \) are the centroids of the triangles \( \Delta_i \). Their \( j \)-th coordinates are the sums of the \( j \)-th column of the matrix \( \Delta_i \). The triangle \( \Delta_G = (G_{ij}) \) is formed by the centroids \( G_i \). Similarly define \( G^i \) and \( \Delta^G = (G^{ij}) \).

**Proposition 1.** The triangles \( \Delta_G \) and \( \Delta^G \) have the same centroid \( G \) as \( \Delta \).

**Proof.** The sums of the \( i \)-th column sums of \( \Delta_1 \), \( \Delta_2 \) and \( \Delta_3 \) resp. \( \Delta^1 \), \( \Delta^2 \) and \( \Delta^3 \) have the same value 3.

The centers of homothety \( P_{ij} \) of the triangle pairs \((\Delta_i, \Delta_j)\) are in detail

\[
P_{12} = (\beta_1(\gamma_1 \alpha_2 - \gamma_3 \alpha_3) : \gamma_2(\alpha_3 \beta_1 - \alpha_2 \beta_2) : \alpha_3(\beta_2 \gamma_3 - \beta_1 \gamma_1)),
\]

\[
P_{23} = (\alpha_1(\beta_3 \gamma_1 - \beta_2 \gamma_2) : \beta_2(\gamma_2 \alpha_3 - \gamma_1 \alpha_1) : \gamma_3(\alpha_1 \beta_2 - \alpha_3 \beta_3)),
\]

\[
P_{31} = (\gamma_1(\alpha_2 \beta_3 - \alpha_1 \beta_1) : \alpha_2(\beta_1 \gamma_2 - \beta_3 \gamma_3) : \beta_3(\gamma_3 \alpha_1 - \gamma_2 \alpha_2)).
\]

These three points are collinear because each triad of homothetic triangles has collinear homothetic centers. The line \( g \) containing them can be written as

\[
g = [\delta_2 - \delta_3 : \delta_3 - \delta_1 : \delta_1 - \delta_2], \tag{16}
\]

with abbreviations \( \delta_i := \alpha_i \beta_i \gamma_i \). Similarly, the homothetic centers \( P^{ij} \) of the pairs \((\Delta^1, \Delta^j)\) lie on the same line \( g \).

**Proposition 2.** The line \( g \) contains the centroid \( G \) of \( \Delta \).

**Proof.** In accordance with (16) \( g \) has a vanishing sum of coordinates.
Proposition 3. The homothetic center \( P_{ij} \) of the pair \((\Delta_i, \Delta_j)\) lies on the side \( G_iG_j \) of \( \Delta_G \) and the homothetic center \( P^{ij} \) on the side \( G^{i}G^{j} \) of \( \Delta^{G} \).

Proof. Verification. \( \square \)

4. Areas

Let \(|UVW|\) be the area of a triangle \( UVW \) and \( \sigma \) the area of the reference triangle \( \Delta \). If the vertices of the triangles are given by their normalized barycentric coordinates, then \( UVW \) has the area \( \sigma \cdot | \det(U,V,W) | \). We shall see below that there is a simple connection between the areas of \( \Delta, \Delta_i, \Delta^i, \Delta_G \) and \( \Delta^{G} \), independent of \( \alpha, \beta \) and \( \gamma \).

Making use of (1) to (6) we find

\[
\det(\Delta) = \mu^2_i, \quad \det(\Delta^i) = \lambda^2_i.
\]

Let \( d_{ij} \) and \( d^{ij} \) be the column sums of the matrices \( \Delta_i \) and \( \Delta^i \), respectively, then we have the (normalized) matrices \( (G_{ij}) = \frac{1}{3} d_{ij} \) and \( (G^{ij}) = \frac{1}{3} d^{ij} \), and it is valid

\[
\det(\Delta_G) = \frac{1}{27} \det(d_{ij}) = \frac{1}{27} \begin{vmatrix} d_{11} & d_{12} & 3 \\ d_{21} & d_{22} & 3 \\ d_{31} & d_{32} & 3 \end{vmatrix} = \frac{1}{27} \begin{vmatrix} d_{11} & d_{12} & 3 \\ d_{21} & d_{22} & 3 \\ 3 & 3 & 9 \end{vmatrix} = \frac{1}{3} \begin{vmatrix} d_{11} - 1 & d_{12} - 1 \\ d_{21} - 1 & d_{22} - 1 \end{vmatrix}.
\]

In accordance with

\[
\begin{align*}
d_{11} - 1 &= \gamma_1 \lambda_1 - \beta_1 \lambda_2 \\
d_{21} - 1 &= \beta_1 \lambda_2 - \alpha_1 \lambda_3 \\
d_{12} - 1 &= \alpha_2 \lambda_2 - \gamma_2 \lambda_3 \\
d_{22} - 1 &= \gamma_2 \lambda_3 - \beta_2 \lambda_1
\end{align*}
\]

it follows that

\[
\det(\Delta_G) = \frac{1}{3} ( \lambda_1^2 + \lambda_2^2 + \lambda_1 \lambda_2 ) = \frac{1}{6} ( \lambda_1^2 + \lambda_2^2 + \lambda_3^2 ).
\]

Similarly,

\[
\det(\Delta^G) = \frac{1}{3} ( \mu_1^2 + \mu_2^2 + \mu_1 \mu_2 ) = \frac{1}{6} ( \mu_1^2 + \mu_2^2 + \mu_3^2 ).
\]

From this we obtain

Proposition 4.

\[
\sum |\Delta_i| = 6 \cdot |\Delta^G| = 6 \cdot (|\Delta_G| + |\Delta|), \quad \sum |\Delta^i| = 6 \cdot |\Delta_G| = 6 \cdot (|\Delta^G| - |\Delta|),
\]

\[
|\Delta^G| - |\Delta_G| = |\Delta|, \quad \sum |\Delta_i| - \sum |\Delta^i| = 6 \cdot |\Delta|.
\]

5. The triads \( T_\Delta, T^\Delta \) and their midway triangles

Given two labeled triangles \( T_1 = X_1Y_1Z_1 \) and \( T_2 = X_2Y_2Z_2 \). Let \( X_{12} \) be the midpoint of the line segment \( X_1X_2 \), \( Y_{12} \) and \( Z_{12} \) the midpoints of the segments \( Y_1Y_2 \) and \( Z_1Z_2 \), respectively. Then the triangle \( T_{12} = X_{12}Y_{12}Z_{12} \) is called the midway triangle of the pair \( (T_1, T_2) \).
The midway triangles of the pairs \((\Delta_i, \Delta_j)\) have obvious the normalized representations:

\[
\Delta_{12} = \begin{pmatrix} U_{12} \\ V_{12} \\ W_{12} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\beta_1 \gamma_3 & -\beta_3 \gamma_2 & \beta_3 \gamma_1 + \beta_2 \gamma_3 \\ \gamma_1 \alpha_2 + \gamma_3 \alpha_1 & -\gamma_2 \alpha_1 & -\gamma_1 \alpha_3 \\ -\alpha_2 \beta_1 & \alpha_2 \beta_3 + \alpha_1 \beta_2 & -\alpha_3 \beta_2 \end{pmatrix},
\]

\[
\Delta_{23} = \begin{pmatrix} U_{23} \\ V_{23} \\ W_{23} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \beta_1 \gamma_2 + \beta_3 \gamma_1 & -\beta_2 \gamma_1 & -\beta_1 \gamma_3 \\ -\gamma_2 \alpha_1 & \gamma_2 \alpha_3 + \gamma_1 \alpha_2 & -\gamma_3 \alpha_2 \\ -\alpha_1 \beta_3 & -\alpha_3 \beta_2 & \alpha_3 \beta_1 + \alpha_2 \beta_3 \end{pmatrix},
\]

\[
\Delta_{31} = \begin{pmatrix} U_{31} \\ V_{31} \\ W_{31} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\beta_2 \gamma_1 & \beta_2 \gamma_3 + \beta_1 \gamma_2 & -\beta_3 \gamma_2 \\ -\gamma_1 \alpha_3 & -\gamma_3 \alpha_2 & \gamma_3 \alpha_1 + \gamma_2 \alpha_3 \\ \alpha_1 \beta_2 + \alpha_3 \beta_1 & -\alpha_2 \beta_1 & -\alpha_1 \beta_3 \end{pmatrix},
\]

and we find by calculation

\[
|\det(\Delta_{12})| = \frac{1}{4} \mu_3^2, \quad |\det(\Delta_{23})| = \frac{1}{4} \mu_1^2, \quad |\det(\Delta_{31})| = \frac{1}{4} \mu_2^2.
\]

**Proposition 5.** For \(i \neq j, k \neq i, j\),

(a) the midway triangle \(\Delta_{ij}\) is the medial image of \(\Delta_k\) and from this congruent with the medial triangle of \(\Delta_k\).

(b) \(G\) is the homothetic center of the pair \((\Delta_k, \Delta_{ij})\).

---

**Proof.** Verification. \(\square\)

For the triangles of the triad \(T^\Delta\) the above proposition is true by analogy.
6. Two triads of homothetic inscribed triangles

The above described construction of two triads of homothetic in \( \Delta \) circumscribed triangles \( \Delta_i \) and \( \Delta^i \) raises the question whether or not exist triads of homothetic in \( \Delta \) inscribed triangles whose sides have the given directions \( \alpha, \beta, \gamma \). For this purpose we form groups of vertices of \( \Delta^i \) and \( \Delta_i \) to new triangles \( \Phi^i \) and \( \Phi_i \), respectively:

\[
\Phi^1 = \begin{pmatrix} U^3 \\ V^1 \\ W^2 \end{pmatrix}, \quad \Phi^2 = \begin{pmatrix} U^2 \\ V^3 \\ W^1 \end{pmatrix}, \quad \Phi^3 = \begin{pmatrix} U^1 \\ V^2 \\ W^3 \end{pmatrix};
\]

\[
\Phi_1 = \begin{pmatrix} U_3 \\ V_1 \\ W_2 \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} U_2 \\ V_3 \\ W_1 \end{pmatrix}, \quad \Phi_3 = \begin{pmatrix} U_1 \\ V_2 \\ W_3 \end{pmatrix}.
\]

Then we intersect the sidelines of these triangles with certain sidelines of the reference triangle (see Figure 4). The points of intersection \( A_i, B_i, C_i \) and \( A^i, B^i, C^i \) form triangles \( \Omega_i \) and \( \Omega^i \), respectively, with following normalized barycentric coordinates:

\[
\Omega_1 = \begin{pmatrix} A_1 \\ B_1 \\ C_1 \end{pmatrix} := \begin{pmatrix} V^1 W^2 \cap a \\ W^2 U^3 \cap b \\ U^3 V^1 \cap c \end{pmatrix} = -\frac{1}{\mu_1} \begin{pmatrix} 0 & \gamma_2 & -\beta_3 \\ -\gamma_1 & 0 & \alpha_3 \\ \beta_1 & -\alpha_2 & 0 \end{pmatrix},
\]

\[
\Omega_2 = \begin{pmatrix} A_2 \\ B_2 \\ C_2 \end{pmatrix} := \begin{pmatrix} U^2 V^3 \cap a \\ V^3 W^1 \cap b \\ W^1 U^2 \cap c \end{pmatrix} = -\frac{1}{\mu_2} \begin{pmatrix} 0 & \beta_2 & -\alpha_3 \\ -\beta_1 & 0 & \gamma_3 \\ \alpha_1 & -\gamma_2 & 0 \end{pmatrix},
\]

\[
\Omega_3 = \begin{pmatrix} A_3 \\ B_3 \\ C_3 \end{pmatrix} := \begin{pmatrix} W^3 U^1 \cap a \\ U^1 V^2 \cap b \\ V^2 W^3 \cap c \end{pmatrix} = -\frac{1}{\mu_3} \begin{pmatrix} 0 & \alpha_2 & -\gamma_3 \\ -\alpha_1 & 0 & \beta_3 \\ \gamma_1 & -\beta_2 & 0 \end{pmatrix},
\]

\[
\Omega^1 = \begin{pmatrix} A^1 \\ B^1 \\ C^1 \end{pmatrix} := \begin{pmatrix} V_1 U_3 \cap a \\ U_3 W_2 \cap b \\ W_2 V_1 \cap c \end{pmatrix} = \frac{1}{\lambda_1} \begin{pmatrix} 0 & \alpha_2 & -\beta_3 \\ -\alpha_1 & 0 & \gamma_3 \\ \beta_1 & -\gamma_2 & 0 \end{pmatrix},
\]

\[
\Omega^2 = \begin{pmatrix} A^2 \\ B^2 \\ C^2 \end{pmatrix} := \begin{pmatrix} U_2 W_1 \cap a \\ W_1 V_3 \cap b \\ V_3 U_2 \cap c \end{pmatrix} = \frac{1}{\lambda_2} \begin{pmatrix} 0 & \gamma_2 & -\alpha_3 \\ -\gamma_1 & 0 & \beta_3 \\ \alpha_1 & -\beta_2 & 0 \end{pmatrix},
\]

\[
\Omega^3 = \begin{pmatrix} A^3 \\ B^3 \\ C^3 \end{pmatrix} := \begin{pmatrix} W_3 V_2 \cap a \\ V_2 U_1 \cap b \\ U_1 W_3 \cap c \end{pmatrix} = \frac{1}{\lambda_3} \begin{pmatrix} 0 & \beta_2 & -\gamma_3 \\ -\beta_1 & 0 & \alpha_3 \\ \gamma_1 & -\alpha_2 & 0 \end{pmatrix}.
\]

Then we have (see Figure 5)

**Proposition 6.** (a) The sidelines of the inscribed triangles \( \Omega_i \) and \( \Omega^i \) have the directions \( \alpha, \beta \) and \( \gamma \).

(b) The sides of \( \Delta_i \) are parallels of the sides \( a_i, b_i, c_i \) of \( \Omega_i \) at \( A, B, C \) (in this order). An analogous statement is true for \( \Delta^i \).
(c) The points in each of the triples \(\{A_1, B_3, C_2\}, \{A_2, B_1, C_3\}, \{A_3, B_2, C_1\}, \{A^1, B^2, C^3\}, \{A^2, B^3, C^1\} \) and \(\{A^3, B^1, C^2\}\) are collinear.

(d) The product of the areas of \(\Delta_i\) and \(\Omega_i\) and of \(\Delta_i\) and \(\Omega^i\) is independent of \(\alpha, \beta, \gamma\):

\[|\Delta_i| \cdot |\Omega_i| = |\Delta^i| \cdot |\Omega^i| = |\Delta|^2.\]

**Proof.** Verification.

(a) For instance the line \(A_1B_1\) intersects the infinite line in accordance with (4) at \(-\mu_1(\gamma_1 : \gamma_2 : \gamma_3)\), therefore it has the direction \(\gamma\).

(b) For instance the translation of the line \(A_1B_1\) at \(C\) yields the line \([-\gamma_2 : \gamma_1 : 0]\) = \(\gamma_C\).

(c) For instance \(\det(A_1, B_3, C_2) = \begin{vmatrix} 0 & \gamma_2 & -\beta_3 \\ -\alpha_1 & 0 & \beta_3 \\ \alpha_1 & -\gamma_2 & 0 \end{vmatrix} = 0.\)

(d) For instance according to (4)

\[\det(\Omega_1) \cdot \det(\Delta_1) = -\frac{1}{\mu_1}(\alpha_3\beta_1\gamma_2 - \alpha_2\beta_3\gamma_1) \cdot \mu_1^2 = 1.\]

\(\square\)

For the comparison of homothetic triangles it is useful to give homologous vertices the same position:

\[\Omega_1 = A_1B_1C_1, \quad \Omega_2 = B_2C_2A_2, \quad \Omega_3 = C_3A_3B_3;\]

\[\Omega^1 = C^1B^1A^1, \quad \Omega^2 = B^2A^2C^2, \quad \Omega^3 = A^3C^3B^3.\]
Then for instance the homothetic center $Q_{12}$ of the pair $(\Omega_1, \Omega_2)$ is the common point of intersection of the lines $A_1B_2$, $B_1C_2$ and $C_1A_2$. By a simple calculation it follows for the homothetic centers $\Omega_{ij}$ and $\Omega^{ij}$

$$Q_{12} = (\beta_1 : \gamma_2 : \alpha_3), \quad Q_{23} = (\alpha_1 : \beta_2 : \gamma_3), \quad Q_{31} = (\gamma_1 : \alpha_2 : \beta_3),$$

$$Q^{12} = (\alpha_1 : \gamma_2 : \beta_3), \quad Q^{23} = (\gamma_1 : \beta_2 : \alpha_3), \quad Q^{31} = (\beta_1 : \alpha_2 : \gamma_3).$$

It is clear, that the homothetic centers $Q_{12}$, $Q_{23}$, $Q_{31}$ and $Q^{12}$, $Q^{23}$, $Q^{31}$ are collinear on lines $g_{13}$ and $g^{\Omega}$, respectively, and these lines have according to (8)-(13) the simple representations

$$g_{13} = [\xi_1 : \xi_2 : \xi_3], \quad g^{\Omega} = [\eta_1 : \eta_2 : \eta_3].$$

The centroids $S_i$ of $\Omega_i$ and $S^i$ of $\Omega^i$, respectively, are

$$S_1 = (\beta_1 - \gamma_1 : \gamma_2 - \alpha_2 : \alpha_3 - \beta_3), \quad S^1 = (\beta_1 - \alpha_1 : \alpha_2 - \gamma_2 : \gamma_3 - \beta_3),$$

$$S_2 = (\alpha_1 - \beta_1 : \beta_2 - \gamma_2 : \gamma_3 - \alpha_3), \quad S^2 = (\alpha_1 - \gamma_1 : \gamma_2 - \beta_2 : \beta_3 - \alpha_3),$$

$$S_3 = (\gamma_1 - \alpha_1 : \alpha_2 - \beta_2 : \beta_3 - \gamma_3), \quad S^3 = (\gamma_1 - \beta_1 : \beta_2 - \alpha_2 : \alpha_3 - \gamma_3).$$

A simple computation proves the following proposition:

**Proposition 7.** The three centroids of the $\Omega_i$ and the three homothetic centers $Q_{ij}$ lie on the line $g_{13}$, the three centroids of the $\Omega^i$ and the three homothetic centers $Q^{ij}$ on the line $g^{\Omega}$ (see Figure 6).

Marginal node: The triangles $\Phi_i$ and $\Phi^i$ have else some peculiarities:
Proposition 8. (a) The triangles $\Phi_i$ and $\Phi^i$ have the same area as the reference triangle (see Figure 7).

(b) The vertices of $\Phi_i$ and $\Phi^i$ lie on a circumconic $C_i$ and $C^i$ of $\Delta$, respectively. The $C_i$ are concurrent at $\eta^*$, the $C^i$ at $\xi^*$ (see Figure 8).

Proof: (a) By use of normalized barycentric coordinates of the vertices of $\Phi_i$ and $\Phi^i$ it is easy to show, that $|\det(\Phi_i)| = |\det(\Phi^i)| = 1$.

(b) It is simple to verify, that the isotomic conjugates of the vertices of each triangle $\Phi_i$ and $\Phi^i$ are collinear and that the concerned lines are concurrent at $\eta$ and $\xi$, respectively. From this follows the assertion directly. □
7. A special direction triple \((\alpha, \beta, \gamma)\)

In the end we consider an interesting special case. Given a direction \(\alpha = (\alpha_1 : \alpha_2 : \alpha_3)\). We choose \(\beta\) and \(\gamma\) as iterate Brocardians: \(\beta = \alpha_{\leftarrow \leftarrow} = (\alpha_3 : \alpha_1 : \alpha_2)\) and \(\gamma = \alpha_{\rightarrow \rightarrow} = (\alpha_2 : \alpha_3 : \alpha_1)\) [1]. The isotomic conjugates \(\alpha^*, \beta^*, \gamma^*\) are points on the Steiner ellipse, they form a Brocardian triple, that is the line at two of these points is the dual of the third point.

Let \(A', B', C'\) be the reflections of \(A, B, C\) in \(G\) and

\[
C_A : x^2 - yz = 0, \quad C_B : y^2 - zx = 0, \quad C_C : z^2 - xy = 0
\]

three ellipses created by translation of the Steiner ellipse with centers \(A', B', C'\) and passing through the common point \(G\).

In the following we mention without proofs some properties of points and triangles defined above.

(1) The line triples \((\gamma_A, \beta_B, \alpha_C), (\beta_A, \alpha_B, \gamma_C)\) and \((\alpha_A, \gamma_B, \beta_C)\) are concurrent at the points

\[
P_1 = (\alpha_3 \alpha_1 : \alpha_2 \alpha_3 : \alpha_1 \alpha_2), \quad P_2 = (\alpha_1 \alpha_2 : \alpha_3 \alpha_1 : \alpha_2 \alpha_3), \quad P_3 = (\alpha_2 \alpha_3 : \alpha_1 \alpha_2 : \alpha_3 \alpha_1),
\]

respectively, that is each of the triangles \(\Delta_i\) degenerates into a point on the Steiner ellipse.

(2) The triangles \(\Delta_i\) have the same centroid \(G\), it is at the same time the homothetic center of each pair \((\Delta_i, \Delta_j)\).

(3) The points of each triple \(\{U_1, U_2, U_3\}, \{V_1, V_2, V_3\}, \{W_1, W_2, W_3\}\) are collinear. The three lines concurrent in \(G\) are the duals of \(\alpha, \beta, \gamma\).
(4) For the vertices of the triangles $\Delta_i$ is valid:

$$U_1, V_3, W_2 \in C_A, \quad U_2, V_1, W_3 \in C_B, \quad U_3, V_2, W_1 \in C_C.$$  

These triangles are homothetic, congruent and equal in area with $\Delta$.

(5) The triangles $\Phi_k$ are homothetic and congruent with $\Delta$. The homothetic centers of the pairs $(\Delta, \Phi_k)$ are the medial images $mP_k$ of $P_k$. They are the midpoints of the segments $P_iP_j$, lie on the Steiner inellipse, have the representations

$$mP_1 = (\alpha_2^2 : \alpha_1^2 : \alpha_3^2), \quad mP_2 = (\alpha_3^2 : \alpha_2^2 : \alpha_1^2), \quad mP_3 = (\alpha_1^2 : \alpha_3^2 : \alpha_2^2)$$  

and form a Brocardian Triple (see Figure 11).

(6) The sum of the areas of $\Delta_i$ is sixfold the area of $\Delta$:

$$\sum |\Delta_i| = 6 \cdot |\Delta|.$$  

(7) The midway triangle of the pair $(\Delta_i, \Delta_j)$ coincides with the medial triangle of $\Delta_k$. 
The triangles $\Omega_i$ have the representations

\[
\Omega_1 = \begin{pmatrix}
0 & \alpha_3 & -\alpha_2 \\
-\alpha_2 & 0 & \alpha_3 \\
\alpha_3 & -\alpha_2 & 0
\end{pmatrix}, 
\quad \Omega_2 = \begin{pmatrix}
0 & \alpha_1 & -\alpha_3 \\
-\alpha_3 & 0 & \alpha_1 \\
\alpha_1 & -\alpha_3 & 0
\end{pmatrix}, 
\quad \Omega_3 = \begin{pmatrix}
0 & \alpha_2 & -\alpha_1 \\
-\alpha_1 & 0 & \alpha_2 \\
\alpha_2 & -\alpha_1 & 0
\end{pmatrix}
\]

and the centroid $G$. The homothetic centers of the pairs $(\Omega_i, \Omega_j)$ coincide with $G$.

(9) The triangles $\Omega_i$ degenerate, their vertices lie on the infinite line.

References


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Symbolic Substitution Has a Geometric Meaning

Manfred Evers

Abstract. By comparing two different metrics in the affine plane, it is shown that symbolic substitution, introduced by Clark Kimberling, has a clear geometric meaning.

1. Introduction

Consider in the real plane \( \mathbb{R}^2 \) a triangle \( ABC \) with vertices

\[
A = \left( \frac{a^2 - b^2 + c^2}{2a}, \frac{\sqrt{\sigma}}{2a} \right), \quad B = (0, 0), \quad C = (a, 0),
\]

where \( a, b \) and \( c \) are positive real numbers with

\[
\sigma := \sigma(a, b, c) = (a + b + c)(-a + b + c)(a - b + c)(a + b - c) > 0.
\]

If in \( \mathbb{R}^2 \) the Euclidean distance \( d \) of two points \( U = (X, Y) \) and \( U' = (X', Y') \) is defined by

\[
d(U, U') = \sqrt{(X - X')^2 + (Y - Y')^2},
\]

then the reference triangle \( ABC \) has side lengths \( a, b, c \). In the following we use Conway’s triangle notation:

\[
S = \frac{1}{2}\sqrt{\sigma}, \quad S_A = \frac{1}{2}(b^2 + c^2 - a^2), \text{ etc.}
\]

Thus, we can write \( A = \left( \frac{S_B}{a}, \frac{S}{a} \right) \).

Besides Cartesian coordinates we also use barycentric coordinates with respect to the reference triangle. The notation \( U = (u, v, w)_{ABC} \) is used if \( (u, v, w) \) are the absolute coordinates of \( U \) with respect to \( ABC \). If \( (u : v : w) \) are homogeneous coordinates of \( U \), then we write \( U = (u : v : w)_{ABC} \). The conversion from Cartesian coordinates \( (X, Y) \) of a point \( U \) to its barycentric coordinates \( (u, v, w) \) can be achieved by taking

\[
u = \frac{aY}{S}, \quad v = -\frac{(X - a)S + YS_C}{aS}, \quad w = \frac{XS - YS_B}{aS} \quad (\ast)
\]

Given two points \( U \) and \( V \) with absolute barycentric coordinates \( (u, v, w) \) and \( (u', v', w') \), the square of the distance between these two points can be calculated

by the formula
\[ d^2(U, U') = S_A(u - u')^2 + S_B(v - v')^2 + S_C(w - w')^2. \]

In addition to the canonical Euclidean metric, we introduce a second metric \( d_g \), which we call \textit{generalized metric} or \textit{g-metric}, for short. This second metric is obtained by giving new lengths to the sides of the reference triangle:
\[ d_g(A, B) = c_g, \quad d_g(B, C) = a_g, \quad d_g(C, A) = b_g. \]

We still demand that these lengths \( a_g, b_g, c_g \) are positive real numbers \(^1\) but do not claim that
\[ \sigma_g := (a_g + b_g + c_g)(-a_g + b_g + c_g)(a_g - b_g + c_g)(a_g + b_g - c_g) \]
is a positive real number.

For different signs of \( \sigma_g \) we get different geometries: \( \sigma_g > 0 \) delivers an affine version of the Euclidean metric, for \( \sigma_g = 0 \) the metric is Galilean, and for \( \sigma_g < 0 \) the metric is a Lorentz-Minkowski metric. With respect to this generalized metric, the square of the distance between two points \( U \) and \( U' \) with absolute barycentric coordinates \((u, v, w)\) and \((u', v', w')\) is given by
\[ d_g^2(U, U') = S_{A,g}(u - u')^2 + S_{B,g}(v - v')^2 + S_{C,g}(w - w')^2 \]
with \( S_{A,g} = -a_g^2 - b_g^2 - c_g^2 \) etc. The point with coordinates \((a_g : b_g : c_g)\) will become the generalized incenter \( I_g \). The new circumcenter \( O_g \) has coordinates \((a_g^2 (-a_g^2 + b_g^2 + c_g^2) : \cdots : \cdots)\). More generally, if \( x(a, b, c) \) is a barycentric center function of a triangle center \( X \), then \( X_g = (x(a_g, b_g, c_g) : \cdots : \cdots)_{ABC} \) is the corresponding \( g \)-center. The point \( G_g \) still agrees with \( G = (1 : 1 : 1)_{ABC} \). The square of the new triangle area is \( \frac{1}{16} \sigma_g \).

![Image](image-url)

**Figure 1.** An equilateral triangle is given new side lengths \((a_g : b_g : c_g) = (3 : 4 : 5)\). The picture shows the circumcircle (blue), the incircle (green) and the nine-point circle (red).

Figures 1 and 2 show the situation for an equilateral triangle \((a = b = c)\) on which was imposed a new metric \((a_g, b_g, c_g) = (3, 4, 5)\) and \((a_g, b_g, c_g) = (1)_{ABC} .

---

\(^1\)Later on, we dismiss this condition, see 2.4.
Symbolic substitution has a geometric meaning

Figure 2. An equilateral triangle is given new side lengths \((a_g : b_g : c_g) = (6 : 9 : 13)\). Besides the circumcircle, the incircle and the nine-point circle, the picture shows the excircles (green) and the polarcircle (brown).

(6, 9, 13), respectively. Considering these two new metrics, the first triangle is right angled, the second obtuse.

A generalization of the incenter, the circumcenter and the orthocenter of a triangle is given by I. Minevich and P. Morton [12, Section 4]. They introduce the name generalized triangle centers. We adopt this terminology.

1.1. Embedding the affine in a projective plane. The affine plane can be embedded in a projective plane \(P\) by adding the line at infinity \(L_\infty\). This line consists of all points \(U = (u : v : w)_{ABC}\) with \(u + v + w = 0\). Points on \(L_\infty\) are called infinite or improper while those in the affine plane are called finite or proper. A line is called proper if it differs from \(L_\infty\).

1.2. \(g\)-orthogonality of lines. If two proper lines are \(g\)-orthogonal, we shall say that their infinite points form a pair of \(g\)-orthopoints. Two infinite points \(U = (u : v : w)_{ABC}\) and \(U' = (u' : v' : w')_{ABC}\) are \(g\)-orthopoints if and only if \(S_{A_g} u u' + S_{B_g} v v' + S_{C_g} w w' = 0\). See [15, p. 55] or [2] for the Euclidean case. The following version of Thales’ theorem is valid in all metric affine geometries: Given two lines \(L\) and \(L'\) which meet in a finite point \(P\), these lines are orthogonal precisely when there exist two (not necessarily finite) points \(Q\) and \(R\), one on \(L\) and the other on \(L'\) and both different from \(P\), so that the center of the (generalized) circle \(PQR\) is a point on the line \(QR\).

In the following, we analyze the three cases \(\sigma_g > 0\), \(\sigma_g = 0\) and \(\sigma_g < 0\).

2. The affine Euclidean case

If \(\sigma_g > 0\), then the \(g\)-incenter \(I_g\) lies inside the medial triangle and the \(g\)-symmedian \(K_g\) lies inside the Steiner inellipse. The metric given by formula (**) is affine Euclidean. Proof: By an affine transformation \(\tau\) with fixed point \(I_g\), the inconic with center \(I_g\) can be mapped onto a circle. Let \(A', B', C'\) be the image points of \(A, B, C\) under \(\tau\). This transformation maps a point \(U = (u : v : w)_{ABC}\) to the point \(U' = (u : v : w)_{A'B'C'}\). \(I_g\) is the incenter of \(A'B'C'\); therefore the ratio of the sidelengths of this triangle is \(a_g : b_g : c_g\). If \(x\) is a barycentric center function, then \(\tau\) maps the \(g\)-center \(X_g = (x(a_g, b_g, c_g) : x(b_g, c_g, a_g) : x(c_g, a_g, b_g))_{ABC}\) of
2.1. Description of the generalized circumcircle. In case of \( \sigma_g > 0 \), the generalized circles are ellipses. We assume that \( I_g \) and \( I \) are different points, so that these generalized circles are ellipses with two foci. The focal axes of these ellipses have all the same infinite point. We will determine this point. Furthermore, for the \( g \)-circumcircle we calculate the lengths of the two principal axes and the distance between the two foci.

The equation of a \( g \)-circle with center \( M = (m_a, m_b, m_c)_{ABC} \) and radius \( \rho \) is

\[
\sum_{\text{cyclic}} S_{A,g}((m_b + m_c)x - m_a(y + z))^2 = \rho^2(m_a + m_b + m_c)^2(x + y + z)^2.
\]

We concentrate on the \( g \)-circumcircle, which is given by the equation

\[
a_g^2yz + b_g^2zx + c_g^2xy = 0.
\]

Using (*), we can derive an equation in Cartesian coordinates,

\[
\alpha(X - X_0)^2 + 2\beta(X - X_0) + \gamma(Y - Y_0)^2 = \left(\frac{\alpha a_g b_g c_g S}{S_g}\right)^2,
\]

with \( \alpha = (2a_g S)^2, \beta = 2S(a_g^2(b^2 - c^2) - a^2(b_g^2 - c_g^2)) \), \( \gamma = a_g^2(b^2 - c^2) - 2a^2(b^2 - c^2)(b_g^2 - c_g^2) - a^4(a_g^2 - 2b_g^2 - 2c_g^2) \), and \((X_0, Y_0)\) the Cartesian coordinates of the \( g \)-circumcenter

\[
O_g = (a_g^2(-a_g^2 + b_g^2 + c_g^2) : \cdots : \cdots)_{ABC}.
\]

The direction vectors of the two principal axes of the circumellipse are the eigenvectors of the matrix \( M = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \). The two eigenvalues are

\[
\lambda_{\pm} = \frac{1}{2} \left( \alpha + \gamma \pm \sqrt{(\alpha - \gamma)^2 + 4\beta^2} \right) = 2a^2(\theta \pm \phi^2)
\]

with

\[
\theta = \frac{\alpha + \gamma}{4a^2} = a^2S_{A,g} + b^2S_{B,g} + c^2S_{C,g} = a_g^2S_A + b_g^2S_B + c_g^2S_C,
\]

\[
\phi = \sqrt{\theta + 2S_g}(\theta - 2S_g) \quad \text{and} \quad S_g = \frac{1}{2}\sqrt{\sigma_g}.
\]

In the affine Euclidean case we have \( \lambda_+ > \lambda_- > 0 \). A calculation shows that the lengths of the major and minor half axes are

\[
r_{\max} = \frac{aa_g b_g c_g S}{S_g \sqrt{\lambda_-}} \quad \text{and} \quad r_{\min} = \frac{aa_g b_g c_g S}{S_g \sqrt{\lambda_+}}.
\]

The distance \( f \) of the foci from the center \( O_g \) is therefore

\[
f = \sqrt{r_{\max}^2 - r_{\min}^2} = \frac{a_g b_g c_g \phi}{\sigma_g}.
\]
Symbolic substitution has a geometric meaning. The eigenspaces of \( \lambda_+ \) and \( \lambda_- \) are \( \mathbb{R}(\lambda_+ - \gamma, \beta) \), so that the focal line is \( (X_0, Y_0) + \mathbb{R}(\lambda_- - \gamma, \beta) \). The infinite point on this line is
\[
(a^2 \beta : -S(\lambda_- - \gamma) - SC \beta : S(\lambda_- - \gamma) - SB \beta)_{ABC}.
\]
The Cartesian coordinates of the two foci \( F_\pm \) are \( (X_0 \pm fv_1, Y_0 \pm fv_2) \), \( (v_1, v_2) \) being a unit eigenvector of \( \lambda_- \). Subsequently, the barycentric coordinates of the two foci can be calculated:
\[
F_\pm = (p \pm p', q \pm q', r \pm r')_{ABC},
\]
where \( (p, q, r) \) are the absolute barycentric coordinates of \( O \).

2.2. Criterion for the \( g \)-orthogonality of lines / Theorem of Brianchon and Poncelet. Let two different lines meet the line at infinity in the points \( U, V \). These lines are \( g \)-orthogonal if and only if there exists a hyperbola through the points \( A, B, C, U, V \) and the \( g \)-orthocenter \( H_g = (1 - a^2 g + b^2 g + c^2 g : \cdots : \cdots) \).

2.3. The \( g \)-angle between two lines. We want to determine the \( g \)-measure \( \phi_g \in [0, \pi/2] \) of the angle \( \angle (\mathcal{L}, \mathcal{L}') \) between the proper lines \( \mathcal{L} : lx + my + nz = 0 \) and \( \mathcal{L}' : l'x + m'y + n'z = 0 \). This measure is fixed as soon as we know the value of \( \sin \phi_g \) or \( \cos \phi_g \). To get these two values, let
\[
k := l(m' - n') + m(n' - l') + n(l' - m'),
\]
and calculate the \( g \)-area \( |PQR|_g \) of a triangle \( PQR \), where
\[
P = \left( \frac{mn' - m'n}{k}, \frac{l'n - ln'}{k}, \frac{ln' - l'm}{k} \right)_{ABC}
\]
is the intersection of \( \mathcal{L} \) and \( \mathcal{L}' \),
\[
Q = \left( \frac{mn' - m'n}{k} + m - n, \cdots, \cdots \right)_{ABC}
\]
is a second point on \( \mathcal{L} \), and
\[
R = \left( \frac{mn' - m'n}{k} + m' - n', \cdots, \cdots \right)_{ABC}
\]
is a second point on \( \mathcal{L}' \). For this,
\[
|PQR|_g = |k| \cdot |ABC|_g = \frac{1}{4} |k| \sqrt{\sigma_g} = \frac{1}{2} |k| S_g.
\]
Thus, we get
\[
\sin \phi_g = \frac{2k \cdot |ABC|_g}{d_g(P, Q)d_g(P, R)},
\]
\[
\cos \phi_g = \frac{S_{A, g}(m - n)(m' - n') + S_{B, g}(n - l)(n' - l') + S_{C, g}(l - m)(l' - m')}{d_g(P, Q)d_g(P, R)},
\]
\[
d_g^2(P, Q) = S_{A, g}(m - n)^2 + S_{B, g}(n - l)^2 + S_{B, g}(l - m)^2,
\]
\[
d_g^2(P, R) = S_{A, g}(m' - n')^2 + S_{B, g}(n' - l')^2 + S_{B, g}(l' - m')^2.
\]

We interpret \( \vec{l} = \begin{pmatrix} m - n \\ n - l \\ l - m \end{pmatrix} \) and \( \vec{l}' = \begin{pmatrix} m' - n' \\ n' - l' \\ l' - m' \end{pmatrix} \) as direction vectors of \( L \) and \( L' \) respectively, and the expression
\[
S_{A, g}(m - n)(m' - n') + S_{B, g}(n - l)(n' - l') + S_{C, g}(l - m)(l' - m')
\]
as a dot product \( \vec{l} \cdot \vec{l}' = <\vec{l}, \vec{l}'>_g \) of these two vectors, so that we can write
\[
\cos(\phi_g) = \frac{|\vec{l} \cdot \vec{l}'|}{\sqrt{\vec{l} \cdot \vec{l}'}}.
\]

2.4. Negative sidelengths \( a_g, b_g, c_g \). There is no reason to forbid negative values for \( a_g, b_g, c_g \). For example, if we choose \( a_g = -a \), \( b_g = b \) and \( c_g = c \) or \( a_g = a \), \( b_g = -b \) and \( c_g = -c \), we just swap the incenter \( I \) with the excenter \( I_a \) and the excenter \( I_b \) with the excenter \( I_c \). If a weak center, consisting of the main point and its three mates, is not considered as a quadruple but as a set of four points, changing signs leaves this weak center invariant. Strong centers keep their positions, anyway. This situation does not change if we dismiss the condition \( \sigma_g > 0 \).

3. The Galilean case

If \( \sigma_g = 0 \), the point \( H_g \) is an infinite point and agrees with \( O_g \) and \( N_g \). \( H_g \) is the absolute pole of the Galilean plane: two finite points lying on the same line through \( H_g \) have Galilean distance 0. A line through the absolute pole is therefore called a null line. In particular, the three generalized triangle altitudes, considered as lines, are null lines. This is in accordance with the area of the triangle: \( \sqrt{\sigma_g} = 0 \). The \( g \)-symmedian point \( K_g \) is a point on the Steiner inellipse. The incenter and the excenters are the barycentric square roots of \( K_g \). One of these roots agrees with \( H_g \), so it is an infinite point. The other three roots lie on the sidelines of the medial triangle of \( ABC \).

3.1. \( g \)-circles in Galilean geometry. Generalized circles in the Galilean plane are parabolas. Some of them may degenerate. For example, the \( g \)-incircle and the \( g \)-excircles are double lines (lines with multiplicity 2), and if the incenter is a vertex of the medial triangle, then the \( g \)-circumcircle and the \( g \)-nine-point circle each consist of two parallel lines. It should be noted that a degenerate \( g \)-circle in Galilean geometry has more than one center. All these centers lie on a “centerline” of this \( g \)-circle. Michel Bataille [2] calculated the barycentric coordinates of the focus \( F \) of
Symbolic substitution has a geometric meaning.

A circumparabola $C$ of the triangle $ABC$: if the infinite point of $C$ has barycentric coordinates $(u:v:w)$, then the focus is

$$F = \left( \frac{u^2}{vw} + \frac{a^2 vw}{a^2 vw + b^2 vw + c^2 vw} : \cdots : \cdots \right)_{ABC}.$$

Figure 3 gives an illustration of a triangle in the Galilean plane.

3.2. $g$-Orthogonality of lines in a Galilean plane. Two lines which meet in a finite point are orthogonal if and only if one of these is a null line.

3.3. The $g$-angle between two lines. It makes sense to define the $g$-measure of the angle between two non-null-lines $L$ and $L'$ to be the $g$-distance of the poles of these lines with respect to the $g$-circumcircle of the triangle $ABC$. Proof: We may assume that the finite part of the $g$-circumcircle of the triangle $ABC$ in $\mathbb{R}^2$ is given by the equation $y = kx^2$, $k \neq 0$, and that $y = mx + n$ and $y = m'x + n'$ are the equations of the lines $L$ and $L'$. The poles of these two lines with respect to the $g$-circumcircle are $P = (\frac{m}{2k}, -n)$ resp. $P' = (\frac{m'}{2k}, -n')$. There exists a real number $k' \neq 0$ so that the Galilean distance between any two points $(x, y)$ and $(x', y')$ is $k'|x - x'|$. Therefore, the distance between $P$ and $P'$ is $\frac{k'}{2k}|m - m'|$. This is consistent with the usual definition of the measure of the angle between the two lines $L$ and $L'$ in Galilean geometry, see for example [7, Chap 4] or [13, Chap 23].

Remarks. (1) Using barycentric coordinates, the pole of the line $L : lx+my+nz = 0$ with respect to the circumcircle $C : a^2 yz + b^2 zx + c^2 xy = 0$ is

$$P = \left( a^2_g(-a^2_g l + b^2_g m + c^2_g n) : \cdots : \cdots \right)_{ABC}.$$

(2) For the reference triangle $ABC$, the ratio of the measure of an angle and the length of its opposite side is $\frac{1}{2}$. (The measure $\alpha$ of the angle between $AB$ and $AC$ is $\frac{1}{2}a$ etc).
(3) For the medial triangle of $ABC$, the measure of an angle equals the measure of its opposite side.

4. The Lorentz-Minkowski case

If $\sigma_g < 0$ then the metric $d_g$ is a Lorentz-Minkowski metric or an LM metric, for short. In this case, generalized circles are hyperbolas which might eventually degenerate. Given a $g$-metric, all the $g$-circles run through the same two points on the line at infinity $L_\infty$. These two points $P_1$ and $P_2$ are the absolute poles of the LM plane. The distance between two different finite points $U$ and $V$ is 0 if and only if (either) $P_1$ or $P_2$ is a point on the line $UV$; in this case $UV$ is called a null line. In LM geometry, not only the distance between two points can be negative but also the square of the distance. Following A. Einstein [6], we will define: Two points $U$ and $V$ are in a spacelike position if $d_g^2(U,V) > 0$, and they are in a timelike position if $d_g^2(U,V) < 0$. If two points $U$, $V$ are in a spacelike (respectively timelike) position, then all finite points $P$, $Q$ on the line $UV$ with $P \neq Q$, are in a spacelike (respectively timelike) position. Therefore, we can call the line $UV$ spacelike (respectively timelike). Obviously, in LM geometry triangles can have real as well as imaginary sidelengths. Triangles having both, a real and an imaginary sidelength $\neq 0$, do not have weak triangle centers. All strong centers still exist.

4.1. Description of the generalized circles. Generalized circles in the LM plane are hyperbolas. The focal axes of two $g$-circles do not have to be parallel but may be orthogonal, as can be seen in Figure 7. The principal axes and the foci can be calculated the same way as in §2.1 for the Euclidean case, except for the distance between the foci and the center $O_g$. This is now

$$f = \sqrt{r_{\text{max}}^2 + r_{\text{min}}^2} = -\frac{a_gb_gc_g\sqrt{|\theta|}}{\sigma_g}.$$ 

The two absolute poles are

$$P_1 = (-a^2(\omega + \beta) : (a^2 + S_B)(\omega + \beta) + S_\gamma : -S_B(\omega + \beta) - S_\gamma)_{ABC},$$

$$P_2 = (-a^2(\omega - \beta) : (a^2 - S_B)(\omega - \beta) + S_\gamma : S_B(\omega - \beta) - S_\gamma)_{ABC}.$$ 

Here we use the notations of §2.1 and $\omega := \sqrt{\beta^2 - \alpha \gamma} = a^2 \sqrt{-\sigma \sigma_g}$.

4.2. $g$-Orthogonality of lines. A pair $(U,V)$ of points on the line at infinity is a pair of $g$-orthopoints if and only if the quadruple $(U,V,P_1,P_2)$ is harmonic. It can be easily checked that if a line is spacelike then all its $g$-orthogonal lines are timelike. Particularly, if a sideline of a triangle is spacelike then the corresponding altitude, considered as a line, is timelike and vice versa. Figure 5 shows a triangle with $d_g(B,C) = 0$, and Figure 6 illustrates the situation for a triangle with $d_g(A,B) = d_g(A,C) = 0$. In both cases, the generalized circumcircle as well as the generalized nine point circle degenerate to a pair of lines. In Figure 7 the generalized symmedian $K_g$ is outside the triangle, therefore weak generalized centers do not exist in the real plane.
Symbolic substitution has a geometric meaning

Figure 4. An equilateral triangle is given new side lengths \((a_g : b_g : c_g) = (3 : 5 : 11)\). The picture shows the \(g\)-circumcircle (blue), the \(g\)-incircle (green), the \(g\)-nine-point circle (red) and the \(g\)-polar circle (brown). All \(g\)-circles go through the same infinite points.

Figure 5. An equilateral triangle is given new side lengths \((a_g : b_g : c_g) = (0 : 1 : 2)\). The picture shows the following generalized circles: the circumcircle (it consists of two blue lines intersecting in \(O_g\)), the incircle (green), the nine-point circle (it consists of two red lines intersecting in \(N_g\)) and the polarcircle (brown). \(I_g, O_g\) and \(K_g\) are points on the sideline \(BC\), \(N_g\) is a point on the \(g\)-incircle.

4.3. \textit{The g-angle between two lines.} We want to calculate the \(g\)-measure \(\phi_g\) of the angle \(\angle_g(L, L')\) between the proper lines \(L : lx + my + nz = 0\) and \(L' : l'x + m'y + n'z = 0\). To get a real value for \(\phi_g\), these lines have to be both spacelike or both timelike. The value of \(\phi_g \geq 0\) is determined by the value of \(\cosh(\phi_g)\). A calculation similar to the one in the Euclidean case gives the following result (we
Figure 6. An equilateral triangle is given new side lengths \((a_g : b_g : c_g) = (0 : 0 : 1)\). The points \(I_g, O_g\) and \(K_g\) agree with the vertex \(C\).

Figure 7. An equilateral triangle is given new side lengths \((a_g : b_g : c_g) = (2i : 2 : 3)\).

use notations defined in §2.3):

\[
\cosh(\phi_g) = \frac{|\vec{l} \cdot \vec{l}'|}{\sqrt{|\vec{l} \cdot \vec{l}| \sqrt{|\vec{l}' \cdot \vec{l}'|}}}
\]

5. Symbolic substitutions

In [10], C. Kimberling gives the following definition of the transfigured plane of a triangle:

Suppose \(a, b, c\) are variables (or indeterminates) over the field of complex numbers and that \(x, y, z\) are homogeneous algebraic functions of \((a, b, c)\): \(x = x(a, b, c), y = y(a, b, c), z = z(a, b, c)\), all of the same degree of homogeneity and not all identically zero. Triples \((x, y, z)\) and \((x', y', z')\) are equivalent if \(xy' = yx'\) and
Symbolic substitution has a geometric meaning

\[ yz' = zy' \]. The equivalence class containing any particular \((x, y, z)\) is denoted by \(x : y : z\) and is a ‘point’. Let \(A = 1 : 0 : 0\), \(B = 0 : 1 : 0\), \(C = 0 : 0 : 1\). These three points define the reference triangle \(ABC\). The set of all points is the transfigured plane.

We denote this transfigured plane by \(\mathcal{P}\). A model of \(\mathcal{P}\) can be obtained by taking \(a, b, c\) as the side lengths of the triangle \(ABC\) and \((x(a, b, c) : y(a, b, c) : z(a, b, c))\) as barycentric coordinates of a point. A second model is constructed by taking \((x : y : z)\) as trilinear coordinates of a point. If \(x', y', z'\) are functions of \(a, b, c\), all of the same degree of homogeneity and not all identically zero, then the substitution \((a, b, c) \mapsto (x'(a, b, c), y'(a, b, c), z'(a, b, c))\) induces a transformation on \(\mathcal{P}\).

While in the first model, in addition to the vertices \(A, B, C\) of the reference triangle, its centroid is a fixed point of the transformations, in the second the vertices and the incenter stay fixed. The advantage of the first model, as compared to the second, is that the transformations leave the line at infinity invariant. Kimberling [9] (see also [4, \S 2.3]) names these substitutions and the hereby induced transformations *symbolic* because, as he says, they suffer from a geometric meaning:

The adjective *symbolic* is applied to substitutions \((\alpha, \beta, \gamma) \mapsto (\alpha', \beta', \gamma')\) in order to distinguish between these and geometric transformations. Consider the substitution \((a, b, c) \mapsto (bc, ca, ab); \) if \(a, b, c\) are the sidelengths 2, 4, 5 of a triangle then \(bc, ca, ab\) are not side-lengths of a triangle. Moreover, for general \(a, b, c\), a geometric construction of a point \((x(a, b, c) : y(a, b, c) : z(a, b, c))\) offers no clues for constructing the point \(x(bc, ca, ab) : y(bc, ca, ab) : z(bc, ca, ab)\).

The first of these two arguments can be dismissed, as in the former sections was shown that every triangle can be given any side lengths \(a_g, b_g, c_g\) except for \(a_g = b_g = c_g = 0\) by taking a suitable generalized metric \(d_g\). Furthermore, this new metric can be used for the construction of new triangle centers. We give two examples:

1. The substitution

\[
(x, y, z) = (-a, b, c) \mapsto (x', y', z')
\]

\[
=(a^2(b^2 + c^2) - (b^2 - c^2)^2, b^2(a^2 + c^2) - (c^2 - a^2)^2, c^2(a^2 + b^2) - (a^2 - b^2)^2)
\]

maps the excenter \(I_a = (-a, b, c)_{ABC}\) to the nine-point center \(N\), the incenter \(I\) to the anticevian point \(aN\) of \(N\), and the symmedian point \(K\) to \(N^2\) (the barycentric square of \(N\)). If we take \(aN\) as the generalized incenter \(I_{a,g}\) then \(N\) is going to be the generalized excenter \(I_{a,g}N\) and \(N^2\) the perspector of the \(g\)-circumcircle.

2. Construction of the contact triangle of the MacBeath inconic, starting from its center \(N\). The substitution

\[
(x, y, z) = (a, b, c) \mapsto (x', y', z')
\]

\[
=(a^2(b^2 + c^2) - (b^2 - c^2)^2, b^2(a^2 + c^2) - (c^2 - a^2)^2, c^2(a^2 + b^2) - (a^2 - b^2)^2)
\]
maps \( I \) to \( N \) and the Gergonne point \( G_e \) to \( G/O \), the isotomic conjugate of the circumcenter \(^2\). \( I \) is the center and \( G_e \) is the perspector of the incircle. Therefore, \( G_e \) is the isotomic conjugate of the anticomplement of \( I \). Thus, we can construct \( G/O \) as isotomic conjugate of the anticomplement of \( N \). The cevian triangle of \( G/O \) is the contact triangle of the MacBeath inconic.

5.1. The geometric meaning of a symbolic substitution. Let \( x(a, b, c) \) be a barycentric center function. If in the affine plane the square of the distance between two points \( U = (u, v, w)_{ABC} \) and \( U' = (u', v', w')_{ABC} \) is given by

\[
d^2(U, U') = \sum_{\text{cyclic}} \frac{1}{2} (-a^2 + b^2 + c^2)(u - u')^2,
\]

the point \( X \) that corresponds to this center function has barycentric coordinates \((x(a, b, c) : x(b, c, a) : x(c, a, b))\). The same point is the incenter of the reference triangle when using a distance function \( d_x \) with

\[
d_x^2(U, U') = \sum_{\text{cyclic}} \frac{1}{2} (-x(a, b, c))^2 + (x(b, c, a))^2 + (x(c, a, b))^2)(u - u')^2.
\]

Let \( x', y', z' \) be functions of \( a, b, c \), all of the same degree of homogeneity and not all identically zero. The point

\[X' = (x'(a, b, c), y'(a, b, c), z'(a, b, c)) : \cdots : \cdots\] \(_{ABC}\]

can be interpreted as a point with center function \( x \), when the square of the distance \( d' \) of two points \( U = (u, v, w) \) and \( U' = (u', v', w') \) is given by

\[
d'^2(U, U') = \sum_{\text{cyclic}} \frac{1}{2} (-x'(a, b, c))^2 + (y'(a, b, c))^2 + (z'(a, b, c))^2)(u - u')^2.
\]

6. Generalized versions of Feuerbach’s conic theorem

Feuerbach’s conic theorem establishes a connection between two different metrics of the affine plane.

6.1. Feuerbach’s conic theorem (see [5, 1]). Each circumconic of a triangle running through the orthocenter \( H \) has its center on the nine point circle.

If we interpret this circumconic as a \( g \)-circumcircle of the reference triangle with center \( O_g \), then the \( g \)-orthocenter \( H'_g \) is a point on the Euclidean circumcircle. Thus, we can formulate the following generalized versions (6.1.1) and (6.1.2) of Feuerbach’s conic theorem:

6.1.1. Let \( \mathcal{C} \) and \( \mathcal{C}' \) be two circumconics of triangle \( ABC \). We interpret \( \mathcal{C} \) and \( \mathcal{C}' \) as generalized circumscribed circles with center \( O_g \) and \( O_{g'} \). Then \( H'_g \) is a point on \( \mathcal{C}' \) if and only if \( H'_g \) is a point on \( \mathcal{C} \).

\(^2\)\( G/O \) is \( X_{284} \) in [9].
6.1.2. A conic runs through all four points of a $g$-orthocentric system if and only if its center lies on the $g$-orthic circle, which is the $g$-circumcircle of the $g$-orthic triangle.

6.2. Corollaries.

6.2.1. Each conic through the vertices of a quadrangle has its center on the nine-point conic of this (complete) quadrangle, see for example [4, Lemma 19.1.2], [12] and [15].

6.2.2. Together with the incenter $I$, the excenters $I_a, I_b, I_c$ of the triangle $ABC$ form an orthocentric system, the reference triangle $ABC$ being the orthic triangle. Each conic running through these four points has its center on the circumcircle of the triangle $ABC$. Taking one of these conics and interpreting its center as a $g$-orthocenter, this conic is the $g$-polar circle of $ABC$. The triangle $ABC$ is self polar with respect to it.

6.2.3. We shall call an LM metric canonical if $H_g$ is a point on the (common) circumcircle of $ABC$. In this case the $g$-circumconic is a rectangular circumhyperbola of $ABC$, thus running through $H$. The center $O_g$ of this conic is the complement of $H_g$ and lies on the nine-point circle. The $g$-nine-point circle is a rectangular circumhyperbola through the vertices of the medial triangle and its orthocenter (= circumcenter of $ABC$). The perspector of the $g$-circumconic, the $g$-symmedian point $K_g$, is a point on the tripolar line of $H$. If $H$ is inside the triangle, $K_g$ has to lie outside, and weak $g$-centers do not exist in the real plane. We now assume that $K_g$ lies inside the triangle $ABC$. In this case $ABC$ is obtuse, the $g$-incenter and the $g$-excenters exist, and these four points lie on the polar circle of $ABC$.

6.2.4. If $P = (p : q : r)$ and $P' = (p' : q' : r')$ are the centers of two different circumconics $C$ and $C'$ of $ABC$, then the perspectors of these conics have coordinates $(u : v : w) = (p(-p + q + r) : \cdots : \cdots)$ respectively $(u' : v' : w') = (p'(-p' + q' + r') : \cdots : \cdots)$. The fourth (nontrivial) intersection point of $C$ and $C'$ has coordinates

\[
\left( \frac{1}{uw' - v'w} : \frac{1}{wu' - w'u} : \frac{1}{vw' - u'v} \right).
\]

In the case of $P' = O$, the fourth intersection point is called the Collings transform of $P$, see [9] for a definition and a bibliography.

6.2.5. Let $ABCD$ be a not degenerate quadrangle, the point $D$ having barycentric coordinates $(d : e : f)$ with respect to $ABC$. Then the centers of circumconics of $ABCD$ lie on the nine-point conic of the complete quadrangle $ABCD$. This conic is also the bicevian conic of $D$ and $G$ with respect to the triangle $ABC$. The barycentric equation of this conic is

\[
e f x(-x + y + z) + f dy(x - y + z) + de z(x + y - z) = 0. \quad (***)
\]
Its center is the midpoint of the Varignon parallelogram of the quadrangle $ABCD$ and has coordinates $(2d + e + f : \cdots : \cdots)$. We will denote this midpoint by $M_D$. If by a homothety $\chi$ with center $G$ and scale factor $\frac{1}{4}$, the quadrangle $ABCD$ is mapped onto a quadrangle $A'B'C'D'$, and $D'$ agrees with $M_D$, as can be proved by a simple calculation. If we now assume that $D$ is a point on a circumconic $\mathcal{C}$ of $ABC$, the point $D' = M_D$ lies on the image $\mathcal{C}'$ of $\mathcal{C}$ under $\chi$. Suppose that the center of $\mathcal{C}$ has coordinates $(k : l : m)$, then the center of $\mathcal{C}'$ has coordinates $(2k + l + m : \cdots : \cdots)$. Proof of $(***)$: Taking $(d : e : f)_{ABC}$ as the $g$-orthocenter $H_g$ of $ABC$, the symmedian point $K_g$ has coordinates $(a^2_g : b^2_g : c^2_g) = (d(e + f) : e(f + d) : f(d + e))$. We can now use the equation of the $g$-nine-point circle $\sum_{\text{cyclic}} S_{A,g}(x^2 - a^2_gyz) = 0$ (see [16] for the ordinary version) to get the equation $(***)$. □

7. Additional change of the centroid

In the plane $\mathcal{P}$ of the reference triangle $ABC$ we choose a point $P = pA + qB + rC$ with $p + q + r = 1$ and $pqr \neq 0$. The mapping, which assigns each point $Q$ the barycentric product $P \cdot Q$, is a bijective projective transformation of the plane. It maps the quadrangle $ABCG$ ($G$ being the centroid) to the quadrangle $ABCP$ and the line at infinity to the tripolar line $T_P$ of $P$. Making $P$ the new centroid $G_n$ of triangle $ABC$ and $T_P$ the new line at infinity, we can define a new metric on $\mathcal{P} - T_P$: The triangle $ABC$ is given side lengths $a_g, b_g, c_g$, where $a_g, b_g, c_g$ may take any real or purely imaginary values except for $a_g = b_g = c_g = 0$. The square of the new distance between two points $U = (u : v : w)_{ABC}$ and $V = (u' : v' : w')_{ABC}$ is now defined by

$$d^2_{\text{new}}(U, U') = \sum_{\text{cyclic}} S_{A,g}((uv' - u'v)r + (uw' - u'w)q)^2((u'qr + v'r + w'pq)^2)$$

If $X_g$ is a triangle center of $ABC$ with respect to the metric $d^2_g$ as defined in Section 1, then $X_g \cdot P$ is the corresponding center of triangle $ABC$ with respect to the new metric. Figure 8 gives an illustration by showing the most important triangle centers and triangle circles.
Symbolic substitution has a geometric meaning

Figure 8. An equilateral triangle is given new side lengths \((a : b : c) = (6 : 9 : 13)\). Here, the new line at infinity is the polar line \(T_P\) of the point \(P = G_n\) with barycentric coordinates \((2 : 3 : 4)\). There exists a point \(Z\) on the new line at infinity with the following properties: The mirror image of \(Z\) in a new circle is its new center, and the polar line of \(Z\) with respect to this new circle is parallel to \(T_P\).

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Quasi-circumcenters and a Generalization of the Quasi-Euler Line to a Hexagon

Michael de Villiers

Abstract. This short note first proves an elementary property of the quasi-circumcenter of a quadrilateral, and then generalizes the quasi-Euler line of a quadrilateral to a hexagon involving its quasi-circumcenter, its quasi-orthocenter and its lamina centroid.

1. Introduction

The term “quasi-circumcenter” of a quadrilateral seems to have first been introduced by Myakishev in [2], where it is defined as follows: Given a quadrilateral $ABCD$, denote by $O_a$ the circumcenter of triangle $BCD$, and similarly, $O_b$, for triangle $ACD$, $O_c$ for triangle $ABD$, and $O_d$ for triangle $ABC$, then the quasi-circumcenter for the quadrilateral is given by $O = O_aO_c \cap O_bO_d$.

From a problem posed in [1] to find the “best” place to build a water reservoir for four villages of more or less equal size, if the four villages are not concyclic, the following theorem was experimentally discovered and proved. It followed from the classroom discussion of a proposed solution by an undergraduate student, Renate Lebleu Davis, at Kennesaw State University during 2006.

![Figure 1](image-url)

**Theorem 1.** For a general quadrilateral $ABCD$, the quasi-circumcenter $O$ is equidistant from $A$ and $C$, and also from $B$ and $D$. (See Figure 1.)
Proof. Since both $O_a$ and $O_c$ lie on the perpendicular bisector of the $BD$, all points on the line $O_aO_c$ are equidistant from $B$ and $D$. Similarly, all points on the line $O_bO_d$ are equidistant from $A$ and $C$. Thus, the intersection $O$ of lines $O_aO_c$ and $O_bO_d$ is equidistant from the two pairs of opposite vertices. 

This result was used in the Kennesaw State Mathematics Competition for High School students in 2007, as well as in the World InterCity Mathematics Competition for Junior High School students in 2009. Of interest too is that an analogous result exists as given below for the “quasi-incenter” of a quadrilateral, defined in the same way as quasi-circumcenter. The proof is left to the reader.

Theorem 2. Given a general quadrilateral $ABCD$, then the quasi-incenter $I$ is equidistant from $AD$ and $BC$, as well as equidistant from $AB$ and $CD$.

2. The quasi-circumcenter of a hexagon

The point of concurrency given in the Theorem 3 below defines the quasi-circumcenter of a hexagon.

Theorem 3. If the quasi-circumcenters $P$, $Q$, $R$, $S$, $T$, and $U$, respectively of the quadrilaterals $ABCD$, $BCDE$, $CDEF$, $DEFA$, $EFAB$, and $FABC$ subdividing an arbitrary hexagon $ABCDEF$ are constructed, then the lines connecting opposite vertices of the hexagon formed by these quasi-circumcenters are concurrent. (See Figure 2).

Proof. The result follows directly from the dual of the theorem of Pappus, which can be conveniently formulated as follows: The diagonals of a plane hexagon whose sides pass alternatively through two fixed points, meet at a point. With reference to Figure 2, note that alternate sides $QR$, $ST$ and $UP$ respectively lie on the perpendicular bisectors of sides $AE$, $EC$ and $CA$ of triangle $ACE$, and are therefore concurrent. Similarly, the other set of alternate sides are concurrent (with respect to triangle $BDF$). Hence, according to the dual of Pappus, the lines connecting the opposite vertices (the main diagonals) are concurrent.
3. The quasi-Euler line of a hexagon

In [2], it is shown how the Euler line for a triangle generalizes to the Ganin - Rideau - Myakishev theorem, e.g. a quasi-Euler line for a general quadrilateral $ABCD$, which involves its lamina centroid $G$, its quasi-circumcenter $O$, and its quasi-orthocenter $H$ (which is defined in the same way as the quasi-circumcenter), and $OH : HG = 3 : -2$. Using the result of Theorem 3, this result generalizes to a hexagon as follows.

**Theorem 4.** In any hexagon, its lamina centroid $G$, its quasi-circumcenter $O$, and its quasi-orthocenter $H$ are collinear, and $OH : HG = 3 : -2$. (See Figure 3).

**Proof.** Subdivide the hexagon $ABCDEF$ into the same six quadrilaterals as in Theorem 3 above, and determine the quasi-circumcenter $O$, the lamina centroid $G$, and the quasi-orthocenter $H$ of each quadrilateral, respectively labelling the formed hexagons as

$\mathcal{P} : PQRSTU$, $\mathcal{P}_G : QG_RG_SG_TG_U$, $\mathcal{P}_H : QH_RH_SH_TH_UH$.

Since the same affine relations hold between $O$, $G$ and $H$ in each quadrilateral, affine mappings exist that will map $\mathcal{P}$ onto $\mathcal{P}_G$ and $\mathcal{P}_H$. But since the diagonals of $\mathcal{P}$ are concurrent, it follows that the diagonals of $\mathcal{P}_G$ and $\mathcal{P}_H$ would also be concurrent. Respectively label and define those two points of concurrency of $\mathcal{P}_G$ and $\mathcal{P}_H$ as the centroid $G$ and quasi-orthocenter $H$ for the whole hexagon. Finally, since affine transformations preserve collinearity as well as ratios into which segments are divided, we note from the affine mappings between the various hexagons that the result holds.

We can similarly define the quasi-ninepoint center of a hexagon in terms of the six quasi-ninepoint centers of the subdividing quadrilaterals, from which it follows by the same affine transformations that the quasi-ninepoint center $N$ bisects the
segment $OH$. An interactive Java applet to illustrate and explore Theorem 4 is available for the reader at:


The result unfortunately does not generalize further to an octagon as subdividing it in the same way into quadrilaterals or hexagons do not produce octagons that generally have concurrent diagonals.

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A Simple Property of Isosceles Triangles with Applications

Surajit Dutta

Abstract. In this paper we prove a simple property of isosceles triangles and give two applications: construction of third proportional line segments and construction of the inverse point with respect to a circle.

1. Introduction

Here we give an interesting property of isosceles triangles. It is known that if $a$ and $b$ are two given line segments, then their third proportional line segment $c$ can be constructed geometrically; see [2, VI.11]. We can more easily construct the third proportional line segment by using the simple geometric property of the isosceles triangles. Two points $A$ and $B$ are inverse points with respect to the inversion circle with center $O$ and radius $r$, if $OA \cdot OB = r^2$; see [3]. We can construct the inverse point with respect to a circle by using the same property of the isosceles triangle.

2. The property

Lemma 1. Let $ABC$ be an isosceles triangle with $AB = BC$. Let $D$ be a point on the ray $BC$ and let $h$ be the ray obtained by reflecting the ray $AD$ in the line $AC$. Then the ray $h$ cuts the ray $BC$ in a point $E$ which lies outside the segment $BC$ if the point $D$ lies inside this segment (see Figure 1) and inside the segment $BC$ if the point $D$ lies outside (see Figure 2).

![Figure 1](image1.png)

![Figure 2](image2.png)

Proof. For Figure 1, let $k$ denote the ray which is the part of the ray $BC$ outside the segment $BC$. Then,

$$\angle(k, CA) + \angle(AC, h) = \pi - \angle ACB + \angle(AC, h)$$

$$= \pi - \angle BAC + \angle(AC, h)$$

$$= \pi - \angle BAC + \angle DAC$$

$$< \pi.$$
The second equality holds since the triangle $ABC$ is isosceles, the third by reflection, and the fourth since $D$ is an interior point of the segment $BC$. Now according to Euclid’s fifth postulate the rays $k$ and $h$ meet in a point $E$. By the properties of reflection it is obvious that this intersection point $E$ must lie on the ray $BC$ outside the segment $BC$.

For Figure 2, $\angle(h, AC) = \angle CAD < \angle ACB = \angle BAC$. The first equality holds by reflection, the inequality by the Exterior Angle Theorem, the second equality since the triangle $ABC$ is isosceles. Thus, the ray $h$ runs first within the triangle $ABC$ and meets the side $BC$ in an interior point $E$. □

**Theorem 2.** If $ABC$ is an isosceles triangle and points $D$, $E$ are given as in Lemma 1, then $BC^2 = BD \cdot BE$, i.e., $BC$ is the geometric mean of $BD$ and $BE$.

**Proof.** Firstly, we assume the point $D$ inside the segment $BC$. The triangles $ABD$ and $EBA$ share the angle at vertex $B$. Now consider the angle sums of the triangles $ABC$ and $ABD$.

\[
\angle ABC + \angle BCA + \angle CAB = \pi, \\
\angle ABD + \angle BDA + \angle DAB = \pi.
\]

Note, that $\angle CAB = \angle CAD + \angle DAB$. Thus, comparison of the two angle sums yields $\angle BDA = \angle BCA + \angle CAD$. But $\angle BCA = \angle CAB$ since the triangle $ABC$ is isosceles and $\angle CAD = \angle EAC$ by reflection. Thus,

\[
\angle BDA = \angle CAB + \angle EAC = \angle EAB.
\]

Therefore the triangles $ABD$ and $EBA$ are inversely similar, and

\[
BC : BD = AB : BD = BE : BA = BE : BC.
\]

From this, $BC^2 = BD \cdot BE$.

Secondly, if the point $D$ lies outside the segment $BC$ then interchanging the roles of the points $D$ and $E$ in the previous argument yields the same result. □

### 3. Applications

3.1. **Construction of third proportional line segments.** Let $a$ and $b$ be the lengths of two line segments, and we have to draw a line segment of length $c$ such that the square of $b$ equals the product of $a$ and $c$.

For this, construct an isosceles triangle $ABC$ with $AB = BC = b$ (see Figures 3 and 4). Let $D$ be a point on the ray $BC$ such that $BD = a$ and let $h$ be the line

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**Figure 3**

**Figure 4**
obtained by reflecting the ray $AD$ in the line $AC$. By Lemma 1 the ray $h$ cuts the ray $BC$ in a point $E$. Theorem 2 implies $BE = c$.

3.2. *Construction of the inverse point with respect to a circle.* Consider a circle $C$ with center $B$ and $D$ a point which may lies inside or outside of the circle $C$. In both cases we can follow the same steps to construct the inverse point of $D$ with respect to the circle $C$, a case distinction as in the usual treatments, see for example [1, pp.108–109], is not needed.

Take the intersection point $C$ of the ray $BD$ with the circle $C$, see Figures 5 and 6. Connect the point $C$ with an arbitrary point $A$ on the circle $C$ (different from $C$) and let $h$ be the ray obtained by reflecting the ray $AD$ in the line $AC$. The ray $h$ cuts the ray $BC$ in a point $E$ by Lemma 1 which is the inverse point of $D$ with respect to the circle $C$ in view of Theorem 2.

Note that the circumcircle $C'$ of the triangle $ADE$ is orthogonal to the circle $C$, since it is invariant under the inversion at the circle $C$ (see Figure 7).
References

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A Note on Haga’s Theorems in Paper Folding

Hiroshi Okumura

Abstract. Haga’s three theorems in the mathematics of square paper folding are unified in a simple way.

1. Introduction

Haga’s famous theorems in the mathematics of square paper folding consists of three main parts [1, 2, 3]. Let us assume that $ABCD$ is a piece of square paper with a point $E$ on the side $AD$. We fold the paper so that the corner $C$ coincides with $E$ and the side $BC$ is carried into $B'E$, which intersects the side $AB$ at a point $F$ (see Figure 1). We call this Haga’s fold of the first kind. Haga discovered if $E$ is the midpoint of $AD$, then $F$ divides $AB$ in the ratio $2:1$ internally (first theorem). Also if $F$ is the midpoint of $AB$, then $E$ divides $AD$ in the ratio $2:1$ internally (third theorem; see Figure 2).

Let $F$ be a point on the side $AB$ such that the reflection of $B$ in the line $CF$ coincides with the reflection of $D$ in the line $CE$ (see Figure 3). This is called Haga’s fold of the second kind, with the crease lines $CE$ and $CF$. He discovered if $F$ is the midpoint of $AB$, then $E$ divides $AD$ in the ratio $2:1$ internally (second theorem). In this note, we show that these three facts are unified in a simple way.

2. Main theorem

It is easy to show $AF$ as a function of $DE$. Indeed, the following fact is given in [1]: If $AB = 1$, then $AF = \frac{2DE}{1+DE}$ holds for the fold of the first kind. Also he pointed out that his fold of the first kind derived from the fold of the second kind, and vice versa. In fact for the fold of the first kind, the reflection of $B$ in the line $CF$ coincides with the reflection of $D$ in the line $CE$ (see Figure 4). Haga’s results are unified as follows.

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**Theorem.** The relation $\frac{AF}{FB} = 2 \cdot \frac{DE}{EA}$ holds for Haga’s folds of the first and second kinds.

![Figure 4](image)

**Proof.** Let $AB = 1$. The theorem can be proved using the relation $AF = 2DE \cdot 1 + DE$. We give a proof using trigonometry. By the above remark, it is sufficient to prove for the fold of the second kind. Let $\theta = \angle DCE$ and $t = \tan \theta$. Then we get $DE = t$ and $EA = 1 - t$ (see Figure 4). This implies $\frac{DE}{EA} = \frac{t}{1-t}$. While $\angle BCF = \frac{\pi}{4} - \theta$ leads to $FB = \tan \left( \frac{\pi}{4} - \theta \right) = \frac{1-t}{1+t}$ and $AF = 1 - FB = \frac{2t}{1+t}$. Hence we get $\frac{AF}{FB} = \frac{2t}{1-t}$. The theorem is now proved. 

By the theorem $AF : FB = k : 1$ is equivalent to $DE : EA = k : 2$ for a positive real number $k$. Haga’s results are obtained when $k = 1$ and $k = 2$.

**References**


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Dao’s Theorem on Six Circumcenters associated with a Cyclic Hexagon

Nikolaos Dergiades

Abstract. We reformulate and give an elegant proof of a wonderful theorem of Dao Thanh Oai concerning the centers of the circumcircles of the six triangles each bounded by the lines containing three consecutive sides of the hexagon.

In slightly different notations Dao Thanh Oai [3] has posed the problem of proving the following remarkable theorem.

Theorem (Dao). Let \(A_i, i = 1, 2, \ldots, 6\), be six points on a circle. Taking subscripts modulo 6, we denote, for \(i = 1, 2, \ldots, 6\), the intersection of the lines \(A_iA_{i+1}\) and \(A_{i+2}A_{i+3}\) by \(B_{i+3}\), and the circumcenter of the triangle \(A_iA_{i+1}B_{i+2}\) by \(C_{i+3}\). The lines \(C_1C_4, C_2C_5, C_3C_6\) are concurrent.

Indeed a proof with tedious computer aided calculations with barycentric coordinates has been given in [4]. In this note we give an elegant proof using complex numbers by considering the given hexagon as inscribed on the unit circle with center 0 in the complex plane.
**Lemma 1.** If \(A, B, C, D\) are points on the unit circle with affixes \(a, b, c, d\) respectively, and the lines \(AB, CD\) intersect at \(E\), then the circumcenter \(P\) of triangle \(ACE\) has affix \(p = \frac{ac(b-d)}{ab-cd}\).

**Proof.** If the oriented angle between \(\overrightarrow{AB}\) and \(\overrightarrow{CD}\) is \(\varphi = \angle AEC\), then \(\angle APC = 2\varphi\). If \(z = \cos \varphi + i \sin \varphi\), then since for a point \(A\) on the unit circle, the conjugate of its affix is \(\overline{a} = \frac{1}{a}\), we conclude that

\[
(c - p) = (a - p)z^2.
\]

Now,

\[
\frac{d - c}{|d - c|} = \frac{b - a}{|b - a|}z \implies \frac{(d - c)^2}{|d - c|^2} = \frac{(b - a)^2}{|b - a|^2}z^2 \implies \frac{d - c}{d - c} = \frac{b - a}{|b - a|^2}z^2 \implies \frac{d - c}{d - c} = \frac{b - a}{|b - a|^2}z^2
\]

This reduces to \(cd = abz^2\), and from (*) \((c - p)ab = (a - p)cd\). From this, \(p = \frac{ac(b-d)}{ab-cd}\). \(\Box\)

To avoid excessive use of subscripts, we reformulate and prove Dao’s Theorem in the following form.

**Theorem 2.** Let \(A, B, C, X, Y, Z\) be arbitrary points on the unit circle with complex affixes \(a, b, c, x, y, z\) respectively. The lines \(ZB, XC, YA\) and \(CY, AZ, BX\) bound the triangles \(A'B'C'\) and \(A''B''C''\). If \(A_1, B_1, C_1, A_2, B_2, C_2\) are the circumcenters of the circles \((A'YC)\), \((B'ZA)\), \((C'XB)\), \((A''BZ)\), \((B''CX)\), \((C''AY)\), then the lines \(A_1A_2, B_1B_2, C_1C_2\) are concurrent.
Proof. From Lemma 1, we have the affixes of the circumcenters:

\[ a_1 = \frac{cy(x - a)}{cx - ay}, \quad a_2 = \frac{bz(a - x)}{az - bx}, \]
\[ b_1 = \frac{az(y - b)}{ay - bz}, \quad b_2 = \frac{cx(b - y)}{bx - cy}, \]
\[ c_1 = \frac{bx(z - c)}{bz - cx}, \quad c_2 = \frac{ay(c - z)}{cy - az}. \]

For every point \( W \) on the line \( A_1A_2 \), the number \( t = \frac{w - a_1}{w - a_2} \) is real. Therefore,

\[ t = \frac{w - a_1}{w - a_2}. \]

This gives the equation of the line \( A_1A_2 \) as

\[ \begin{vmatrix} w & w & 1 \\ a_1 & a_1 & 1 \\ a_2 & a_2 & 1 \end{vmatrix} = 0 \]

(see [1]).

Since \( a, b, c, x, y, z \) are unit complex numbers,

\[ \overline{a_1} = \frac{cy(x - \overline{a})}{cx - ay} = \frac{1}{c} \cdot \frac{1}{y} \left( \frac{1}{x} - \frac{1}{a} \right) = \frac{x - a}{cx - ay}. \]

Similarly, \( \overline{a_2} = \frac{a - \overline{x}}{az - bz} \). From these we obtain the equation of the line \( A_1A_2 \), and likewise those of \( B_1B_2 \) and \( C_1C_2 \). These are

\[ (az + cx - ay - bx)w + (bcxy + abyz - cayz - bcxz)w + (a - x)(cy - bz) = 0, \]
\[ (bx + ay - bz - cy)w + (cayz + bcxz - abxz - caxy)w + (b - y)(az - cx) = 0, \]
\[ (cy + bz - cx - az)w + (abzx + caxy - bcxy - abyz)w + (c - z)(bx - ay) = 0. \]
The three lines are concurrent if and only if the determinant
\[
\begin{vmatrix}
az + cx - ay - bx & bcxy + abyz - cayz - bczx & (a - x)(cy - bz) \\
px + ay - bx - cy & cayz + bczx - abzx - caxy & (b - y)(az - cx) \\
py + bx - cx - az & abzx + eaxy - bexy - abzy & (c - z)(bx - ay)
\end{vmatrix} = 0.
\]

This clearly is true since each column sum is equal to 0. □

From the equations of the lines it is clear that the point of concurrency is \( O \) if and only if
\[
(a - x)(cy - bz) = (b - y)(az - cx) = (c - z)(bx - ay) = 0.
\]

Assume the points \( A, B, C, X, Y, Z \) distinct. This condition is satisfied precisely when the unit complex affixes satisfy \( \frac{x}{a} = \frac{y}{b} = \frac{z}{c} \). From this we conclude that \( XYZ \) is obtained from \( ABC \) by a rotation.

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Two Tangent Circles from Jigsawing Quadrangle

Tran Quang Hung

Abstract. We establish the tangency of two circles associated with the jigsawing quadrangle of Floor van Lamoen.

Floor van Lamoen [1] has given a construction, for an acute angled triangle $ABC$, a pair of isotomic points $QP$, $Q$ on $BC$ such that when perpendiculars at $P$, $Q$ to $BC$ are constructed to intersect $AB$ and $AC$ at $P'$ and $Q'$ respectively, the quadrangle $PP'Q'Q$ satisfies $PP' = PP' + QQ'$ (see Figure 1). If the triangles $BPP'$ and $CQQ'$ are rotated about $P'$ and $Q'$ respectively, so that the images of $P$ and $Q$ coincide at a point on $P'Q'$, then the images of $B$ and $C$ coincide at a point $A'$ such that the quadrangle $AP'A'Q'$ is cyclic. van Lamoen also showed that $AA'$ passes through the circumcenter $O$ of triangle $ABC$. Denote by $S$ the center of the circle through the four points. We show that this circle is tangent at $A'$ to another circle naturally associated with the triangle. Let the tangents at $B$ and $C$ to the circumcircle of $ABC$ intersect at $T$.

Theorem. The circles through $A$, $P'$, $A'$, $Q'$ is tangent at $A'$ to the circle, center $T$, passing through $B$ and $C$.

Proof. (1) We first show that $A'$ also lies on the circle, center $T$, passing through $B$ and $C$. Note that triangles $P'BA'$ and $Q'CA'$ are isosceles (see Figure 1). Therefore,

$$
\angle BA'C = 360^\circ - \angle BA'P' - \angle P'A'Q' - \angle Q'A'C
= 360^\circ - \frac{180^\circ - \angle BP'A'}{2} - (180^\circ - \angle BAC) - \frac{180^\circ - \angle CQ'A'}{2}
= \frac{\angle BP'A' + \angle CQ'A'}{2} + \angle BAC
= \frac{\angle AQ'A' + \angle CQ'A'}{2} + \angle BAC
= 90^\circ + \angle BAC
= 180^\circ - \frac{1}{2} \angle BTC.
$$

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This shows that $A'$ lies on the circle, center $T$, passing through $B$ and $C$.

![Figure 1](image-url)

(2) We show that the points $S$, $A'$, and $T'$ are collinear.

\[
\angle SA'Q' + \angle Q'A'C + \angle CA'T' = (90° - \angle A'P'Q') + \angle Q'CA' + (90° - \angle A'BC)
\]
\[
= 180° - \angle BP'P + \angle ACB - (180° - \angle BA'C)
\]
\[
= 180° - (90° - \angle ABC) + \angle ACB - (90° - \angle BAC)
\]
\[
= \angle ABC + \angle ACB + \angle BAC
\]
\[
= 180°.
\]

Therefore, $A'$ lies on the segment $ST$.

It follows that the two circles are tangent externally at $A'$.

\[\square\]

Reference

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Two More Pairs of Archimedean Circles in the Arbelos

Tran Quang Hung

Abstract. We construct two more pairs of Archimedean circles in the arbelos. One of them is a pair constructed by Floor van Lamoen in another way.

In addition to the two pairs of Archimedean circles associated with the arbelos constructed by Dao Thanh Oai [1], we construct two more pairs. Given a segment $AB$ with an interior point $C$, consider the semicircles $(O)$, $(O_1)$, $(O_2)$ with diameters $AB$, $AC$, and $CB$, all on the same side of $AB$. The perpendicular to $AB$ at $C$ intersects $(O)$ at $D$. Let $a$ and $b$ be the radii of the semicircles $(O_1)$ and $(O_2)$ respectively. The Archimedean circles have radii $\frac{ab}{a+b}$.

**Theorem 1.** Let the perpendiculars to $AB$ at $O_1$ and $O_2$ intersect $(O)$ at $E$ and $F$ respectively. If $AF$ intersects $(O_1)$ at $H$ and $BE$ intersects $(O_2)$ at $K$, then the circles tangent to $CD$ with centers $H$ and $K$ are Archimedean circles.

**Proof.** Let $M$ and $N$ be the orthogonal projections of $H$ and $K$ on $CD$ respectively. Since $CH$ and $BF$ are both perpendicular to $AF$, the right triangles $CHM$ and $FBO_2$ are similar (see Figure 1).

\[
\frac{HM}{BO_2} = \frac{CH}{FB} = \frac{AC}{AB} \implies HM = BO_2 \cdot \frac{AC}{AB} = b \cdot \frac{2a}{2a + 2b} = \frac{ab}{a + b}.
\]

Therefore the circle $H(M)$ is Archimedean; similarly for $K(N)$. \hfill \Box

Floor van Lamoen has kindly pointed out that this pair has appeared before in a different construction, as $(K_1)$ and $(K_2)$ in [3] (see also $(A_{25a})$ and $(A_{25b})$ in [4]). We show that $H$ and $K$ are intersections of $(O_1)$ and $(O_2)$ with the mid-semicircle with diameter $O_1O_2$. It is enough to show that $\angle O_1HO_2 = \angle O_1KO_2 = 90^\circ$.

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In Figure 2, $O_2$ is the midpoint of $BC$, and $BF$, $CH$ are parallel. The parallel through $O_2$ to these lines is the perpendicular bisector of $FH$. This means that $O_2F = O_2H$, and

$$\angle O_1HO_2 = 180^\circ - \angle O_1HA - \angle O_2HF$$

$$= 180^\circ - \angle O_1AH - \angle O_2FH$$

$$= \angle AO_2F = 90^\circ.$$  

Similarly, $\angle O_1KO_2 = 90^\circ$.

**Theorem 2.** Let $P$ be the intersection of $AD$ with the semicircle with diameter $AO_2$, and $Q$ that of $BD$ with the semicircle with diameter $BO_1$. The circles tangent to $CD$ with centers $P$ and $Q$ are Archimedean.

**Proof.** Let $X$ and $Y$ be the orthogonal projections of $P$ and $Q$ on $CD$ (see Figure 3). Since $BD$ and $O_2P$ are both perpendicular to $AD$, they are parallel.

$$\frac{PX}{AC} = \frac{DP}{DA} = \frac{BO_2}{BA} \implies PX = AC \cdot \frac{BO_2}{BA} = 2a \cdot \frac{b}{2a + 2b} = \frac{ab}{a + b}.$$  

Therefore, the circle $P(X)$ is Archimedean; similarly for $Q(Y)$.  

We show that $PQ$ is a common tangent to the semicircles with diameters $AO_2$ and $BO_1$ (see [5]). In Figure 4, these two semicircles intersect at a point $Z$ on $CD$ satisfying $CZ^2 = 2a \cdot b = a \cdot 2b$. Now, $DP \cdot DA = DZ(DC + ZC) = DQ \cdot DB$.

It follows that $\frac{DP}{DQ} = \frac{DB}{DA}$, so that the right triangles $DPQ$ and $DBA$ are similar. Now, if $O_1'$ is the midpoint of $AO_2$, then

$$\angle O_1'PQ = 180^\circ - \angle O_1'PA - \angle DPQ$$

$$= 180^\circ - \angle BAD - \angle DBA$$

$$= \angle ADB = 90^\circ.$$  

Therefore, $PQ$ is tangent to the semicircle on $AO_2$ at $P$. Similarly, it is also tangent to the semicircle on $BO_1$ at $Q$. It is a common tangent of the two semicircles.
Two more pairs of Archimedean circles in the arbelos

References


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A Special Point in the Arbelos Leading to a Pair of Archimedean Circles

Floor van Lamoen

Abstract. In 2011 Quang Tuan Bui found a beautiful and simple pair of Archimedean circles, which were published on a website. From this pair we find a special point in the Arbelos leading to a related pair of Archimedean circles.

In 2011 Quang Tuan Bui found a beautiful and elegant pair of Archimedean circles. These were published by Alexander Bogomolny on his website [1]. In 2013 the circle pair was found independently by Hiroshi Okumura [2, 3].

Consider an arbelos with $(O)$ being the semicircle with diameter $AB$, while the point $C$ on $AB$ defines the smaller semicircles $(O_1)$ and $(O_2)$ on $AC$ and $BC$ respectively. Let the perpendiculars to $AB$ from $O_1$ and $O_2$ meet $(O)$ in $D$ and $E$ respectively. The segments $DA$ and $DC$ meet $(O_1)$ in two points $F$ and $G$. The segment $FG$ is the diameter of an Archimedean circle (see Figure 1). Likewise an Archimedean circle is found from $E$.

To prove the correctness of the finding of Bui, we let $r$, $r_1$ and $r_2$ be the radii of $(O)$, $(O_1)$ and $(O_2)$ respectively. Note that $AF : AD = r_1 : r$, so that $AD : FD = r : r_2$. Of course $G$ divides $CD$ in the same ratio. So, by similarity $FG = \frac{r_2}{r} \cdot AC = 2 : \frac{r_1 + r_2}{2}$, the Archimedean diameter.

Now one may wonder what the locus of points $P$ is such that $PA$ and $PC$ cut a chord $ST$ off $(O_1)$ congruent to $FG$. See Figure 2.

For $ST$ to be congruent to $FG$, it is clear that arcs $FS$ and $GT$ must be congruent. From this the angles $DAP$ and $DCP$ must be congruent, and we conclude that $ACDP$ is cyclic. The locus of $P$ is thus the circumcircle of triangle $ACD$.

Figure 1.
Similarly, the locus of $P$ for $PB$ and $PC$ to cut congruent to the one cut out by $EB$ and $EC$ is the circumcircle $BCE$. Now the circumcircles of $ACD$ and $BCE$ intersect, apart from $C$, in a point $L$. This point is thus the only point leading to an Archimedean circle on each of the semicircles $(O_1)$ and $(O_2)$. A notable point (see Figure 3).

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Three Constructions of Archimedean Circles in an Arbelos

Paul Yiu

Abstract. We give ruler and compass constructions of three Archimedean circles in an arbelos, each with the endpoints of a diameter on the smaller semicircles. In the first case, the diameter contains the intersection of the defining smaller semicircles of the arbelos. In the second case, these endpoints are the intersections of the smaller semicircles with the lines joining the endpoints of the base of the arbelos to a fixed point on the dividing perpendicular line. In the third case, the diameter containing these endpoints is parallel to the base line of the arbelos.

1. Introduction

We consider three constructions of Archimedean circles in an arbelos. Given a segment $AB$ with an interior point $C$, the semicircles $(O), (O_1), (O_2)$ with diameters $AB, AC, CB$ on the same side of $AB$ bound the arbelos, with dividing line $CD$ perpendicular to $AB$. Let $a$ and $b$ be the radii of the semicircles $(O_1)$ and $(O_2)$. Circles with radius $t := \frac{ab}{a+b}$ are called Archimedean. They are congruent to the Archimedean twin circles [1, 2, 3]. We shall make use of the Archimedean circles with centers $O_1$ and $O_2$ respectively.

In particular, we shall encounter below lines making an angle $\theta$ with $AB$ defined by

$$\sin \theta = \frac{t}{a+b}. \quad (1)$$

Figure 1.

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The author thanks Floor van Lamoen for his excellent suggestion of searching for the Archimedean circle in the last section of this paper.
Such lines are parallel to the tangents from $O_1$ to the Archimedean circle with center $O_2$ or vice versa. The points of tangency are the intersections with the circle with diameter $O_1O_2$ and center $O'$ (see Figure 1). We adopt a Cartesian coordinate system with origin at $C$, so that the points $A$ and $B$ have coordinates $(-2a, 0)$ and $(2b, 0)$ respectively. The equations of the circles $(O_1)$ and $(O_2)$ are

$$(x + a)^2 + y^2 = a^2,$$

$$(x - b)^2 + y^2 = b^2.$$  

2. Archimedean circles with diameter through $C$ and endpoints on $(O_1)$ and $(O_2)$

Consider the construction of a line $L'$ through $C$ intersecting the circles $(O_1)$ and $(O_2)$ at $A'$ and $B'$ respectively so that the segment $A'B'$ has length $2t$, and the circle with diameter $A'B'$ is Archimedean. If $L'$ has slope $m$, then these intersections are $A' = \left( -\frac{2a}{1+m^2}, -\frac{2am}{1+m^2} \right)$ and $B' = \left( \frac{2b}{1+m^2}, \frac{2bm}{1+m^2} \right)$. Since the difference between the $x$-coordinates is $\frac{2(a+b)}{1+m^2}$, $A'B'^2 = \frac{4(a+b)^2}{1+m^2}$. This is equal to $(2t)^2$ if and only if

$$1 + m^2 = \frac{(a + b)^4}{a^2b^2} = \frac{(a + b)^2}{t^2} = \csc^2 \theta$$

for the angle $\theta$ defined by (1). It follows that the slope $m = \pm \cot \theta$, and the line $L'$ is perpendicular to a tangent from $O_1$ to the Archimedean circle at $O_2$.

**Theorem 1.** A line through $C$ intersecting $(O_1)$ and $(O_2)$ at the endpoints of a diameter of an Archimedean circle if and only if it is perpendicular to a tangent from $O_1$ to the Archimedean circle with center $O_2$. 

![Figure 2(a)](image1)

![Figure 2(b)](image2)
3. Archimedean circle from intersections of $QA, QB$ with $Q$ on $CD$

We construct a point $Q$ on the line $CD$ such that the intersections of $AQ$ with $(O_1)$ and $BQ$ with $(O_2)$ are the endpoints of a diameter of an Archimedean circle (see Figure 3). Let $Q = (0, q)$. These intersections are $A'' = \left(\frac{-2aq^2}{q^2 + 4a^2}, \frac{4a^2 q}{q^2 + 4a^2}\right)$ and $B'' = \left(\frac{2bq^2}{q^2 + 4b^2}, \frac{4b^2 q}{q^2 + 4b^2}\right)$. From these coordinates,

$$A''B'' = 4\left(\frac{(a + b)^2 q^4}{(q^2 + 4a^2)(q^2 + 4b^2)}\right).$$

This is equal to $(2t)^2$ if and only if $(a + b)^4 q^4 - a^2 b^2 (q^2 + 4a^2)(q^2 + 4b^2) = 0$. Rewriting this as

$$(a + b)^4 q^4 - a^2 b^2 (q^2 + 4a^2)(q^2 + 4b^2) = 0,$$  

(2)

we see that there is a unique positive root.

**Theorem 2.** There is a unique point $Q$ on the dividing line $CD$ such that the intersections of $QA$ with $(O_1)$ and $QB$ with $(O_2)$ are the endpoints of a diameter of an Archimedean circle.

![Figure 3](image)

From (2), we obtain explicitly

$$q^2 = \frac{a^2 b^2}{(a + b)^4 - a^2 b^2 (2(a^2 + b^2) + 2(a + b)\sqrt{(a - b)^2 + 4(a + b)^2})}. $$

Now, $\frac{a^2 b^2}{(a+b)^4-a^2b^2} = \frac{l^2}{(q+b)^2-t^2} = \tan^2 \theta$ for $\theta$ defined by (1). It is enough to construct a segment $CX$ on $AB$ with

$$CX^2 = 2(a^2 + b^2) + 2(a + b)\sqrt{(a - b)^2 + 4(a + b)^2}. $$

(3)

Let $M$ be the “highest” point of $(O)$, i.e., the intersection of $(O)$ with the perpendicular to $AB$ at $O$ (see Figure 4). For the construction of $X$, we make use of the following.

(i) $2(a^2 + b^2) = (a + b)^2 + (a - b)^2 = OM^2 + OC^2 = CM^2$,

(ii) $(a - b)^2 + 4(a + b)^2 = 4(O'O^2 + OM^2) = 4 \cdot O'M^2$. 

Construction. (1) On different sides of $AB$ on the perpendicular at $C$, choose points $Y_1$ and $Y_2$ such that $CY_1 = OM$ and $CY_2 = O'M$. Construct the circle with diameter $Y_1Y_2$, to intersect the line $AB$ in a segment $X_1X_2$.

(2) On the perpendicular to $MC$ at $M$, choose a point $X_0$ such that $MX_0 = X_1X_2$.

(3) Let $X$ be a point on $AB$ such that $CX = CX_0$. The segment $CX$ has length given by (3) above.

(4) Construct a parallel through $X$ to a tangent from $O_1$ to the Archimedean circle with center $O_2$, to intersect $CD$ at $Q$. This is the unique point $Q$ in Theorem 2.

4. Archimedean circle with a diameter parallel to $AB$ and endpoints on $(O_1)$ and $(O_2)$

We consider the possibility of an Archimedean circle with a diameter parallel to $AB$ having its endpoints one on each of the semicircles $(O_1)$ and $(O_2)$. If the diameter is at a distance $d$ from $AB$, its endpoints are among the points

$$X_- = (-a - \sqrt{a^2 - d^2}, d), \quad X_+ = (-a + \sqrt{a^2 - d^2}, d);$$
$$Y_- = (b - \sqrt{b^2 - d^2}, d), \quad Y_+ = (b + \sqrt{b^2 - d^2}, d).$$

The differences between the lengths of the various segments and the diameter of an Archimedean circle are

$$X_-Y_+ - 2t = a + b - 2t + \sqrt{a^2 - d^2} + \sqrt{b^2 - d^2};$$
$$X_-Y_- - 2t = a + b - 2t + \sqrt{a^2 - d^2} - \sqrt{b^2 - d^2};$$
$$X_+Y_+ - 2t = a + b - 2t - \sqrt{a^2 - d^2} + \sqrt{b^2 - d^2};$$
$$X_+Y_- - 2t = a + b - 2t - \sqrt{a^2 - d^2} - \sqrt{b^2 - d^2}.$$
The condition
\[(X+Y-2t)(X-Y-2t)(X+Y+2t)(X-Y+2t) = 0\]
simplifies into
\[4t(a-t)(b-t)(a+b-t)-(a+b-2t)^2d^2 = 0.\] (4)
This clearly has a unique positive root \(d\).

**Theorem 3.** There is a unique Archimedean circle with a diameter parallel to \(AB\), having endpoints one on each of the semicircles \((O_1)\) and \((O_2)\).

Now, by Heron’s formula, \(t(a-t)(b-t)(a+b-t)\) is the square of the area of a triangle with sides \(a, b,\) and \(a+b-2t\). From (4), \(d\) is the altitude of the triangle on the side \(a+b-2t\). This leads to the following simple construction.

**Construction.** Let the Archimedean circle with center \(O_1\) intersect \(O_1C\) at \(X\) and that with center \(O_2\) intersect \(CO_2\) at \(Y\). Construct a point \(Z\) (on the same side of the arbelos) such that \(XZ = a\) and \(YZ = b\). The parallel to \(AB\) through \(Z\) is the line which intersects \((O_1)\) and \((O_2)\) at two points at a distance \(2t\) apart.

The point \(Z\) is indeed the center of the Archimedean circle in question.

Assume \(a \geq b\) without loss of generality. Note that \(O_1X_+ = XZ = a\), and they are parallel since
\[
\sin X_+O_1C = \frac{d}{a} = \sin ZXY.
\]
This means that \(O_1XZX_+\) is a parallelogram, and \(ZX_+ = XO_1 = t\). The circle, center \(Z\), passing through \(X_+\) is Archimedean. The other end of the the diameter is \(Y_+\) or \(Y_-\) according as \(a^3\) is less than or greater than \(a^2b + ab^2 + b^3\) (see Figures 5a and 5b). This follows from the simple fact
\[
\sqrt{b^2-d^2} = \begin{cases} 
-\frac{a^3b+a^2b^2+ab^3+b^4}{(a+b)(a^2+b^2)}, & \text{if } a^3 < a^2b + ab^2 + b^3; \\
\frac{a^3b-a^2b^2-ab^3-b^4}{(a+b)(a^2+b^2)}, & \text{if } a^3 > a^2b + ab^2 + b^3.
\end{cases}
\]
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A Purely Synthetic Proof of Dao’s Theorem on Six Circumcenters Associated with a Cyclic Hexagon

Telv Cohl

Abstract. We present a purely synthetic proof of Dao’s theorem on six circumcenters associated with a cyclic hexagon.

Nikolaos Dergiades [4] has given an elegant proof using complex numbers of the following theorem.

Theorem (Dao [2]). Let six points $A, B, C, D, E, F$ lie on a circle, and $U = AF \cap BC$, $V = AB \cap CD$, $W = BC \cap DE$, $X = CD \cap EF$, $Y = DE \cap FA$, $Z = EF \cap AB$. Denote by $O_1, O_2, O_3, O_4, O_5, O_6$ the circumcenters of the six triangles $ABU$, $BCV$, $CDW$, $DEX$, $EFY$, $FAZ$. The three lines $O_1O_4$, $O_2O_5$, $O_3O_6$ are concurrent.

In this note we present a purely synthetic proof.

Lemma 1. Let $A, B, C, A', B', C'$ be six points (in cyclic order) on a circle $(O)$, and $X = AB \cap A'C$, $X' = A'B' \cap AC'$. Let $O_1, O_1'$ be the circumcenters of $(XBC)$, $(X'B'C')$ respectively. The lines $O_1O_1'$, $BB'$, $CC'$ are concurrent (see Figure 2).

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Lemma 1. Denote \(O\) the point which is the center of the circle \((O_1)\). Hence, \(O\) is the circumcenter of triangle \(XYB\). Let \(O\)' be the center of circle \((O_1')\). Clearly, \(O\)' coincides with \(O\) because they are both antiparallels to \(BB'\). By Desargues’ theorem, the lines \((X'Y'B')\) and \((XYB)\) are perspective. Hence, \(BB'\) passes through \(O\). We reformulate and prove Dao’s theorem in the following form.

**Theorem 2.** Divide a circle in six consecutive arcs \(c_2, a_1, b_2, c_1, a_2, b_1\) with the arbitrary points \(A, B, C, A', B', C'\). Let the chords of the arcs \(a_2, b_2, c_2\) bound a triangle \(A_1B_1C_1\), and those of the arcs \(a_1, b_1, c_1\) bound a triangle \(A_2B_2C_2\). If \(O_1, O_1', O_2, O_2', O_3, O_3'\) are the circumcenters of the circles \((A_1BC), (A_2B'C'), (B_1C'A), (B_2CA'), (C_1A'B'), (C_2AB)\) respectively, then the lines \(O_1O_1', O_2O_2', O_3O_3'\) are concurrent (see Figure 3).

**Proof.** Let \(A_3 = BB' \cap CC'\), \(B_3 = CC' \cap AA'\), and \(C_3 = AA' \cap BB'\). By Lemma 1 the points \(A_3, B_3, C_3\) lie on the lines \(O_1O_1', O_2O_2', O_3O_3'\) respectively. Denote

\[
\begin{align*}
\angle O_1BA_3 &= A_b, & \angle O_2C'B_3 &= B_c, & \angle O_3A'C_3 &= C_a, \\
\angle O_1CA_3 &= A_c, & \angle O_2AB_3 &= B_a, & \angle O_3B'C_3 &= C_b.
\end{align*}
\]
We have $A_b = \angle O_1 BC + \angle CBA_3 = 90^\circ - \angle CA_1 B + \angle CBB'$ or

$$A_b = 90^\circ \frac{a_2 + b_1 + c_1 - a_1}{2} + \frac{b_2 + c_1}{2} = 90^\circ + \frac{a_1 + b_2 - a_2 - b_1}{2}.$$  

Similarly,

$$B_a = 90^\circ \frac{b_2 + c_1 + a_1 - b_1}{2} + \frac{a_2 + c_1}{2} = 90^\circ - \frac{a_1 + b_2 - a_2 - b_1}{2}.$$  

From these, $A_b + B_a = 180^\circ$, and $\sin A_b = \sin B_a$. Similarly, $\sin B_c = \sin C_b$ and $\sin C_a = \sin A_c$.

Consider $O_1 A_3 O'_1$, $O_2 B_3 O'_2$, and $O_3 C_3 O'_3$ as lines through the vertices of triangle $A_3 B_3 C_3$. Let $R_1$ be the radius of the circle $(O_1)$. Since

$$\frac{\sin A_b}{\sin C_3 A_3 O'_1} = \frac{\sin A_b}{\sin B A_3 O_1} = \frac{O_1 A_3}{R_1} = \frac{\sin A_c}{\sin O_1 A_3 C} = \frac{\sin A_c}{\sin O'_1 A_3 B_3},$$

we have $\frac{\sin C_3 A_3 O'_1}{\sin O'_1 A_3 B_3} = \frac{\sin A_b}{\sin A_c}$. Similarly, $\frac{\sin A_b}{\sin B_3 O'_2} = \frac{\sin B_c}{\sin B_a}$, and $\frac{\sin B_3 C_3 O'_3}{\sin O'_3 C_3 A_3} = \frac{\sin C_a}{\sin C_b}$. Therefore,

$$\frac{\sin C_3 A_3 O'_1}{\sin O'_1 A_3 B_3} \cdot \frac{\sin A_3 B_3 O'_2}{\sin O'_2 B_3 C_3} \cdot \frac{\sin B_3 C_3 O'_3}{\sin O'_3 C_3 A_3} = \frac{\sin A_b}{\sin B_c} \cdot \frac{\sin B_c}{\sin C_a} \cdot \frac{\sin C_a}{\sin C_b} = 1.$$  

By the converse of Ceva’s theorem, we conclude that the lines $O_1 O'_1$, $O_2 O'_2$, $O_3 O'_3$ are concurrent. \[\square\]
References


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The Triangle of Reflections

Jesus Torres

Abstract. This paper presents some results in triangle geometry discovered with the aids of a dynamic software, namely, the Geometer’s Sketchpad, and confirmed with computations using Mathematica 9.0. With the method of barycentric coordinates, we study geometric problems associated with the triangle of reflections $T^\dagger$ of a given triangle $T$ (obtained by reflecting the vertices in their opposite sides), resulting in interesting triangle centers and simple loci such as circles and conics. These lead to some new triangle centers with reasonably simple coordinates, and also new properties of some known, classical centers. In particular, we show that the Parry reflection point (reflection of circumcenter in the Euler reflection point) is the common point of two triads of circles, one associated with the tangential triangle, and another with the excentral triangle. More interestingly, we show that a certain rectangular hyperbola through the vertices of $T^\dagger$ appears as the locus of the perspector of a family of triangles perspective with $T^\dagger$, and in a different context as the locus of the orthology center of $T^\dagger$ with another family of triangles.

1. Introduction

This paper is a revision of the author’s master thesis [14]. We present some results in triangle geometry discovered with the aids of a dynamic software, namely, the Geometer’s Sketchpad©, and confirmed with computations using Mathematica 9.0. With the method of barycentric coordinates, we study geometric problems associated with the triangle of reflections $T^\dagger$ of a given triangle $T$ (obtained by reflecting the vertices in their opposite sides). We use the notations and basic formulas in triangle geometry as presented in [15]. In particular, coordinates of triangle centers are expressed in the Conway notation, so as to reduce the degrees of polynomials involved. We obtain a number of interesting triangle centers with reasonably simple coordinates, and also new properties of some known, classical centers.

1.1. Summary. Let $T$ be a given triangle. The triangle of reflections $T^\dagger$ is the one whose vertices are the reflections of the vertices of $T$ in their opposite sides. This is introduced in CHAPTER 2. Propositions 2.1 and 2.2 explain the significance of the nine-point center of $T$ in the geometry of $T^\dagger$. The homogeneous barycentric coordinates of a few classical centers on the Euler line are computed. While the calculations for the centroid and the circumcenter are easy (Proposition 2.4), the
coordinates of the orthocenter and nine-point center can only be computed with the aids of Mathematica. These two centers will feature in Chapter 7.

Chapters 3 and 4 give a number of simple results related to perspectivity and orthology with $T^\dagger$. In §4.1, we give a simple computational proof of Sondat’s theorem (Theorem 4.1) which states that if two nondegenerate triangles are both perspective and orthologic, then the perpector and the two orthology centers are collinear. This applies to $T^\dagger$ and the orthic triangle of $T$ (Theorem 4.3). The line containing these centers has a remarkably simple equation. This line also appears as a locus discussed in §7.2. Also, the orthology center $\text{cev}(H) \perp (T^\dagger)$ in Theorem 4.3 is a new center which reappears in a number of places in later chapters.

In Chapters 5 and 6 we construct a number of circles associated with $T^\dagger$. In §5.3 we construct a triad of circles in relation to the tangential triangle of $T$, and show that they are concurrent at the Parry reflection point (which is the reflection of the circumcenter of $T$ in its Euler reflection point). Another triad of circles is constructed in §5.4, this time in connection with the excentral triangle of $T$. This triad of circles are also concurrent at the same Parry reflection point. A new Tucker circle (through the pedals of the vertices of $T^\dagger$ on the sidelines of $T$) is constructed in §6.3. The center of this circle bears a very simple relationship with the Parry reflection point and the Hatzipolakis reflection point in §6.1.

In Chapter 7 we present two locus problems related to $T^\dagger$ and resulting in conic loci, which can be easily identified as rectangular hyperbolas. Specifically, we show that the rectangular circum-hyperbola through the vertices of $T^\dagger$ and the orthocenter of $T$ arises as the locus of the perspector of a family of triangles perspective with $T^\dagger$ (Theorem 7.1), and also as the locus of the orthology center of $T^\dagger$ with another family of triangles (Theorem 7.4(a)). Some of the triangle centers and lines constructed in earlier chapters also feature in the solutions of the loci problems discussed in this chapter.

Appendix A lists a number of triangle centers catalogued in ETC [7] that feature in this paper with properties related to $T^\dagger$. Appendix B is a summary of new triangle centers appearing in this thesis, listed in order of their search numbers in ETC.

2. The triangle of reflections and the nine-point center

Given a reference $T := ABC$, consider the reflections of the vertices in the respective opposite sides. In homogeneous barycentric coordinates, these are the points:

$$A' = (-S_B + S_C) : 2S_C : 2S_B,$$
$$B' = (2S_C : -(S_C + S_A) : 2S_A),$$
$$C' = (2S_B : 2S_A : -(S_A + S_B)).$$

The triangle $T^\dagger := A'B'C'$ is called the triangle of reflections of $T$. It is the main object of study of this paper.

2.1. Perspectivity with $T$. Clearly, $T$ and $T^\dagger$ are perspective at the orthocenter $H$. 
Proposition 2.1. The perspectrix of $T$ and $T^\dagger$ is the trilinear polar of the nine-point center $N$.

\[ (-3S_{AA} + S^2)x + 2(S^2 + S_{AB})y + 2(S^2 + S_{CA})z = 0. \]

It is clear that $B'C' \cap BC = (0 : -(S^2 + S_{CA}) : S^2 + S_{AB})$. The equations of the lines $C'A'$ and $A'B'$, and their intersections with the corresponding sidelines, can be written down easily by cyclic permutations of parameters. These are

\[ C'A' \cap CA = (S^2 + S_{BC} : 0 : -(S^2 + S_{AB})), \]
\[ A'B' \cap AB = (-(S^2 + S_{BC}) : S^2 + S_{CA} : 0). \]

The line containing these points is

\[ \frac{x}{S^2 + S_{BC}} + \frac{y}{S^2 + S_{CA}} + \frac{z}{S^2 + S_{AB}} = 0, \]

the trilinear polar of the nine-point center $N$. $\square$

2.2. Homothety between $T^\dagger$ and the reflection triangle of $N$.

Proposition 2.2. The triangle of reflections $T^\dagger$ is the image of the reflection triangle of the nine-point center $N$ under the homothety $h(O, 2)$.

Proof. If $D$ is the midpoint of $BC$, it is well known that $2 \cdot OD = AH = H_\perp A'$ (see Figure 2.2). If $N_\perp a$ is the reflection of $N$ in $BC$, then

\[ NN_\perp a = OD + HH_a = \frac{1}{2}(2 \cdot OD + 2HH_a) = \frac{1}{2}(HH_a + H_aH_a^\perp + H_a^\perp A') = \frac{1}{2} HA'. \]

Since $N$ is the midpoint of $OH$, it follows that $N_\perp a$ is the midpoint of $OA'$. A similar reasoning shows that $N_\perp b$ and $N_\perp c$ are the midpoints of $OB'$ and $OC'$. $\square$
Proposition 2.3. The medial triangles of $T$ and $T^\dagger$ are perspective at $N$.

Proof. The midpoint of $B'C'$ is the point

$$X' = \frac{1}{2}(B' + C')$$

$$= \frac{1}{2} \left( \frac{(2S_C, -(S_C + S_A), 2S_A)}{S_C + S_A} + \frac{(2S_B, 2S_A, -(S_A + S_B))}{S_A + S_B} \right)$$

$$= \frac{1}{2} \left( \frac{(2S^2 + S_{BC}), (S_A - S_B)(S_C + S_A), (S_A - S_C)(S_A + S_B)}{2(S_C + S_A)(S_A + S_B)} \right).$$

In homogeneous barycentric coordinates, this is

$$X' = (2(S^2 + S_{BC}) : (S_A - S_B)(S_C + S_A) : (S_A - S_C)(S_A + S_B)).$$

The line joining $X'$ to the midpoint of $BC$ has equation

$$\begin{vmatrix}
2(S^2 + S_{BC}) & (S_A - S_B)(S_C + S_A) & (S_A - S_C)(S_A + S_B) \\
0 & 1 & 1 \\
x & y & z
\end{vmatrix} = 0,$$

or

$$S_A(S_B - S_C)x + (S^2 + S_{BC})(y - z) = 0.$$ 

This line clearly contains the nine-point center $N$, since

$$S_A(S_B - S_C)(S^2 + S_{BC}) + (S^2 + S_{BC})((S^2 + S_{CA}) - (S^2 + S_{AB})) = 0.$$ 

Similarly, the lines joining the midpoints of $C'A'$ and $A'B'$ to those of $CA$ and $AB$ also contain $N$. We conclude that the two medial triangles are perspective at $N$ (see Figure 2.3).
2.3. The Euler line of $T^\dagger$.

**Proposition 2.4.** (a) The centroid, circumcenter, and orthocenter of $T^\dagger$ are the points

\[ G' = (a^2(S_{AA} - S_A(S_B + S_C) - 3S_{BC}) : \cdots : \cdots), \]
\[ O' = (a^2(-3S_A^3(S_B + S_C) + S_{AA}(5S_{BB} + 6S_{BC} + 5S_{CC})
+ 9S_{ABC}(S_B + S_C) + 4S_{BB}S_{CC})) : \cdots : \cdots), \]
\[ H' = (a^2(2(S_A + S_B + S_C)S^6 + S_{BC}(2(-S_A + 4S_B + 4S_C)S^4
+ (S_{AA} - 3S^2)((7S_A + 5S_B + 5S_C)S^2 + S_{ABC}))) : \cdots : \cdots), \]
\[ N' = (a^2(3a^4S_A^5 + 2a^2(a^4 + 9S_{BC})S_A^4 - (a^8 + 26a^4S_{BC} - 16S_{BC}^2)S_A^3
- 4a^2S_{BC}(4a^4 + 19S_{BC})S_{AA} - S_A S_{BB} S_{CC}(29a^4 + 48S_{BC}) - 14a^2(S_{BC})^3)
: \cdots : \cdots). \]

(b) The equation of the Euler line of $T^\dagger$:

\[ \sum_{\text{cyclic}} (S_C + S_A)(S_A + S_B)(S_B - S_C)f(S_A, S_B, S_C)x = 0, \]

where

\[ f(S_A, S_B, S_C) = 2(8S_A + S_B + S_C)S^4 - S_A S_B + S_C)((5S_A + 7S_B + 7S_C)S^2 + S_{ABC}). \]

**Remarks.** (1) The centroid $G'$ is $X(3060)$ in ETC, defined as the external center of similitude of the circumcircle of $T$ and the nine-point circle of the orthic triangle.

(2) The circumcenter $O'$ is the reflection of $O$ in $N^\ast$.

(3) The orthocenter $H'$ has ETC (6-9-13)-search number 31.1514091170 \cdots.

(4) The nine-point center $N'$ has ETC (6-9-13)-search number 5.99676405896 \cdots.

(5) The Euler line of $T^\dagger$ also contains the triangle center $X(156)$, which is the nine-point center of the tangential triangle.

2.4. Euler reflection point of $T^\dagger$. A famous theorem of Collings [2] and Longuet-Higgins [8] states that the reflections of a line $L'$ in the sidelines of $T$ are concurrent if and only if $L'$ contains the orthocenter $H$. If this condition is satisfied, the point of concurrency is a point on the circumcircle.

Applying this to the Euler line of $T$, we obtain the Euler reflection point

\[ E = \left( \frac{a^2}{b^2 - c^2} : \frac{b^2}{c^2 - a^2} : \frac{c^2}{a^2 - b^2} \right). \]

The Euler reflection point of $T^\dagger$ is a point $E'$ on the circumcircle of $T^\dagger$ (see Figure 2.4). Its coordinates involve polynomial factors of degree 12 in $S_A$, $S_B$, $S_C$; it has ETC (6-9-13)-search number $-1.94515015138 \cdots$. 
3. Perspectivity

3.1. Triangles perspective with $T^\dagger$.

3.1.1. The excentral triangle. The excentral triangle of $T$ has the excenters as vertices. L. Evans [3] has shown that this is perspective with $T^\dagger$ at the triangle center $X(484) = \left( a^3 + a^2(b + c) - a(b^2 + bc + c^2) - (b + c)(b - c)^2 \right) : \cdots : \cdots$.

This triangle center is often called the Evans perspector. It is the inverse of $I$ in the circumcircle of the excentral triangle, and divides $OI$ in the ratio $OX(484) : X(484)I = R + 2r : -4r$.

3.1.2. The Fermat triangles. Hatzipolakis and Yiu [5] have shown that the only Kiepert triangles perspective with $T^\dagger$ are the Fermat triangles, consisting of vertices of equilateral triangles erected on the sides of $T$. The perspectors are the isodynamic points, $X(16)$ or $X(15)$ according as the vertices of the equilateral triangles are on the same or opposite sides of the vertices of $T$.

3.2. Triangles bounded by reflections of the sidelines of $T$ and $T^\dagger$ in each other.

Let $a, b, c$ be the sidelines $BC, CA, AB$ of triangle $T := ABC$, and $a', b', c'$ those of $T^\dagger$. The reflections of these lines in $a, b, c$ (and vice versa) give rise to interesting examples of perspective triangles. Let $L_a$ be the reflection of $a$ in $a'$, and $L'_a$ that of $a'$ in $a$. Similarly, define $L_b, L'_b$, and $L_c, L'_c$.

Since $a, a'$ intersect at $X$, the lines $L_a, L'_a$ intersect $BC$ at the same point. Similarly, the reflections of $b$ and $b'$ in each other intersect $b$ at $Y$; so do those of $c$ and $c'$ at $Z$. By Proposition 2.1, $X, Y, Z$ define the trilinear polar of $N$. 
The triangle of reflections

Therefore, \( T, T^\dagger \), the triangles \( T^* \) bounded by \( L'_a, L'_b, L'_c \) (see Figure 3.1), and \( T'' = A''B''C'' \) bounded by \( L_a, L_b, L_c \) are line-perspective to each other, all sharing the same perspectrix \( XYZ \). They are also vertex-perspective.

The following table gives the \((6 - 9 - 13)\)-search numbers of the perspectors, with the highest degree of the polynomial factors (in \( S_A, S_B, S_C \)) in the coordinates.

<table>
<thead>
<tr>
<th>( T )</th>
<th>( T^* )</th>
<th>( T'' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( H )</td>
<td>3.99180618013 \ldots (5)</td>
</tr>
<tr>
<td>( T^\dagger )</td>
<td>-9.04876879620 \ldots (11)</td>
<td>-7.90053389552 \ldots (16)</td>
</tr>
<tr>
<td>( T^\ast )</td>
<td>-0.873191727540 \ldots (26)</td>
<td></td>
</tr>
</tbody>
</table>

Here is the perspector of \( T \) and \( T^* \) in homogeneous barycentric coordinates:

\[
\left( \frac{a^2}{(3S_A - S_B - S_C)S^4 + S_{BC}((6S_A + 5S_B + 5S_C)S^2 + 7S_{ABC})} : \ldots : \ldots \right).
\]

4. Orthology

4.1. Orthology and perspectivity.

**Theorem 4.1** (Sondat [12]; see also [13, 9]). If two nondegenerate triangles are both perspective and orthologic, then the perspector and the two orthology centers are collinear.

**Proof.** Assume triangles \( ABC \) and \( XYZ \) are perspective at a point \( P = (u : v : w) \) and have perpendiculars from \( X \) to \( BC \), \( Y \) to \( CA \), and \( Z \) to \( AB \) concurrent at a point \( Q = (u' : v' : w') \). Now the point \( X \) is the intersection of the line \( AP \) and the perpendicular from \( Q \) to \( BC \). It has coordinates

\[
(v(S_B + S_C)w' + S_Bw') - w(S_Cu' + (S_B + S_C)v') : v(S_Bv' - S_Cw') : w(S_Bv' - S_Cw').
\]

Similarly, \( Y \) is the intersection of \( BP \) and the perpendicular from \( Q \) to \( CA \), and \( Z \) is that of \( CP \) and the perpendicular from \( Q \) to \( AB \). Their coordinates can be written down from those of \( X \) by cyclic permutations of parameters. Since the
triangles are orthologic, we find the second orthology center. The perpendiculars from $A$ to $YZ$, $B$ to $ZX$, and $C$ to $AB$ are the lines

$$(S_A u - S_B v)(u'w - w'u)y - (S_C w - S_A u)(v'u - u'v)z = 0,$$

$$(S_B v - S_C w)(v'u - u'v)z - (S_A u - S_B v)(w'v - v'w)x = 0,$$

$$(S_C w - S_A u)(w'v - v'w)x - (S_B v - S_C w)(u'w - w'u)y = 0.$$

These lines are concurrent at the point

$$Q' = \left( \frac{S_B v - S_C w}{w'v - v'w} : \frac{S_C w - S_A u}{u'w - w'u} : \frac{S_A u - S_B v}{v'u - u'v} \right).$$

This clearly lies on the line $PQ$:

$$(w'v - v'w)x + (u'w - w'u)y + (v'u - u'v)z = 0.$$

Therefore the perspector $P$ and the orthology centers $Q$ and $Q'$ are collinear. □

Here is an illustrative example. Let $T^\perp(P)$ be the pedal triangle of a point $P$. It is clear that the perpendiculars from the vertices of $T^\perp(P)$ to $T$ are concurrent at $P$. Therefore, the perpendiculars from $A$, $B$, $C$ to the corresponding sides of the pedal triangle are also concurrent. This is the isogonal conjugate of $P$.

Since the reflection triangle is homothetic to the pedal triangle. The same result holds, namely,

$$(T^\perp(P))^\perp(T) = P \quad \text{and} \quad T^\perp(T^\perp(P)) = P^*.$$

4.2. Orthology with $T$. Clearly the perpendiculars from $A'$, $B'$, $C'$ to the sidelines of $T$ are concurrent at $H$. We find the other orthology center.

**Proposition 4.2.** The orthology center $T^\perp(T^\perp)$ is $N^*$. 

![Figure 4.1](image-url)
**Proof.** The perpendiculars from $A$ to $B'C'$, $B$ to $C'A'$, and $C$ to $A'B'$ are the lines

\[
\begin{align*}
-(S_A + S_B)(S^2 + S_{BC})x &\quad - (S_B + S_C)(S^2 + S_{AB})z = 0, \\
(S_C + S_A)(S^2 + S_{BC})x &\quad - (S_B + S_C)(S^2 + S_{CA})y = 0.
\end{align*}
\]

These are concurrent at

\[
\left( \frac{S_B + S_C}{S^2 + S_{BC}} : \frac{S_C + S_A}{S^2 + S_{CA}} : \frac{S_A + S_B}{S^2 + S_{AB}} \right),
\]

which is the isogonal conjugate of the nine-point center $N$. □

### 4.3. Orthic triangle.

**Theorem 4.3.** The triangle of reflections $T^\dagger$ is orthologic to the orthic triangle.

![Figure 4.2](image)

**Proof.** The perpendiculars from $H_a$ to $B'C'$, $H_b$ to $C'A'$, $H_c$ to $A'B'$ are the lines

\[
\begin{align*}
2S^2(S_B - S_C)S_Ax + (S^2(3S_A + S_B + S_C) + S_{ABC})(S_{BY} - S_{CZ}) &= 0, \\
2S^2(S_C - S_A)S_{BY} + (S^2(S_A + 3S_B + S_C) + S_{ABC})(S_{CZ} - S_{AX}) &= 0, \\
2S^2(S_A - S_B)S_{CZ} + (S^2(S_A + S_B + 3S_C) + S_{ABC})(S_{AX} - S_{BY}) &= 0.
\end{align*}
\]

These are concurrent at

\[Q := (S_{BC}(S_B + S_C)(S_{AA} - 3S^2)(S^2(3S_A + S_B + S_C) + S_{ABC})) : \cdots : \cdots.\]

The perpendiculars from $A'$ to $H_bH_c$, $B'$ to $H_cH_a$, $C'$ to $H_aH_b$ are the lines
\[
2S^2(S_B - S_C)(x + y + z) = (S_B + S_C)(-(S_A + S_B)S_C y + (S_C + S_A)S_B z),
\]
\[
2S^2(S_C - S_A)(x + y + z) = (S_C + S_A)(-(S_B + S_C)S_A z + (S_A + S_B)S_C x),
\]
\[
2S^2(S_A - S_B)(x + y + z) = (S_A + S_B)(-(S_C + S_A)S_B x + (S_B + S_C)S_A y).
\]

These lines are concurrent at
\[
Q' = ((S_B + S_C)(3S_A^3(S_B + S_C) - S_A(S_B - S_C)^2 - 5S_{ABC}(S_B + S_C) - 4S_{BC}^2): \cdots : \cdots).\]

\[\square\]

**Remarks.** (1) The orthology center \(Q := \text{cev}(H)^\perp(T^\dagger)\) has ETC (6-9-13)-search number 12.4818250323 \(\cdots\). This also appears in Proposition 6.3 and §7.1.1 below.

(2) The orthology center \(Q' := T^\dagger^\perp(\text{cev}(H))\) has ETC (6-9-13)-search number -8.27009636449 \(\cdots\).

(3) Since the two triangles are perspective at \(H\), the line joining these two orthology centers contains \(H\). This is the line
\[
\sum_{cyclic} a^2 S_A(S_B - S_C)(3S^2 - S_{AA})x = 0.
\]

See Theorem 7.4(b) below.

4.4. **Tangential triangle.** Since the tangential triangle is homothetic to the orthic triangle, the results of §4.2 also shows that the triangle of reflections is orthologic to the tangential triangle. Clearly, \((T^\dagger)^\perp(\text{cev}^{-1}(K)) = (T^\dagger)^\perp(\text{cev}(H)) = Q'\).

**Proposition 4.4.** The orthology center \(\text{cev}^{-1}(K)^\perp(T^\dagger)\) is the circumcenter of \(T^\dagger\).
Proof. The vertex \( K^a = (-S_B + S_C) : S_C + S_A : S_A + S_B \) is equidistant from \( B' \) and \( C' \). In fact,
\[
K^a B'^2 = K^a C'^2 = \frac{S_A(9S^2 + S_{BB} + S_{BC} + S_{CC}) + (S_B + S_C)S_{BC}}{4S_A}.
\]
Therefore, \( K^a \) lies on the perpendicular bisector of \( B'C' \). Similarly, \( K^b \) and \( K^c \) lie on the perpendicular bisectors of \( C'A' \) and \( A'B' \) respectively. From this the result follows.

5. Triads of circles

In this chapter we consider triads of circles related to \( T^\dagger \). The circumcircle of the reflection flanks are considered in \( \S \) 5.1. In \( \S \) 5.2, we construct a triad of coaxial circles associated with pedals and with the line \( HK \) as axis. In \( \S \) 5.3,4, we show that a common triangle center, the Parry reflection point \( X(399) \), the reflection of the circumcenter in the Euler reflection point of \( T \), occurs as the point of concurrence of two triads of circles, one associated with the tangential triangle (Proposition 5.7), and another with the excentral triangle (\( \S \) 5.4).

We shall make frequent use of the following fundamental theorem.

Theorem 5.1 ([5, Proposition 18]). If the circles \( XBC, AYC, ABZ \) have a common point, so do the circles \( AYZ, XBZ, XYC \).

5.1. The reflection flanks and their circumcircles. We shall refer to the triangles \( T^\dagger a := AB'C', T^\dagger b := A'BC', \) and \( T^\dagger c := A'B'C \) as the reflection flanks.

Proposition 5.2. The reflection flank \( T^\dagger a \) is degenerate if and only if \( A = \frac{\pi}{3} \) or \( 2\pi \frac{2}{3} \).

Proof. The line \( B'C' \) has equation
\[
(-3S_A + S^2) x + 2(S^2 + S_{AB}) y + 2(S^2 + S_{CA}) z = 0.
\]
(See \( \S \) 2.1). This contains the vertex \( A \) if and only if \( 3S_A = S^2, \cot^2 A = \frac{1}{3} \), i.e., \( A = \frac{\pi}{3} \) or \( 2\pi \frac{2}{3} \).

The circumcircle of \( T^\dagger a \):
\[
(S^2 - 3S_A)(a^2yz + b^2zx + c^2xy) - 2(x + y + z)(c^2(S^2 + S_{CA})y + b^2(S^2 + S_{AB})z) = 0.
\]
Its center is the point
\[
O_a = 2S^2(3S^2 - S_A)(1, 0, 0) + b^2c^2(S^2 + S_{BC}, S^2 + S_{CA}, S^2 + S_{AB}).
\]

Proposition 5.3. (a) The circumcenters of the reflection flanks form a triangle perspective with \( ABC \) at the nine-point center \( N \).

(b) The orthocenters of the reflection flanks form a triangle perspective with \( ABC \) at \( N^* \).
Proof. From the coordinates of $O_a$, we note that the line $AO_a$ contains the nine-point center $N = (S^2 + S_{BC} : S^2 + S_{CA} : S^2 + S_{AB})$. Similarly, for the circumcenters $O_b$ and $O_c$ of $T^b$ and $T^c$ the lines $BO_b$ and $CO_c$ also contain the nine-point center.

(b) is equivalent to the orthology of $T$ and $T^\dagger$. It follows from Proposition 4.2.

□

Since the circles $(A'BC)$, $(AB'C)$, $(ABC')$ are concurrent at $H$, the circumcircles of the reflection flanks are also concurrent.

Proposition 5.4. The circumcircles of the reflection flanks are concurrent at $X(1157)$, the inverse of $N^*$ in the circumcircle of $T$.

![Figure 5.1](image_url)

Proof. The point of concurrency is necessarily the radical center of the circles. From the equations of the circumcircles of the reflection flanks:

\[(S^2 - 3S_{AA})(a^2yz + b^2zx + c^2xy) - 2(x + y + z)(c^2(S^2 + S_{CA})y + b^2(S^2 + S_{AB})z) = 0,\]
\[(S^2 - 3S_{BB})(a^2yz + b^2zx + c^2xy) - 2(x + y + z)(a^2(S^2 + S_{AB})z + c^2(S^2 + S_{BC})x) = 0,\]
\[(S^2 - 3S_{CC})(a^2yz + b^2zx + c^2xy) - 2(x + y + z)(b^2(S^2 + S_{BC})x + a^2(S^2 + S_{CA})y) = 0.\]
we obtain the radical center as the point \((x : y : z)\) satisfying
\[
\frac{c^2(S^2 + S_{CA})y + b^2(S^2 + S_{AB})z}{S^2 - 3S_{AA}} = \frac{a^2(S^2 + S_{AB})z + c^2(S^2 + S_{BC})x}{S^2 - 3S_{BB}} = \frac{b^2(S^2 + S_{BC})x + a^2(S^2 + S_{CA})y}{S^2 - 3S_{CC}}.
\]

Rewriting this as
\[
\frac{(S^2+S_{CA})y + (S^2+S_{AB})z}{b^2(S^2 - 3S_{AA})} = \frac{(S^2+S_{AB})z + (S^2+S_{BC})x}{a^2(S^2 - 3S_{BB})} = \frac{(S^2+S_{BC})x + (S^2+S_{CA})y}{c^2(S^2 - 3S_{CC})},
\]
we have
\[
\frac{(S^2+S_{CA})y + (S^2+S_{AB})z}{a^2(S^2 - 3S_{AA})} = \frac{(S^2+S_{AB})z + (S^2+S_{BC})x}{b^2(S^2 - 3S_{BB})} = \frac{(S^2+S_{BC})x + (S^2+S_{CA})y}{c^2(S^2 - 3S_{CC})}.
\]

From these,
\[
\frac{(S^2+S_{BC})x}{a^2} = -a^2(S^2 - 3S_{AA}) + b^2(S^2 - 3S_{BB}) + c^2(S^2 - 3S_{CC})
\]
\[
= a^2(S^2 - 3S_{AA}) - b^2(S^2 - 3S_{BB}) + c^2(S^2 - 3S_{CC})
\]
\[
= \frac{(S^2+S_{AB})z}{c^2} = a^2(S^2 - 3S_{AA}) + b^2(S^2 - 3S_{BB}) - c^2(S^2 - 3S_{CC}),
\]
and
\[
x : y : z = \frac{a^2(-a^2(S^2 - 3S_{AA}) + b^2(S^2 - 3S_{BB}) + c^2(S^2 - 3S_{CC}))}{S^2 + S_{BC}} : \cdots : \cdots.
\]

This gives the triangle center \(X(1157)\) in ETC, the inverse of \(N^*\) in the circumcircle of \(T\).

\[\text{\textbf{5.2. Three coaxial circles.}}\]

Let \(H_aH_bH_c\) be the orthic triangle, and \(H'_a, H'_b, H'_c\) the pedal of \(A\) on \(B'C', B\) on \(C'A'\), and \(C\) on \(A'B'\) respectively.

\[\textbf{Proposition 5.5.} \text{The lines } H_aH'_a, H_bH'_b, H_cH'_c \text{ are concurrent at}
\]
\[
(a^2S_{BC}((5S_A + S_B + S_C)S^4 + S_{ABC}(S^2 - 2S_{AA})): \cdots : \cdots).
\]

\[\textbf{Remark.} \text{This has ETC (6-9-13)-search number 3.00505308538} \cdots.\]

\[\textbf{Theorem 5.6.} \text{The three circles } AH_aH'_a, BH_bH'_b, CH_cH'_c \text{ are coaxial with radical axis } HK.\]

\[\textbf{Proof.} \text{The centers of the circles } AXX' \text{ are the point}
\]
\[
O'_a = (S_A(S_B - S_C) : -(S^2 + S_{CA}) : S^2 + S_{AB}),
\]
\[
O'_b = (S^2 + S_{BC} : S_B(S_C - S_A) : -(S^2 + S_{AB}),
\]
\[
O'_c = -(S^2 + S_{BC}) : S^2 + S_{CA} : S_C(S_A - S_B)).
\]
These centers lie on the line
\[ a^2 S_A x + b^2 S_B y + c^2 S_C z = 0. \]

The circles have equations
\[
\begin{align*}
S_A(S_B - S_C)(a^2yz + b^2zx + c^2xy) - (x + y + z)(S_B(S^2 + S_{AB})y - S_C(S^2 + S_{CA})z) &= 0, \\
S_B(S_C - S_A)(a^2yz + b^2zx + c^2xy) - (x + y + z)(S_C(S^2 + S_{BC})z - S_A(S^2 + S_{AB})x) &= 0, \\
S_C(S_A - S_B)(a^2yz + b^2zx + c^2xy) - (x + y + z)(S_A(S^2 + S_{CA})x - S_B(S^2 + S_{BC})y) &= 0.
\end{align*}
\]

The radical axis is
\[
(S_B - S_C)S_{AA}x + (S_C - S_A)S_{BB}y + (S_A - S_B)S_{CC}z = 0,
\]
which clearly contains \( H \) and \( K \) (see Figure 5.2).

\[ \square \]

Remark. The radical axes of the circumcircle with these three circles are concurrent at
\[
X(53) = \left( \frac{S^2 + S_{BC}}{S_A} : \frac{S^2 + S_{CA}}{S_B} : \frac{S^2 + S_{AB}}{S_C} \right).
\]

5.3. \( T^\dagger \) and the tangential triangle. Let \( \text{cev}^{-1}(K) := K^aK^bK^c \) be the tangential triangle. The centers of the circles \( K^aB'C', K^bC'A', K^cA'B' \) are the points
The triangle of reflections

\[ O''_a = (S_{ABC} + (3S_A - S_B - S_C)S^2 : b^2c^2S_C : b^2c^2S_B), \]
\[ O''_b = (c^2a^2S_C : S_{ABC} + (3S_B - S_C - S_A)S^2 : c^2a^2S_A), \]
\[ O''_c = (a^2b^2S_B : a^2b^2S_A : S_{ABC} + (3S_C - S_A - S_B)S^2). \]

The triangle \( O''_aO''_bO''_c \) is perspective with \( T \) at \( H \). Therefore, the two triangles are orthologic. The perpendiculars from \( O''_a, O''_b, O''_c \) to \( BC, CA, AB \) respectively are concurrent at

\[ X(265) = \left( \frac{S_A}{3S_{AA} - S^2} : \frac{S_B}{3S_{BB} - S^2} : \frac{S_C}{3S_{CC} - S^2} \right). \]

**Proposition 5.7** ([5, §5.1.2]). The circles \( K^aB'C', K^bC'A', K^cA'B' \) are concurrent at the Parry reflection point \( X(399) \).

**Proof.** The equations of the circles are

\[ 2S_A(a^2yz + b^2zx + c^2xy) + (x + y + z)(b^2c^2x + 2c^2S_Cy + 2b^2S_Bz) = 0, \]
\[ 2S_B(a^2yz + b^2zx + c^2xy) + (x + y + z)(2c^2S_Cx + c^2a^2y + 2a^2S_Az) = 0, \]
\[ 2S_C(a^2yz + b^2zx + c^2xy) + (x + y + z)(2b^2S_Bx + 2a^2S_Ay + a^2b^2z) = 0. \]
The radical center of the three circles is the point \((x : y : z)\) satisfying
\[\frac{b^2c^2x + 2c^2S_{Cy} + 2b^2S_{Bz}}{S_A} = \frac{2c^2S_{C}x + c^2a^2y + 2a^2S_{Az}}{S_B} = \frac{2b^2S_{B}x + 2a^2S_{Ay} + a^2b^2z}{S_C}.\]
This is
\[(x : y : z) = (a^2(-8S^4 + 3b^2c^2(S^2 + 3S_{BC})) : \cdots : \cdots),\]
the triangle center \(X(399)\), the Parry reflection point, which is the reflection of \(O\) in the Euler reflection point \(E\) (see Figure 5.3).

The circles \(A'K^bK^c, B'K^cK^a, C'K^aK^b\) are also concurrent (see \([11]\)). The point of concurrency is a triangle center with coordinates
\[(a^2f(S_A, S_B, S_C) : b^2f(S_B, S_C, S_A) : c^2f(S_C, S_A, S_B)),\]
with \(\text{ETC} (6-9-13)-\text{search number} \ 1.86365616601 \cdots\). The polynomial \(f(S_A, S_B, S_C)\) has degree 10.

5.4. \(T^t\) and the excentral triangle.

**Proposition 5.8** ([5, \(\S\ 5.1.3\)]). The circles \(A'I^bI^c, I^aB'I^c, I^aI^bC'\) have the Parry reflection point as a common point.

The centers of these circles are perspective with the excentral triangle at a point with \(\text{ETC} (6-9-13)-\text{search number} \ -27.4208873972 \cdots\).
On the other hand, the circles \( I^a B'C', A'I^b C' \) and \( A'B'I^c \) have a common point with ETC (6-9-13)-search number 7.08747856659 \ldots \). Their centers are perspective with \( ABC \) at the point \( X(3336) \) which divides \( OI \) in the ratio \( OX(3336) : X(3336)I = 2R + 3r : -4r \).

### 6. Pedals of vertices of \( T^\uparrow \) on the sidelines of \( T \)

#### 6.1. The triad of triangles \( AB_a C_a, BC_b A_b, CA_c B_c \)

Consider the pedals of \( A', B', C' \) on the sidelines of \( T \). These are the points

<table>
<thead>
<tr>
<th>( BC )</th>
<th>( CA )</th>
<th>( AB )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A' )</td>
<td>( B_a = (S_{CC} - S^2 : 0 : 2S^2) ), ( C_a = (S_{BB} - S^2 : 2S^2 : 0) )</td>
<td></td>
</tr>
<tr>
<td>( B' )</td>
<td>( A_b = (0 : S_{CC} - S^2 : 2S^2) ), ( C_b = (2S^2 : S_{AA} - S^2 : 0) )</td>
<td></td>
</tr>
<tr>
<td>( C' )</td>
<td>( A_c = (0 : 2S^2 : S_{BB} - S^2) ), ( B_c = (2S^2 : 0 : S_{AA} - S^2) )</td>
<td></td>
</tr>
</tbody>
</table>

#### Proposition 6.1. The Euler lines of the triangles \( AB_a C_a, BC_b A_b, CA_c B_c \) are concurrent at the Hatzopolakis reflection point \( X(1986) \).

**Proof.** The circumcenter of \( AB_a C_a \) is \( H_a = (0 : S_C : S_B) \). The centroid is the point

\[
2S^2(S_B + S_C, S_C + S_A, S_A + S_B) - (3S_{ABC} + S^2(S_A + S_B + S_C))(1, 0, 0).
\]

From these we find the equation of its Euler line; similarly for the other two triangles. The Euler lines of the triangles are the lines

\[
2S^2 \cdot S_A(S_B - S_C)x + (3S_{ABC} - S^2(-S_A + S_B + S_C))(S_{By} - S_{Cz}) = 0,
\]

\[
2S^2 \cdot S_B(S_C - S_A)y + (3S_{ABC} - S^2(S_A - S_B + S_C))(S_{Cz} - S_{Ax}) = 0,
\]

\[
2S^2 \cdot S_C(S_A - S_B)z + (3S_{ABC} - S^2(S_A + S_B - S_C))(S_{Ax} - S_{By}) = 0.
\]

These three lines are concurrent at a point with coordinates given above. It is the triangle center \( X(1986) \) (see Figure 6.1). \( \square \)

**Remark.** Here is a definition of the Hatzopolakis reflection point \( X(1986) \) equivalent to the one given in ETC. Let \( H_a H_b H_c \) be the orthic triangle. \( X(1986) \) is the common point of the reflections of the circles \( AB_b H_c \) in \( H_b H_c, BH_c H_a \) in \( H_c H_a \), and \( CH_c H_b \) in \( H_a H_b \).

#### Proposition 6.2. The circumcircles of the triangles \( AB_a C_a, BC_b A_b, CA_c B_c \) have radical center \( X(68) \).

**Proof.** These are the circles with centers \( H_a, H_b, H_c \), passing through \( A, B, C \) respectively.

\[
a^2(a^2yz + b^2zx + c^2xy) - (x + y + z)((S_{BB} - S^2)y + (S_{CC} - S^2)z) = 0,
\]

\[
b^2(a^2yz + b^2zx + c^2xy) - (x + y + z)((S_{CC} - S^2)z + (S_{AA} - S^2)x) = 0,
\]

\[
c^2(a^2yz + b^2zx + c^2xy) - (x + y + z)((S_{AA} - S^2)x + (S_{BB} - S^2)y) = 0.
\]
The radical center is the point defined by
\[
\frac{(S_{BB} - S^2)y + (S_{CC} - S^2)z}{a^2} = \frac{(S_{CC} - S^2)z + (S_{AA} - S^2)x}{b^2} = \frac{(S_{AA} - S^2)x + (S_{BB} - S^2)y}{c^2}.
\]
From these,
\[
\frac{(S_{AA} - S^2)x}{b^2 + c^2 - a^2} = \frac{(S_{BB} - S^2)y}{c^2 + a^2 - b^2} = \frac{(S_{CC} - S^2)z}{a^2 + b^2 - c^2}
\]
and
\[
x : y : z = \frac{S_A}{S_{AA} - S^2} : \frac{S_B}{S_{BB} - S^2} : \frac{S_C}{S_{CC} - S^2}.
\]
This is the triangle center $X(68)$.

6.2. The triad of triangles $A'B_aC_a$, $B'C_bA_b$, $C'A_cB_c$.

**Theorem 6.3.** The Euler lines of the triangles $A'B_aC_a$, $B'C_bA_b$, $C'A_cB_c$ are concurrent at
\[
Q = (S_{BC}(S_B + S_C)(S_{AA} - 3S^2)(S_{ABC} + S^2(3S_A + S_B + S_C)) : \cdots : \cdots).
\]

**Proof.** The circumcenter of $A'B_aC_a$ is $H_a = (0 : S_C : S_B)$, the same as $AB_aC_a$.
The centroid is the point
\[
-(S_{ABC} + S^2(3S_A + S_B + S_C)) : 2(S_C + S_A)(2S^2 - S_{AB}) : 2(S_A + S_B)(2S^2 - S_{CA})).
\]
From these we find the equation of its Euler line; similarly for the other two triangles. The Euler lines of the triangles are the lines
The triangle of reflections

2S^2 \cdot S_A(S_B - S_C)x + (S_{ABC} + S^2(3S_A + S_B + S_C))(S_{BY} - S_{CZ}) = 0,
2S^2 \cdot S_B(S_C - S_A)y + (S_{ABC} + S^2(S_A + 3S_B + S_C))(S_{CZ} - S_{AX}) = 0,
2S^2 \cdot S_C(S_A - S_B)z + (S_{ABC} + S^2(S_A + S_B + 3S_C))(S_{AX} - S_{BY}) = 0.

These three lines are concurrent at a point with coordinates given above.

\[ \Box \]

Remark. This is the same as the orthology center \( \text{cev}(H)^\perp(T^\perp) \) in Theorem 4.3.

6.3. A Taylor-like circle. It is well known that the six pedals of \( H_a, H_b, H_c \) on the sidelines of \( T \) are concyclic. The circle containing them is the Taylor circle (see, for example, [6, §9.6]). The center of the circle, called the Taylor center, is the triangle center

\[ X(389) = (S^4 - S_{AA}S_{BC} : S^4 - S_{BB}S_{CA} : S^4 - S_{CC}S_{AB}). \]

Analogous to the Taylor circle, the pedals of \( A', B', C' \) on the sidelines of \( T \) are also concyclic (see [1]). In fact, the circle containing them is a Tucker circle (Figure 6.3), since

(i) the segments \( B_cC_b, C_aA_c, A_bB_a \) are parallel to \( BC, CA, AB \) respectively, and
(ii) the segments \( B_aC_a, C_bA_b, A_cB_c \) are antiparallel to \( BC, CA, AB \) respectively, i.e., they are parallel to the corresponding sides of the orthic triangle \( H_aH_bH_c \).

The equation of the circle containing these six pedals is

\[ a^2b^2c^2(2a^2yz + b^2zx + c^2xy) - 2S^2(x + y + z)((S_{AA} - S^2)x + (S_{BB} - S^2)y + (S_{CC} - S^2)z) = 0. \]

The center of the circle is the point

\[ (a^2(S_{AA} - S^2)(S^2 + S_{BC}) : b^2(S_{BB} - S^2)(S^2 + S_{CA}) : c^2(S_{CC} - S^2)(S^2 + S_{AB})). \]
This is the triangle center $X(52)$. It is the reflection of $O$ in the center of the Taylor circle. It is also the orthocenter of the orthic triangle (see Figure 6.3).

7. Some locus problems leading to conics

The website [4] CATALOGUE OF TRIANGLE CUBICS of B. Gibert contains a vast number of cubic and higher degree curves arising from locus problems in triangle geometry. In this chapter we consider a few loci related to perspectivity and orthology with $T^\dagger$ which are conics. To avoid presenting excessively complicated algebraic manipulations, we present two problems in which the conic loci can be easily identified as rectangular hyperbolas. For this we recall a basic fact in triangle geometry: A (circum-)conic passing through the vertices of a triangle is a rectangular hyperbola if and only if it also passes through the orthocenter of the triangle.

7.1. Reflection of $T$ in a point. Let $P$ be a point with homogeneous barycentric coordinates $(x : y : z)$. The reflections of $T$ in $P$ is the triangle $T^\dagger_P$ with vertices

$$
A^\dagger_P = (x - y - z : 2y : 2z),
B^\dagger_P = (2x : y - z - x : 2z),
C^\dagger_P = (2x : 2y : z - x - y).
$$

7.1.1. Perspectivity of $T^\dagger$ with reflection of $T$ in a point.

**Theorem 7.1.** The locus of $P$ for which $T^\dagger_P$ is perspective with $T^\dagger$ is the rectangular circum-hyperbola of the orthic triangle containing the orthocenter $H$. 
Proof. The equation of the line $A'A^\perp$ is
\[
\begin{vmatrix}
-(S_B + S_C) & 2S_C & 2S_B \\
x - y - z & 2y & 2z \\
X & Y & Z
\end{vmatrix} = 0
\]
or
\[2(S_BY - S_CZ)X - (S_Bx - S_BY + S_Cz)Y + (S_Cx - S_BY + S_Cz)Z = 0.\]

Similarly, we have the equations of the lines $B'B^\perp$ and $C'C^\perp$. These three lines are concurrent if and only if
\[
\begin{vmatrix}
2(S_BY - S_CZ) & -(S_Bx - S_BY + S_Cz) & S_Cx - S_BY + S_Cz \\
S_Ax + S_Ay - S_Cz & 2(S_Cz - S_Ax) & -(S_Ax + S_CY - S_Cz) \\
-(-S_Ax + S_BY + S_Az) & -S_Ax + S_BY + S_Bz & 2(S_Ax - S_BY)
\end{vmatrix} = 0.
\]

Expanding this determinant, we obtain
\[
(x + y + z) \left( \sum_{\text{cyclic}} (S_B - S_C)(S_Ax^2 + S_BCy) \right) = 0.
\]

Since $P$ is a finite point, $x + y + z \neq 0$. Therefore $(x : y : z)$ must satisfy
\[
\sum_{\text{cyclic}} (S_B - S_C)(S_Ax^2 + S_BCy) = 0.
\]

The clearly defines a conic. By setting $x = 0$, we obtain
\[
(S_BY - S_CZ)(S_B(S_C - S_A)y - S_C(S_A - S_B)z) = 0.
\]

It is clear that the conic contains $H_a = (0 : S_C : S_B)$. Similarly, it also contains $H_B$ and $H_C$. Therefore, it is a circumconic of the orthic triangle $cev(H) = H_aH_bH_c$. We also verify that the conic contains the following two points:
(i) the orthocenter $H = \left( \frac{1}{S_A} : \frac{1}{B} : \frac{1}{S_C} \right)$ (easy),
(ii) the triangle center
\[
X(52) = \left( (S_B + S_C)(S^2 - S_A)(S^2 - S_AA)(S^2 + S_BC) : \cdots : \cdots \right)
\]
(with the help of Mathematica). This latter, according to ETC, is the orthocenter $H^\perp$ of the orthic triangle (see Figure 7.1).

It follows that the locus of $P$ is the rectangular circum-hyperbola of the orthic triangle containing $H$. □

Remarks. (1) For $P = H^\perp = X(52)$, the perspector of $T^\perp$ and $T_P^\perp$, is the orthology center $Q = (T^\perp)^\perp(cev(H))$ in Proposition 4.3.

(2) Since the conic intersects the sidelines of $T$ at the traces of $H$, the second intersections with the sidelines are the traces of another point. This is $X(847)$.

(3) The center of the conic is the point $X(1112)$. 

7.1.2. Locus of perspector.

**Theorem 7.2.** For $P$ on the rectangular circum-hyperbola of the orthic triangle containing $H$, the locus of the perspector of $T_T$ and $T_P$ is the rectangular circum-hyperbola of $T_T$ containing the orthocenter $H$.

**Proof.** Rearranging the equation of $A'A_P^1$ in the form

$$(S_BY - SCZ)x - (2X + Y + Z)(SBY - SCz) = 0,$$

and likewise those of $B'B_P^1$ and $C'C_P^1$, we obtain the condition for concurrency:

$$\begin{vmatrix}
S_BY - SCZ & -S_B(2X + Y + Z) & S_C(2X + Y + Z) \\
S_A(X + 2Y + Z) & S_CZ - SAx & -S_C(X + 2Y + Z) \\
-S_A(X + Y + 2Z) & SB(X + Y + 2Z) & SAx - SBY
\end{vmatrix} = 0.$$

Expanding the determinant, we obtain

$$(X + Y + Z) \left( \sum_{\text{cyclic}} (S_B - SC)(2S_{AA}X^2 - (SA(S_B + SC) - 2S_{BC})YZ) \right) = 0.$$

Since this perspector cannot be an infinite point, it must lie on the conic

$$\sum_{\text{cyclic}} (S_B - SC)(2S_{AA}X^2 - (SA(S_B + SC) - 2S_{BC})YZ) = 0. \quad (7.1)$$

Construct the parallels to $A'C'$ and $A'B'$ through $H_c$ and $H_b$ respectively to intersect at a point $X$. The reflections of $B$ and $C$ in $X$ lie respectively on $A'C'$...
and $A'B'$ (Figure 7.2). Therefore, $T_X^+$ is perspective to $T^+$ at $A'$. This shows that the conic (7.1) contains $A'$. The same reasoning shows that it also contains $B'$ and $C'$. It is a circumconic of $T^+$.

Now we claim that this conic contains the following two points:
(i) the orthocenter $H$ (easy, take $P = H$),
(ii) the orthocenter $H'$ of $T^+$.

In Proposition 4.3, we have constructed the orthology center $Q = (\text{cev}(H))^\dagger(T^+)$. We claim that the reflection $T_Q^+$ is perspective with $T^+$ at the orthocenter of $T^+$. Note that in triangle $AA'A_Q^+$, $Q$ and $H_a$ are the midpoints of $AA_Q$ and $AA'$. Therefore, $A'A_Q^+$ is parallel to $H_aQ$. Since $H_aQ$ is perpendicular to $B'C'$, $A'A_Q^+$ is the altitude of $T^+$ (through $A'$). Similarly, $B'B_Q^+$ and $C'C_Q^+$ are also altitudes of the same triangle (see Figure 7.3). The three lines are concurrent at $H'$, the orthocenter of $T^+$, which therefore lies on the conic defined by (7.1).

It follows that (7.1) defines the rectangular circum-hyperbola of $T^+$ containing $H$. □

7.2. Orthology of $T^+$ with reflection triangle of $P$. For a given point $P$, the reflection triangle $T^+P$ has vertices the reflections of $P$ in the sidelines:

$$P_a^+ = (-S_B + S_C)x : 2S_Cx + (S_B + S_C)y : 2S_Bx + (S_B + S_C)z),$$
$$P_b^+ = (2SCy + (S_C + S_A)x : -(S_C + S_A)y : 2S_Ay + (S_C + S_A)z),$$
$$P_c^+ = (2SBz + (S_A + S_B)x : 2S_Az + (S_A + S_B)y : -(S_A + S_B)z).$$

It is well known that the reflection triangle $T^+P$ is degenerate if and only if $P$ lies on the circumcircle. It is clear that the perpendiculars from $A$, $B$, $C$ to the
line containing $P_a^1$, $P_b^1$, $P_c^1$ are parallel. However, the perpendiculars from these points to the sidelines of $T$ are concurrent if and only if $P$ is an intersection of the Euler line with the circumcircle. Henceforth, we shall consider $P$ not on the circumcircle, so that its reflection triangle is nondegenerate. We study the locus of $P$ for which $T^\dagger(P)$ is orthologic to $T^\dagger$.

7.2.1. Locus of $P$ whose reflection triangle is orthologic to $T^\dagger$.

**Theorem 7.3.** The reflection triangle of $P$ is orthologic to $T^\dagger$ if and only if $P$ lies on the Euler line.

**Proof.** Let $P$ be a point outside the circumcircle, and with homogeneous barycentric coordinates $(x : y : z)$. The perpendiculars from the vertices of $T^\dagger$ to the sidelines of $T^\dagger(P)$ are the lines

\[
2(-c^2S_{Cy} + b^2S_{Bz})X + (c^2(S_B - S_C)y + 2b^2S_Bz)Y + (-2c^2S_{Cy} + b^2(S_B - S_C)z)Z = 0,
\]

\[
(-2a^2S_{Az} + c^2(S_C - S_A)x)X + 2(-a^2S_{Az} + c^2S_Cx)Y + (a^2(S_C - S_A)z + 2c^2S_{Cx})Z = 0,
\]

\[
(b^2(S_A - S_B)x + 2a^2S_{Ay}x)X + (-2b^2S_Bx + a^2(S_A - S_B)y)Y + 2(-b^2S_Bx + a^2S_{Ay})Z = 0.
\]

These are concurrent if and only if

\[
\begin{vmatrix}
2(-c^2S_{Cy} + b^2S_{Bz}) & (c^2(S_B - S_C)y + 2b^2S_Bz) & (-2c^2S_{Cy} + b^2(S_B - S_C)z) \\
(-2a^2S_{Az} + c^2(S_C - S_A)x) & 2(-a^2S_{Az} + c^2S_Cx) & (a^2(S_C - S_A)z + 2c^2S_{Cx}) \\
(b^2(S_A - S_B)x + 2a^2S_{Ay}x) & (-2b^2S_Bx + a^2(S_A - S_B)y) & 2(-b^2S_Bx + a^2S_{Ay})
\end{vmatrix} = 0.
\]
Expanding this determinant, we obtain
\[-a^2b^2c^2(a^2yz+b^2zx+c^2xy)(S_A(S_B-S_C)x+S_B(S_C-S_A)y+S_C(S_A-S_B)z) = 0.\]
Apart from the circumcircle, this defines the Euler line. \(\square\)

7.2.2. Loci of orthology centers.

**Theorem 7.4.** Let \(P\) be a point on the Euler line.

(a) The locus of the orthology center \(T^\top\perp(T^\top(P))\) is the rectangular circum-hyperbola of \(T^\top\) containing the orthocenter \(H\) of \(T\).

(b) The locus of the orthology center \(T^\top(P)\perp(T^\top)\) is the line joining the orthocenter \(H\) of \(T\) to the nine-point center of \(T^\top\).

**Proof.** Let \(P = (S_{BC}+t, S_{CA}+t, S_{AB}+t)\) be a point on the Euler line.

(a) The perpendiculars from \(A'\) to the line \(P_b^\top P_c^\top\) is
\[
2S_A(S_B-S_C)(S^2+t)x \\
+ (S_A(S_A(2S_{BB}+S_{BC}-S_{CC})+S_{BC}(3S_B-S_C)) \hspace{1em} + (S_A(3S_B-S_C)+S_B(S_B+S_C)t)y \\
+ (S_A(S_A(S_{BB}-S_{BC}-2S_{CC})+S_{BC}(S_B-3S_C)) \hspace{1em} + (S_A(S_B-3S_C)-S_C(S_B+S_C)t)z \\
= 0.
\]

The coefficients are linear in \(t\); similarly for the equations of the perpendiculars from \(B'\), \(C'\) to \(P_b^\top P_c^\top\) and \(P_a^\top P_c^\top\). The point of concurrency of the three perpendiculars therefore has coordinates given by quadratic functions in \(t\). Therefore, the locus of the orthology center \(T^\top\perp(T^\top(P))\) is a conic. Note that for \(P = O\), this orthology center is \(H\). Also, for \(P = N\), this is the orthocenter of \(T^\top\). (Proof: Since the reflection triangle of \(N\) is homothetic to \(T^\top\) (Proposition 2.2), they are orthologic, and \((T^\top)\perp(T^\top(N)) = H',\) the orthocenter of \(T^\top\).)

We claim that \(A', B', C'\) are also three such orthology centers. Consider the Jerabek hyperbola (the rectangular circum-hyperbola of \(T^\top\) through \(O\) and \(H\). This is the isogonal conjugate of the Euler line. It also contains the symmedian point \(K\). If the lines \(A'K, B'K, C'K\) intersect the hyperbola again at \(X'^*, Y'^*, Z'^*\) respectively, then their isogonal conjugates \(X, Y, Z\) are on the Euler line. If \(X_a^\top X_b^\top X_c^\top\) is the reflection triangle of \(X\), then the lines \(X_a^\top X_b^\top\) and \(X_a^\top X_c^\top\) are perpendicular to \(A'C'\) and \(A'B'\) respectively. This shows that \(A'\) is the orthology center \((T^\top)\perp(T^\top(X))\), and it lies on the conic locus. The same reasoning shows that \(B' = (T^\top)\perp(T^\top(Y))\) and \(C' = (T^\top)\perp(T^\top(Z))\) are also on the same conic. This shows that the conic is a circumconic of \(T^\top\), containing its orthocenter \(H'\) and \(H\).

(b) On the other hand, we compute the equations of the lines from \(P_a^\top\) to \(B'C'\), \(P_b^\top\) to \(C'A'\), and \(P_c^\top\) to \(A'B'\). These three lines are concurrent at a point whose coordinates are linear functions of \(t\). Therefore, the locus of the orthology center is a line. This line has equation
\[
\sum_{\text{cyclic}} a^2 S_A(S_B-S_C)(3S^2-S_{AA})X = 0.
\]
It is routine to check that this line contains the orthocenter $H$. It also contains the nine-point center $N'$ of $T^\dagger$ (with coordinates given in Proposition 2.4).

Remarks. (1) This is the same line joining the orthology centers of $T^\dagger$ and the orthic triangle in Theorem 4.3.

(2) The orthology center $T^\dagger(P)^-T^\dagger$ is

(i) the orthocenter $H$ for $P = X(3520) = (a^2S_{BC}(5S_{AA} + S^2)) : : : ;$, which can be constructed as the reflection of $H$ in $X(1594)$, which is the intersection of the Euler line with the line joining the Jerabek and Taylor centers,

(ii) the nine-point center of $T^\dagger$ for $P = (-S^4(S_A + S_B + S_C) + S_{BC}((16S_A + 7S_B + 7S_C)S^2 + S_{ABC})) : : : ;$

with ETC (6-9-13)-search number 7.47436627857... .

8. Epilogue

In two appendices we list the triangle centers encountered in this paper. Appendix A lists those catalogued in ETC that feature in this paper with properties related to $T^\dagger$. Appendix B lists new triangle centers in order of their search numbers in ETC. Here we present two atlases showing the positions of some of the centers in Appendix A in two groups.

Figure 8.1 shows a number of centers related to $N$ and its isogonal conjugate $N^*$. On the line $ON^*$ there are $X(195)$ (the circumcenter of $T^\dagger$) and $X(1157)$ (the common point of the circumcircles of the reflection flanks (Proposition 5.4)). The line $NN^*$ intersects the circumcircle at $X(1141)$, which also lies on the rectangular
circum-hyperbola of $T$ containing $N$. The center of the hyperbola is $X(137)$ on the nine-point circle.

The line joining $X(1141)$ to $X(1157)$ is parallel to the Euler line of $T$. This line intersects the hyperbola at the antipode of $N$, which is the triangle center $X(1263)$.

Figure 8.2 shows the Euler line and its isogonal conjugate, the Jerabek hyperbola with center $X(125)$. The four triangle centers $O$, $X(125)$, $X(52)$ (center of the Taylor-like circle in §6.3), and $X(1986)$ (the Hatzopolakis reflection point) form a parallelogram, with center $X(389)$, the center of the Taylor circle. The line joining $O$ to $X(125)$ intersects the hyperbola at $X(265)$ (the reflection conjugate of $O$), which appears several times in this paper.

It is a well known fact that the line the hyperbola intersects the circumcircle at $X(74)$, the antipode of $H$. The antipode of $X(74)$ on the circumcircle is the Euler reflection point $E$. The Parry reflection point $X(399)$ is the reflection of $O$ in $E$.

Figure 8.2 also shows the construction given in §7.2.2 of $X(3520)$ on the Euler line.

**Appendix A: Triangle centers in ETC associated with $T^1$.**

1. $X(5) = N$: nine-point center.
   - Reflection triangle homothetic to $T^1$ (Proposition 2.2).
   - Perspector of the medial triangles of $T^1$ and $T$ (Proposition 2.3).
   - Perspector of the circumcenters of $AB'C'$, $A'B'C'$, $A'B'C$ (Proposition 5.3(a)).

2. $X(15)$: isodynamic point.
Figure 8.2

- Perspector of negative Fermat triangle with $T \dagger$ (§3.1.2).
  - $X(16)$: isodynamic point.
    - Perspector of positive Fermat triangle with $T \dagger$ (§3.1.2).
  - $X(52)$: orthocenter of orthic triangle.
    - Center of the circle through the 6 pedals of $A'$ on $CA$, $AB$ etc. (§6.3).
  - $X(53)$: symmedian point of orthic triangle.
    - Point of concurrency of the radical axes of the circumcircle with the circles $AH_aH'_a$, $BH_bH'_b$, $CH_cH'_c$, where $H_aH_bH_c$ is the orthic triangle, and $H'_a$, $H'_b$, $H'_c$ are the pedals of $A$ on $B'C'$ etc (Theorem 5.6).
  - $X(54) = N^*$.
    - Orthology center from $ABC$ to $T \dagger$ (Proposition 4.2).
    - Perspector of the orthcenters of $A'B'C'$, $A'B'C'$, $A'B'C$ (Proposition 5.3(b)).
    - Perspector of the orthcenters of the reflection flanks (Proposition 5.3(b)).
    - Radical center of the nine-point circles of $A'B'C$, $AB'C$ and $ABC'$.
    - Perspector of the triangle bounded by the radical axes of the circumcircle with the circles each through one vertex and the reflections of the other two in their opposite sides. (The triangle in question is indeed the anticevian triangle of $N^*$.)
  - $X(68)$:
    - Radical center of the circles $AB_aC_aA'$, $BC_bA_bB'$, $CA_tA_tC''$ (Proposition 6.2).
  - $X(195)$: reflection of $O$ in $N^*$.
    - The circumcenter of $T \dagger$ (§2.3).
    - Orthology center from tangential triangle to $T \dagger$ (Proposition 4.4).
  - $X(265)$: reflection conjugate of $O$. 
The triangle of reflections

- Orthology center from circumcenters of $K^aB'C'$, $K^bC'A'$, $K^cA'B'$ to $T$ (§5.3).
- Point of concurrency of parallels through $A$ to Euler line of $A'B'C$ etc.

(10) $X(399)$: Parry reflection point (reflection of $O$ in Euler reflection point):
- Common point of circles $K^aB'C'$, $K^bB'C'$, $K^cB'C'$ (Proposition 5.7).
- Common point of circles $A'I_bI_c$, $B'I_aI_c$, $C'I_aI_b$ (Proposition 5.8).

(11) $X(484)$ Evans perspector
- Perspector of $T^1$ and the excentral triangle (§3.1.1).

(12) $X(1141)$: intersection of circumcircle with the rectangular circum-hyperbola through $N$.
- Perspector of the reflection triangle of $X(1157)$.
- Cevian triangle perspective with $T^1$ at $X(1157)$.

(13) $X(1157)$: inverse of $N^*$ in circumcircle.
- Common point of the circles $A'B'C'$, $A'B'C'$ and $A'B'C'$ (Proposition 5.4).
- Perspector of $T^1$ with the anticevian triangle of $N^*$.
- Perspector of $T^1$ with the cevian triangle of $X(1141)$.

(14) $X(1986)$: Hatzopolakis reflection point.
- Reflection of Jerabek point in Taylor center.
- Common point of the Euler lines of $AB_aC_a$ etc., where $B_a$ and $C_a$ are the pedals of $A'$ on $AC$ and $AB$ respectively (Theorem 6.1).

(15) $X(3060)$
- Centroid $T^1$ (§2.3).

(16) $X(3336)$
- Perspector of centers of circles $I^aB'C'$, $I^bC'A'$, $I^cA'B'$ (§5.4).

(17) $X(3520)$
- Point on the Euler line with orthology center $(T^1)^-|T^1(X(3520)) = H$ (§7.2.2, Remark 2(i)).

Appendix B: Triangle centers not in ETC.

(1) $-27.4208873972\ldots$: perspector of excentrical triangle with centers of circles $A'I_bI_c$, $B'I_cI_a$, $C'I_aI_b$ (§5.4).
(2) $-9.04876879620\ldots$: perspector of $T^1$ and triangle bounded by reflections of $a'$ in $a$ etc (§3.2).
(3) $-8.27009636449\ldots$: orthology center from $T^1$ to orthic triangle (§4.3).
(4) $-7.90053389552\ldots$: perspector of $T^1$ and triangle bounded by reflections of $a'$ in $a$ etc (§3.2).
(5) $-5.94879118842\ldots$: symmedian point of $T^1$.
(6) $-1.94515015138\ldots$: Euler reflection point of $T^1$ (§2.4).
(7) $-0.873191727540\ldots$: perspector of the triangles bounded by the reflections of $a'$ in $a$ etc and those of $a$ in $a'$ etc (§3.2).
(8) $1.8635616601\ldots$: common point of circles $A'K^bK^c$, $B'K^cK^a$, $C'K^aK^b$ (§5.4).
(9) $2.99369649092\ldots$: radical center of pedal circles of $A'$, $B'$, $C'$.
(10) $3.00505308538\ldots$: point of concurrency of $H_aH'_a$, $H_bH'_b$, $H_cH'_c$, where $H'_a$, $H'_b$, $H'_c$, $H'_d$ are the pedals of $A$ on $B'C'$, $B$ on $C'A'$, and $C$ on $A'B'$ respectively (Proposition 5.5).
(11) $3.99186168013\ldots$: perspector of $T$ and triangle bounded by reflections of $a'$ in $a$ etc (§3.2).
(12) $4.229247808831\ldots$: perspector of $T$ and the reflection triangle of $X(1263)$ (§4.1).
(13) $5.99676405896\ldots$: nine-point center of $T^1$ (§2.3).
(14) 7.08747856659 ···: common point of circles $I^aB'C'$, $A'I^bC'$ and $A'B'I^c$ (§5.4).
(15) 7.47436627857 ···: $P$ for which the orthology center $T^1(P) = T^1 = N^1$ (§7.2.2, Remark 2(ii)).
(16) 8.27975385194 ···: perspector of $T$ and triangle bounded by reflections of $a$ in $a'$ etc (§3.2).
(17) 12.48182503 ···: $Q :=$ orthology center from orthic triangle to $T^1$ (§4.3).
  • orthology center from reflection triangle of $X(1594)$ to $T^1$,
  • point of concurrency of the Euler lines of triangles $A'B_aC_a$, $B'C_bA_b$, $C'A_cB_c$
    (Proposition 6.3),
  • reflection of $T$ in this point is perspective with $T^1$ (at the orthocenter of $T^1$)
    (Proof of Theorem 7.3).
(18) 31.1514091170 ···:
  • orthocenter of $T^1$ (§2.3).
  • perspector of $T^1$ and reflection of $T$ in $Q$ (§7.1.2).

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A Gallery of Conics by Five Elements

Paris Pamfilos

Abstract. This paper is a review on conics defined by five elements, i.e., either lines to which the conic is tangent or points through which the conic passes. The gallery contains all cases combining a number \((n)\) of points and a number \((5-n)\) of lines and also allowing coincidences of some points with some lines. Points and/or lines at infinity are handled as particular cases.

1. Introduction

In the following we review the construction of conics by five elements: points and lines, briefly denoted by \((\alpha P\beta T)\), with \(\alpha + \beta = 5\). In these it is required to construct a conic passing through \(\alpha\) given points and tangent to \(\beta\) given lines. The six constructions, resulting by giving \(\alpha, \beta\) the values 0 to 5 and considering the data to be in general position, are considered the most important ([18, p. 387]), and are to be found almost on every book about conics. It seems that constructions for which some coincidences are allowed have attracted less attention, though they are related to many interesting theorems of the geometry of conics. Adding to the six main cases those with the projectively different possible coincidences we land to 12 main constructions figuring on the first column of the classifying table below. The six added cases can be considered as limiting cases of the others, in which a point tends to coincide with another or with a line. The twelve main cases are the projectively inequivalent to which every other case can be reduced by means of a projectivity of the plane. There are, though, interesting classical theorems for particular euclidean inequivalent cases worth studying, as, for example, the much studied case of parabolas tangent to four lines (case \((0P5T_1)\) in \(\S 7.2\)). In this review the frame is that of euclidean geometry and consequently a further distinction of ordinary from points and lines at infinity is taken into account. All of the \((50)\) constructions classified below are known and can be found in the one or the other book on conics (e.g. [7, pp. 136]), but nowhere come to discussion all together, so far I know. In a few cases (e.g. \(\S 11.1, \S 3.2\)) I have added a proof, which seems to me interesting and have not found elsewhere.
Throughout the text, number of solutions or non existence in each case, presuppose such a restriction. Two points is not parallel to the tangent or the asymptote, etc. The statements on the asymptote and the tangent are not collinear to the two points, that the line of the section numbers and the column labels the subsection numbers. The column with label 1 lists the symbols of the twelve projectively inequivalent cases. Each row of the table comprises the cases, which are projectively equivalent to the one of the first column. The notation used is a slight modification of the one introduced by Chasles ([3, p. 304]). The symbol \( P_n \) means that \( n \) of the given points are at infinity and \( T_1 \) means that one of the tangent lines is the line at infinity, later meaning that the conic, to be constructed, is a parabola. The indices, which adhere to the right parenthesis are optional. When absent, it means that the configuration is in general position, i.e. none of the ordinary points is coincident with an ordinary line. When present it means that one or two of the points are correspondingly coincident with one or two tangents. The indices \( i, 2i \) mean that one/two ordinary lines are correspondingly coincident with one/two points at infinity. The index \( 1i \) means that an ordinary point coincides with a line and also a point at infinity coincides with the point at infinity of an ordinary line.

Except for the coincidence suggested by the corresponding symbol, the other data are assumed to be in general positions in the projective sense. For example, the symbol \( (3P_2T)_1 \) stands for the construction of a hyperbola given two points, an asymptote and a tangent. These four elements are assumed in general position, implying that no further coincidences are present, that the intersection of the asymptote and the tangent are not collinear to the two points, that the line of the two points is not parallel to the tangent or the asymptote, etc. The statements on the number of solutions or non existence in each case, presuppose such a restriction. Throughout the text, existence is meant in the real plane.

The following notation is also used: \( t_X \) denotes the tangent at \( X \), \( Y = X(A, B) \) denotes the harmonic conjugate of \( X \) with respect to \( (A, B) \), \( X = (a, b) \) denotes the intersection of lines \( a, b \). Points at infinity are occasionally denoted by \( [A] \), the same symbol indicating also the direction determined by that point at infinity. For a line \( e \) the symbol \([e]\) denotes its point at infinity. For a point at infinity \( A \) the symbol \(XA\) denotes the parallel from \( X \) to the direction determined by \( A \).

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The above table serves as the table of contents of this paper, the row labels are the section numbers and the column labels the subsection numbers. The column with label 1 lists the symbols of the twelve projectively inequivalent cases. Each row of the table comprises the cases, which are projectively equivalent to the one of the first column. The notation used is a slight modification of the one introduced by Chasles ([3, p. 304]). The symbol \( P_n \) means that \( n \) of the given points are at infinity and \( T_1 \) means that one of the tangent lines is the line at infinity, later meaning that the conic, to be constructed, is a parabola. The indices, which adhere to the right parenthesis are optional. When absent, it means that the configuration is in general position, i.e. none of the ordinary points is coincident with an ordinary line. When present it means that one or two of the points are correspondingly coincident with one or two tangents. The indices \( i, 2i \) mean that one/two ordinary lines are correspondingly coincident with one/two points at infinity. The index \( 1i \) means that an ordinary point coincides with a line and also a point at infinity coincides with the point at infinity of an ordinary line.

Except for the coincidence suggested by the corresponding symbol, the other data are assumed to be in general positions in the projective sense. For example, the symbol \( (3P_2T)_1 \) stands for the construction of a hyperbola given two points, an asymptote and a tangent. These four elements are assumed in general position, implying that no further coincidences are present, that the intersection of the asymptote and the tangent are not collinear to the two points, that the line of the two points is not parallel to the tangent or the asymptote, etc. The statements on the number of solutions or non existence in each case, presuppose such a restriction. Throughout the text, existence is meant in the real plane.

The following notation is also used: \( t_X \) denotes the tangent at \( X \), \( Y = X(A, B) \) denotes the harmonic conjugate of \( X \) with respect to \( (A, B) \), \( X = (a, b) \) denotes the intersection of lines \( a, b \). Points at infinity are occasionally denoted by \( [A] \), the same symbol indicating also the direction determined by that point at infinity. For a line \( e \) the symbol \([e]\) denotes its point at infinity. For a point at infinity \( A \) the symbol \(XA\) denotes the parallel from \( X \) to the direction determined by \( A \).
Degenerate conics, consisting of a product of two lines $a, b$, are represented by $a \cdot b$. Regarding hyperbolas, asymptotics are distinguished from asymptotes. The first term gives the direction only, the second denotes the precise line.

Regarding the organization of the material, there follow three preliminary sections on the background, which comprise: (a) Involutions ($\S$1.1), (b) Pencils and Families of conics ($\S$1.2), (c) The great theorems ($\S$1.3). Then follow fifty sections handling the inequivalent euclidean constructions. The sections are divided in twelve groups, each group headed by the problem to which all other of the group are projectively equivalent.

1.1. **Involutions.** Involutions are homographies of projective lines with the property $f^2 = I \iff f^{-1} = f$. Using proper coordinates, involutions are described by functions of the form

$$y = \frac{ax + b}{cx - a} \iff x = \frac{ay + b}{cy - a},$$

whose graphs are rectangular hyperbolas symmetric with respect to the diagonal line $y = x$. Such functions are completely determined by prescribing their values on two elements $(X, f(X)), (Y, f(Y))$ and they have either two fixed points or none. Figure 1 shows a case in which there are two fixed points. When the hyperbola has the two branches totally contained in the two sides of the line $y = x$, the corresponding function has no fixed points. An important property of a pair $(X, f(X))$, of related points of an involution, frequently used below, is that it consists of harmonic conjugates with respect to the fixed points of $f$, when these exist ([7, p. 100], [22, I, p.102], [17, p. 167], [9, p. 35], [2, I, p. 137]).

The practical issue of finding the fixed points of involutions is related to the idea of *common harmonics* of two pairs $(A, B), (C, D)$ of collinear points. This
is another pair \((E, F)\) of points, which are, as the name suggests, simultaneously harmonic conjugate with respect to \((A, B)\) and with respect to \((C, D)\) (see Figure 2). If such a pair \((E, F)\) exists, then it is easily seen that every circle \(d\) passing through \(E, F\) is orthogonal to the circles \(c_1, c_2\) with corresponding diameters \(AB\) and \(CD\) (see Figure 3). Thus, in order to find the common harmonics geometrically, set two circles \(c_1, c_2\) on diameters, respectively, \(AB\) and \(CD\) and draw a circle \(d\) simultaneously orthogonal to \(c_1\) and \(c_2\) (see Figure 4). In case one of the points, \(D\) say, is a point at infinity then \(c_2\) is the line orthogonal to \(AB\) at \(C\) and we can take the circle \(d\) to be the one centered at \(C\) and orthogonal to \(c_1\), points \(E, F\) lying then symmetric with respect to \(C\) (see Figure 4).

The common harmonics are precisely the limiting points of the coaxal system (pencil) of circles determined by \(c_1\) and \(c_2\) ([16, p. 118]). They exist, precisely when the circles \(c_1, c_2\) are non-intersecting.

1.2. Pencils of conics. Pencils of conics are lines in the five-dimensional projective space of conics. This is reflected in the generation of a pencil as the set \(D\) of linear combinations \(c = \alpha \cdot c_1 + \beta \cdot c_2\), where \(c_1 = 0, c_2 = 0\) are the equations of any two particular members of the pencil. Then \(c = 0\) represents the equation of the general member of the pencil for arbitrary real values of \(\alpha, \beta\) with \(|\alpha| + |\beta| \neq 0\). The basic pencil, called of type-I, is that of all conics passing through four points (see Figure 5). There are five projectively inequivalent pencils, characterized by the fact that all their members intersect by two at the same points, real or imaginary, with the same multiplicities. These are referred to as types I to V pencils ([22, I, p. 128]) and can be considered as limiting cases of the type I pencil. Type II, for instance, results by fixing line \(e = AD\) and letting \(D\) coincide with \(A\). The resulting pencil
A gallery of conics by five elements

Figure 5. The pencil of conics through $A, B, C, D$

(seen in §9.1) consists of all conics passing through $A, B, C$ and tangent to $e$ at $A$. Another type of pencil is obtained from a type $I$ pencil by fixing lines $a = AB$ and $c = CD$ and letting point $B$ converge along $a$ to $A$ and $D$ converge along $c$ to $C$. The resulting pencil, referred to as type $IV$ pencil (seen in §10.1), consists of all conics tangent to lines $a, c$ correspondingly at their points $A$ and $C$.

Pencils of conics contain degenerate members, at most three ([2, II, p. 124]). In Figure 5 the degenerate members are visible, consisting of the pairs of lines $AB \cdot CD$, $AD \cdot BC$ and $AC \cdot BD$. In the analytic description of pencils $c = \alpha \cdot c_1 + \beta \cdot c_2$, the conics $c_1, c_2$ can be degenerate members, and this is often convenient and extensively used below.

To every type of pencil corresponds an analogous range of conics or tangential pencil of conics ([2, II, p. 199], with a notation slightly different from that of Veblen). For example to type $I$ pencils corresponds the range of type $I^*$ of conics, which are tangent to four lines in general positions (see §7.1). Ranges are pencils of conics in the dual projective plane $P^*$ consisting of all lines of the projective plane $P$. To each pencil of type $X$ corresponds its dual range $X^*$ with properties resulting from those of $X$ by duality.

A particular property of pencils, together with its dual for ranges, is of interest for our subject. For instance, in the case of a pencil $\mathcal{D}$ of conics through points $A, B, C, D$, it is known ([2, II, p. 197]) that the polars of a fixed point $X$ with respect to all members of the pencil pass through a point $Y$. This defines a quadratic transformation $Y = f(X)$, which in the coordinates with respect to the projective base $\{E, F, G, A\}$, with $E = (AB, CD)$, $F = (AD, BC)$, $G = (AC, BD)$, is represented by

$$x' = \frac{1}{x}, \quad y' = \frac{1}{y}, \quad z' = \frac{1}{z}.$$
The image of a line $e$ under this transformation is a conic $k_e$ circumscribing triangle $EFG$ and passing through eight additional points, therefore called an eleven point conic ([1, p. 97], [9, p. 66]). Six of the points are the harmonic conjugates $W = V(X,Y)$ of the intersection point $V = (XY,e)$, where $X, Y$ are taken from $\{A, B, C, D\}$. The two remaining points, if real, are the intersection points of $k_e$ with line $e$ and simultaneously the contact points of two members of the pencil $D$, which are tangent to $e$ (a case handled in §8.1).

The dual to the previous property relates to the range $D^*$ of conics $k$ tangent to four lines $a, b, c, d$ (see Figure 7). According to this, the poles of a line $h$ with respect to the members of $D^*$ lie on a line $h'$ and the transformation $h' = F^*(h)$ is a quadratic one of the same nature as the previous one, differing only in that it operates on the dual projective plane $P^*$. Line $h'$ can be found by a simple criterion, resulting by considering the triangle of diagonals (efg in Figure 7). Lines $h, h'$ intersect each side $s$ of this triangle at points $X, Y$, which are harmonic conjugate with respect to $(U, V)$, where $U, V$ are the vertices of the quadrilateral lying on $s$. The images $h'$ under $F^*$ of all lines $h$ passing through a fixed point $Q$ are the tangents of a conic $k_Q$ inscribed in the triangle $efg$ and tangent to eight additional lines, therefore called an eleven tangents conic. Six of these lines are the harmonic conjugates $Q(s, s')$ of $Q$ with respect to all pairs $(s, s')$ taken from $\{a, b, c, d\}$. The two remaining tangents, if real, are the tangents through $Q$ of the members of $D^*$ passing through $Q$ (a case handled in §6.1).

Roughly described, a standard method of constructing a conic by five elements is to find a pencil or range satisfying four of the given conditions, and then use the fifth condition to locate the particular member(s) of the pencil satisfying it. In the case of type I pencils, any fifth point $E$, different from $A, B, C, D$, determines exactly one conic of the pencil containing it (a case handled in §2.1).

Pencils of conics, passing through two different points $Q, R$ can be transformed to pencils of circles by a complex projective map, which sends $Q, R$ to the circular
points at infinity $I, J$ ([9, p. 71]). By such a map a type $I$ pencil can be transformed to a pencil of intersecting circles and all related conic construction problems reduce to corresponding Apollonius circle construction problems ([6]). This method is concisely expounded in [10].

1.3. The great theorems. The basic tool in the context of the present subject is Desargues’ theorem, for various types of pencils and ranges, as neatly described in [22, p. 128]. The theorem asserts that a pencil intersects on a fixed line $e$, through its members, pairs of points $(X, Y)$ in involution, later meaning, that there is an involution $f$ on $e$, such that $Y = f(X) \iff X = f(Y)$. The interesting fact is that $f$ is completely determined by the intersections of $e$ with the degenerate members of the pencil, which are products of lines. Line $e$ is assumed to be different from the lines contained in degenerate members of the pencil.

In its dual form, Desargues’ theorem, referred to also as Plücker’s theorem ([4, p. 25], [2, II, p. 202]) states, that the pairs of tangents $(x, y)$ from a fixed point $Q$ to the members of a range, are in involution, later meaning, that there is an involution $f^*$ on the pencil $Q^*$ of lines through $Q$, such that $y = f^*(x) \iff x = f^*(y)$. This involution is determined again by the degenerate members of the range, which are pairs of points. In the case of ranges of conics tangent to four lines, for instance, the degenerate members are pairs of intersection points of the four lines and the involution on $Q^*$ is determined by two pairs of lines joining $Q$ to two such pairs of points ([22, I, p. 129], [9, p. 50], [2, II, p. 197]). Point $Q$ is assumed to be different from the points of degenerate members.

The somewhat difficult to visualize involution on $Q^*$ can be represented by intersecting the rays through $Q$ with a fixed line $e \not\ni Q$. In this way the involution $f^*$ on $Q^*$ defines an involution $f$ on $e$ and the fixed points (rays) of $f^*$ trace on $e$ the fixed points of $f$. Quite typically for our subject, the requested conics are

Figure 7. The eleven tangents conic $k_Q$ of $a, b, c, d$ and point $Q$
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intimately related to the fixed elements of such involutions ([20, p. 69], [19, p. 365], [2, p. 198, II]).

Pascal’s theorem, that the opposite sides of a hexagon, inscribed in a conic, intersect on a line, is used in the present context in order to find additional points on the requested conics. The theorem is used also in its various versions for inscribed pentagons, quadrangles and triangles ([22, I, p. 111], [17, p. 156], [2, II, p. 176]).

Brianchon’s theorem, which is dual to Pascal’s, asserts that the lines through opposite vertices of a circumscribed to a conic hexagon pass through a fixed point. Again the theorem and its versions for pentagons, quadrangles and triangles is used in order to find additional points on the requested conics.

2. Five points

2.1. Conic through five points (5P0T). Construct a conic passing through five points \( A, B, C, D, E \). This is the basic construction, to which, all other constructions may be reduced. Analytically this can be done easily by considering the equations of two line-pairs defined by the five points ([18, p. 232]). Let, for example, the line-pairs \((AB, CD), (AC, BD)\) be given correspondingly by equations \((f = 0, g = 0), (h = 0, j = 0)\). Then the equation

\[
k_{\lambda,\mu} = \lambda \cdot (f \cdot g) + \mu \cdot (h \cdot j) = 0,
\]

for variable \( \lambda \) and \( \mu \), represents the pencil \( D \) of all conics passing through \( A, B, C, D \). The requirement, for such a conic, to pass through \( E \), leads to an equation for \( \lambda, \mu \):

\[
k_{\lambda,\mu}(E) = 0,
\]

from which \( \lambda, \mu \) are determined up to a multiplicative constant, and through this a unique conic is defined as required.

Geometrically, one can use Pascal’s theorem to produce, from the given five, arbitrary many other points \( P \) lying on the conic. Figure 8 illustrates the way this

![Figure 8. Pascal’s theorem producing more points on the conic](image)

is done. Start with an arbitrary point \( X \) on line \( DE \) and define the intersection point \( Y = (CD, AX) \). Join this point to the intersection point \( F = (AB, DE) \) and extend the line to find the intersection point \( Z = (YF, BC) \). By Pascal’s
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Theorem, the intersection point \( P = (ZE, AX) \) is on the conic passing through \( A, B, C, D \) and \( E \). The books of Russell [17, p. 229] and, for more cases Yiu [24, p. 144], contain many useful constructions, which determine various elements of the conic, such as the intersections with a line, the center, the axes, the foci etc. out of the five given points.

2.2. **Conic through five points, one at infinity** (5P10T). Construct a conic passing through five points \( A, B, C, D, [E] \) in general position. The conic is a hyperbola and in some cases a parabola. Additional points can be constructed as in the previous section. Start with a point \( X \) on line \( e = DE \), find the intersections \( Y = (AX, CD) \), and \( Z = (FY, BC) \), where \( F = (AB, DE) \) (see Figure 9). Point \( P = (ZE, XA) \) is on the requested conic, which can be constructed to pass through the five points \( A, B, C, D, P \). There is always a unique solution.

**Remark.** In general the conic is a hyperbola, and \([E]\) represents the direction of one of its asymptotes. Fixing points \( A, B, C, D \), there are either none or two directions \([E]\), determined by the four points, for which the conic passing through \( A, B, C, D, [E] \) is a parabola. This is the case (4P1T1) of §3.2.

2.3. **Hyperbola from asymptotics and three points** (5P20T). Construct a conic passing through five points \( A, B, C, [D], [E] \), thus a hyperbola with asymptotic directions given by \([D], [E]\). The pentagon \( ABCDE \) is infinite with \( DE \) the line

![Figure 9. Finding additional points on the conic through A, B, C, D, [E]](image)

![Figure 10. Pascal’s theorem with D, E at infinity](image)
at infinity and its intersection $F$ with $AB$ is also at infinity. An arbitrary ray of the pencil $A^*$ of lines through $A$, and its intersection $Y$ with $CD$ defines the point $Z = (YF, BC)$ and $P = (ZE, AY)$. Last is a point on the requested conic (see Figure 10). There is always one solution. Figure 11 shows a related pencil consisting of all conics passing through $A, B, [D], [E]$, i.e., all hyperbolas with asymptotic directions $[D], [E]$ and passing through the points $A, B$.

3. Four points and one tangent

3.1. Conic by four points and one tangent ($4P1T$). Construct a conic passing through four points $A, B, C, D$ and tangent to one line $e$. By Desargues’ theorem (see §1.3), each member of the pencil $D$ of conics through $A, B, C, D$ (seen in §1.1) intersects line $e$ to a pair of points in involution. The contact points $K, L$ of the requested conic with $e$ are the fixed points of this involution. Two pairs of points, defining the involution, are the intersections of $e$ with two degenerate members $c_1, c_2$ of the pencil $D$, consisting of the pairs of lines $c_1 = AB \cdot CD$ and $c_2 = AD \cdot BC$, intersecting the line respectively in $(E, F)$ and $(I, J)$ (see Figure 12). The fixed points of the involution are the common harmonics $(K, L)$ of the pairs $(E, F), (I, J)$.

An alternative construction for this case relates to the eleven points conic of four points and a line (see §1.2), consisting of the poles of line $e$ with respect to all
conics of the pencil \( D \). This conic intersects line \( e \) precisely at its contact points \( K, L \) with the requested conics.

Next two figures display the domains of existence of solutions for variable \( D \), assuming given the positions of \( A, B, C \) and line \( e \).

3.2. *Parabola through four points* \( (4P1T_1) \). Construct a conic tangent to the line at infinity, i.e. a parabola, passing through four points \( A, B, C, D \). The only difference from the previous case is that line \( e \) is now at infinity. The involution on \( e \) can be represented on the pencil \( O^* \) of lines through the arbitrary but fixed point \( O \) ([4, p. 40], [20, p. 69], [17, p. 180]). In fact, draw from \( O \) parallels to the lines joining all possible pairs of the four points \( A, B, C, D \). They result in three
pairs of lines $(BD, CA), (AB, CD), (DA, BC)$, which are in involution. This, by intersecting the pencil with an arbitrary line $g$ defines an involution in $g$ (see Figure 16). The corresponding pairs of points $(G, H), (I, J), (K, L)$ of $g$ are in involution. The common harmonics $M, N$ of these pairs, if they exist, define two directions $OM$ and $ON$, i.e. two points at infinity, which represent the directions of the axis of the requested parabolas, passing through $A, B, C, D$. Thus, there are either two or none parabola passing through four points $A, B, C, D$ in general position. Figure 17 shows how the construction of one of these two parabolas can be done. Use is made of one of the chords $BD$ of the requested parabola. From the middle $B'$ of $BD$ we draw a parallel $h$ to the direction $OM$. Then we project $C$ on $C''$ and extend $CC''$ to its double $CC'''$. The projection is by parallels to $BD$. By a well known property of parabolas, point $C'''$ will also belong to the parabola under construction. Thus, taking $C'''$ as the fifth point we define the parabola as the conic passing through $A, B, C, D$ and $C'''$. An analogous construction can be carried out for the second parabola, whose axis is parallel to the direction $ON$. The parabolas exist if none of the four given points is contained in the triangle of the other three.

Besides this construction of the two parabolas, which is considered the standard one, there is another approaching the problem from a different point of view. For
this, consider the line $e$ parallel to the diagonal $EF$ of the quadrangle $ACBD$ at half the distance of $G = (AB, CD)$ from $EF$ (see Figure 18). This line is the common tangent to the two requested parabolas. This follows, for example, by considering the conic passing through the eight contact points of the common tangents of two conics ([18, p. 345]) or the properties of the so-called harmonic locus of two conics, specialized for two parabolas ([14, II, p. 121]). The contact points $V, W$ of the two parabolas with $e$ are the common harmonics of the point-pairs $(E_1, E_2), (F_1, F_2)$, where $E_1 = (AD, e), E_2 = (BC, e), F_1 = (AC, e), F_2 = (BD, e)$. Once $V, W$ are constructed, the parabolas can be defined to pass through the corresponding fivetuples of points.

Notice that every conic of the pencil $\mathcal{D}$ has a pair of conjugate diameters parallel to the axes of the two parabolas ([19, p. 292]).

3.3. Conic by 3 points, 1 at infinity, 1 tangent ($4P_11T$). Construct a conic passing through four points $A, B, C, [D]$, and tangent to line $e$. The conic is either a hyperbola with one asymptotic direction $[D]$ or a parabola with axis parallel to $[D]$. Using the method of §3.1 we construct the fixed points $D_1, D_2$ of the involution, defined on line $e$ by its intersections with the line pairs $(AB, CD), (AC, BD), (AD, BC)$. The common harmonics $D_1, D_2$ of the point-pairs $(E, F), (G, H)$ are the contact points with line $e$ (see Figure 19). Adding one of the $D_i$ to the three points $A, B, C$ we can, as in §8.1, construct a fifth point and pass a conic through the five points. Fixing $A, B, C$ and the position of line $e$, there are some directions.

Figure 18. The two parabolas through $A, B, C, D$.

Figure 19. The two conics through $A, B, C, [D]$ tangent to $e$. 
[D] for which no solutions exist. Figure 20 shows a case, in which \( e \) does not intersect the interior of \( ABC \) and the two angular domains for the direction \( AD \) for which there are solutions of the problem.

In general, the conics are two hyperbolas with one asymptotic direction determined by the point at infinity \( D \), or a pair of a hyperbola, as before, and a parabola with axis direction \([D]\). If the line \( e \) does not intersect the interior of the triangle \( ABC \), then, for four particular directions \([D]\), there are corresponding parabolas passing through \( A, B, C \), axis direction \([D]\) and tangent to \( e \). The directions \([D]\), for which this happens, can be determined from the triangle \( ABC \) and the line \( e \). This is the case \((3P2T1)\), handled in \(\S4.2\). The problem is related to the pencil of conics through \( A, B, C, [D] \). This is a specialization of the one in \(\S1.1\), resulting from it by sending \( D \) at infinity (see Figure 21). In this pencil, all members except one are hyperbolas with one asymptotic direction \([D]\). The one exceptional member is the parabola constructed in \(\S8.2\).

3.4. Hyperbola by 2 points, 2 asymptotics, 1 tangent \((4P21T)\). Construct a conic passing through four points \( A, B, [C], [D] \) and tangent to a line \( e \). This is a hyperbola passing through the points \( A, B \), having directions of asymptotes \([C], [D]\) and being tangent to line \( e \). This can be reduced to the case \((5P02T)\) of \(\S2.3\) by locating the contact point \( S \) of \( e \) with the conic. This is done as in \(\S3.1\): Find the intersections \((H, G), (E, F)\) with \( e \) of line-pairs \((BD, AC), (BC, AD)\) respectively (see Figure 22). The contact points of the conics with \( e \) are the common harmonics \( S, S' \) of these two pairs of points.

An alternative solution results by using the eleven points conic of \( A, B, [C], [D] \) and \( e \) as in \(\S3.1\), defined as the locus \( k \) of poles \( P \) of line \( e \) with respect to the
conics $e$ passing through $A, B, [C], [D]$ (their pencil shown in §2.3). This conic intersects $e$ precisely at the points $S, S'$ (see Figure 23).

Fixing $A, B, [C], [D]$ the lines $e$ for which there are solutions are those defining non-separating segments $EF, GH$, where $E = (BC, e), F = (AD, e), G = (AC, e), H = (BD, e)$ (see Figure 24). These are all lines except those, which separate points $A, B$ and their parallels from $O = (AG, BH)$ fall outside the angular domain $AOB$.

4. Three points and two tangents

4.1. Conic by three points and two tangents ($3P2T$). Construct a conic passing through three points $A, B, C$ and tangent to two lines $d, e$. The construction can be reduced to that of a conic passing through five points ($5P1T$) by locating the points of tangency $G, H$ of the two tangents. This can be done by finding the two
intersection points $A', C'$ of line $GH$ respectively with the known lines $BC$ and $AB$. The key fact here is, that in all cases of existence of solutions, there is a cevian triangle $A'B'C'$ with respect to $ABC$, with corresponding tripolar $A''B''C''$, such that the contact points of each one of the requested conics are the intersections of lines $e, d$ with some side of this triangle or its tripolar ([11], [23], [15]). In Figure 25, for example, appears one of the requested conics, tangent to $d, e$, respectively,

![Figure 25. Conic through $A, B, C$, tangent to $d, e$](image)

at the points $G, H$, which are on the side $A'C'$ of a certain cevian triangle of $ABC$. The conic is led to pass through the five points $A, B, C, G$ and $H$. Analogously are constructed three other conics. The determination of the cevian triangle $A'B'C'$ is done again through the construct of common harmonics. For example, points $B', B''$ are the common harmonics of the point pairs $(AC, B_1B_2)$. These represent the fixed points of an involution, defined, by Desargues’ theorem ([17, p. 204]), on line $AC$, by the intersections with members of the pencil $\mathcal{D}$ of all conics, which are tangent to $d, e$ respectively at $G, H$ (seen in §10.1). Regarding the existence, there are four solutions in the case none of the lines $d, e$ passes through the interior of the triangle $ABC$ (see Figure 26), or both of them intersect the interior of the same couple of sides of this triangle. In all other cases there are no solutions.

An intuitive way to answer, \textit{why four}, offers Figure 27, displaying a cone and a plane $\varepsilon$ on which the cone is projected. Plane $\varepsilon$ is the one containing the lines $e, d
intersecting at \(O\). Plane \(\varepsilon'\) is parallel to \(\varepsilon\) from an arbitrary point \(O'\) projecting orthogonally to \(O\). The circular cone is constructed so that the parallels \(d', e'\) from \(O'\) are generators and its axis is contained in \(\varepsilon'\). The lines orthogonal to \(\varepsilon\) at the three points \(A, B, C\) intersect the cone respectively at pairs of points \((A', A''), (B', B''), (C', C'')\). The plane through \((A', B', C')\) intersects the cone along a conic, which projects to one of the conics solving the construction problem. The same is true with the triples of points on the conic \((A'', B', C'), (A', B'', C'), (A', B', C'')\). They define respectively, a plane, a conic on the cone, and its projection on \(\varepsilon\), representing a solution of the construction problem. The other possible triples of points (e.g. such as the triple \((A'', B'', C'')\)), because of the reflective symmetry of the cone with respect to the plane \(\varepsilon'\), deliver conics on the cone, which are reflections of the previous four (e.g. \((A'', B'', C'')\) is symmetric to \((A', B', C'')\)), hence by the projection falling onto the same four solutions of the construction problem.

**Remark.** To handle this, most interesting case and rich in structure of our constructions, one could consider the set of conics passing through three points \(A, B, C\) and tangent to one line \(d\), as seen in Figure 28, and attempt to find the members
of this set satisfying the fifth condition of tangency with line $e$. Unfortunately this set of conics is not a pencil, and Desargues’ theorem does not apply to it to produce the solutions as usual. The same is true for the set of conics passing through two points $A, B$ and tangent to two lines $d, e$ seen in Figure 29. This set of conics admits also a spacial interpretation, as the set of projections of intersections of a cone with all planes passing through points $X, Y$ on the cone, where $X \in \{A', A''\}, Y \in \{B', B''\}$, the points of the two sets projecting respectively on $A$ and $B$.

4.2. Parabola by three points and a tangent (3P2T1). Construct a conic passing through three points $A, B, C$ and tangent to line $d$ and the line at infinity, thus a parabola. For this, projectively equivalent to the previous, case, the process of determination of the cevian triangle and the tripolar, used there, is even simpler, since line $e$ is now at infinity. The segments $A'A'', B'B'', C'C''$ of common harmonics, determined on each side of $ABC$, are now bisected by the tangent $d$ and the chords of contact points with lines $d, e$ are parallel to the axes of the parabolas (see Figure 30). Figure 31 displays all four parabolas passing through $A, B, C$ and tangent to line $d$. There are four solutions if line $d$ does not intersect the interior of triangle $ABC$ and no solution if it does.
4.3. *Conic by 2 points, 1 infinity, 2 tangents* \((3P_12T)\). Construct a conic passing through points \(A, B, [C]\) and tangent to lines \(d, e\). Triangle \(ABC\) is infinite with two sides parallel to the direction \([C]\). Points \((C', C'')\) are the common harmonics of \((A, B)\) and of the pair of intersections of \(AB\) with lines \(d, e\). Analogously are defined the pairs of points \((B', B'')\) on \(AC\) and \((A', A'')\) on \(BC\). \(A'B'C'\) is a cevian triangle of \(ABC\) and points \(A'', B'', C''\) are on the corresponding tripolar.

Each one of the requested conics passes through \(A, B, C\) and is tangent to \(d, e\) at their intersection points with one side of the cevian triangle or the tripolar. In Figure 32 only two, out of the four, conics are shown. Additional points on the conics can be found by taking harmonic conjugates with respect to the polar of \(P = (d, e)\). The construction of conics can be also completed by using the remark in \((3P2T)_2\) of §10.1. In general the conics are hyperbolas with one asymptotic direction parallel to \([C]\). Fixing \(A, B\) and \(d, e\), there are four directions \([C]\) for which the corresponding conic is a parabola. These are determined in \((2P3T_1)\) of §5.2. There are four solutions if the lines \(d, e\) either do not intersect the interior of \(ABC\) or they intersect the interior of the same pair of sides of this triangle.
4.4. **Hyperbola 1 point 2 asymptotics 2 tangents** \((3P_22T)\). Construct a conic, passing through three points \(A, [B], [C]\) and tangent to two lines \(d, e\). This is a hyperbola with asymptotic directions \([B], [C]\). There are again four solutions determined by the sides and corresponding tripolar of a cevian triangle \(A'B'C'\) of \(ABC\). The triangle \(ABC\) has one side-line at infinity. The corresponding cevian triangle is determined as in §4.1, but now one of its vertices is at infinity. The four chords joining contact points of the same conic with lines \(d, e\) shown in Figure 33 are denoted by \((1), (2), (3), (4)\). The conic touching \(d, e\) at the endpoints of chord \((1)\) is drawn. There are four solutions if the lines \(d, e\) do not intersect the interior of \(ABC\) or both intersect the interior of the same couple of sides of the triangle. In all other cases there are no solutions.

5. **Two points and three tangents**

5.1. **Conic by two points and three tangents** \((2P3T)\). Construct a conic passing through two points \(D, E\) and tangent to three lines \(a, b, c\). The structure of the solution rests upon the dual theorem of Desargues ([17, p. 215], [9, p. 51], [19, p. 229]) and can be described as follows ([4, p. 58], [11], [23]). The three lines in general positions define a triangle \(ABC\) \((A\) opposite to \(a\) etc.) and the two given points \(D, E\) determine a third point \(F\), with the following properties. \(A, B, C, F\) define a projective base ([2, p. 95, I]) and in the coordinates with respect to this base the quadratic transformation ([21, p. 127])

\[
f : (x, y, z) \mapsto \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right), \quad \text{maps } f(D) = E.
\]

There are four conics with the prescribed properties (see Figure 34). Each one of them is tangent to the three sides of the triangle \(ABC\) and to two additional lines. The pairs of additional lines corresponding to the four conics are \((FD, FE), (F_1D, F_1E), (F_2D, F_2E), (F_3D, F_3E)\), where \(F_1, F_2, F_3\) the harmonic associates ([24, p. 100]) of \(F\). Figure 35 shows the domains for which there are solutions for the \((2P3T)\) problem. The two points \(D, E\) must lie, both, in the domain with the same label. Otherwise there are no solutions.
This case is the dual of the previous one and is reducible to that by taking poles and polars with respect to a fixed conic. For example, taking poles and polars with respect to the conic \( k \) with perspector \( F \) ([24, p. 115]), for each conic \( c \) tangent to the sides of \( ABC \) and passing through \( D, E \), we obtain a conic \( c' \) passing through the vertices of the cevian \( A'B'C' \) of \( F \) and tangent to \( d, e \), which are the polars of \( D, E \) with respect to \( k \) (see Figure 36). By this polarity ([22, p. 263, I]) the tangents \( t_D, t_E \) to \( c \) at \( D, E \) map to the contact points \( D_1, E_1 \) of \( c' \) with lines \( d, e \) and the intersection point \( F = (t_D, t_E) \) maps to line \( f = D_1E_1 \).

Using a polarity, as before, we could reduce the cases to the half; but it is not the purpose of the present review to produce a least number of pictures.
Remark. In this case, as noticed also in the previous one, some difficulty in handling the construction lies on the fact that the set of conics tangent to \(a, b, c\) and passing through \(D\) is not a proper pencil of conics (see Figure 37). This set of conics admits a spacial interpretation, as in §4.1, representing the conics as projections of intersections of a cone with planes. The cone is constructed as in that section, and the planes intersecting the cone are defined by pairs \((X, \alpha)\) of points and lines. Point \(X \in \{D', D''\}\), the two points of the cone being those which project on \(D\). Line \(\alpha\) is a tangent to the conic defined on the cone by the plane orthogonal to the plane \(\varepsilon\) of \(b, c\) and intersecting it along line \(a\).

5.2. Parabola by 2 points, 2 tangents (2P3T). Construct a conic passing through two points \(A, B\) and tangent to two lines \(c, d\) and the line at infinity, thus, a parabola. Figure 38 shows the process of determination of the point \(F\) and its harmonic associates \(F_1, F_2, F_3\) stepping on the previous section. Points \(B_1, B_2\) are the common harmonics of the point-pairs \((A', B')\) and \((G, [d])\), where \(A', B'\) are the parallel to \(c\) projections of \(A, B\) on \(d\) and \(G = (c, d)\). Analogously are defined on \(c\) the common harmonics \(C_1, C_2\) of two similar point-pairs on \(c\). Point \(F\) is the
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Figure 38. Searching for parabolas through $A, B$ and tangent to $c, d$

intersection of the parallels from $B_1, C_1$ respectively to $c, d$. Figure 39 shows the

Figure 39. The four parabolas through $A, B$ and tangent to $c, d$

four parabolas solving the problem. Each of them is tangent to the two given lines $c, d$ and the pair of lines $F_i A, F_i B$, where $F_i$ are either $F$ or one of its harmonic conjugates (with respect to the triangle with infinite sides $G[c][d]$). There are four solutions if points $A, B$ are in the same or opposite angular domains of lines $c, d$. Otherwise there are no solutions.

Remark. When line $FF_3$ passes through the middle $M$ of $AB$, the two corresponding parabolas, tangent respectively to the line-pairs $(FA, FB), (F_3A, F_3B)$, are homothetic with respect to $G$ and solve problem $(3P_12T_1)$ of §9.3, line $FF_3$ being then parallel to the axis of the two parabolas and also being harmonic conjugate of $AB$ with respect to lines $(c, d)$.

5.3. Conic by 1 point, 1 infinity, 3 tangents $(2P_13T)$. Construct a conic passing through two points $A, [B]$ and tangent to three lines $c, d, e$. The construction can be done by adapting the one of §5.1. By that method, we first find the cevians of $A, B$ with respect to the triangle $A'B'C'$, whose sides are $c, d, e$. Then we find the common harmonics $A_0, A_1$ on line $c$ of the point-pair consisting of the traces of the cevians from $A, B$ and $(B', C')$. Analogously are defined points $B_0, B_1$ on $d$ and $C_0, C_1$ on $e$ (see Figure 40). The six points thus determined define a cevian triangle with perspector $F$ and the corresponding tripolar. Then we define the three harmonic associates $F_1, F_2, F_3$ of $F$. Each one of the requested conics is tangent to the three lines $c, d, e$ and also tangent to the two lines $F_i A, F_i B$, joining some of the points $F_i$ with $A$ and $B$. There are four solutions if $A$ is on the exterior of
the triangle $A'B'C''$ and in the angular domain containing the parallel to $[B]$ from a vertex. Otherwise there are no solutions. The conics are in general hyperbolas. Fixing the lines $c, d, e$ and point $A$, there are two directions $[B]$ for which the conics are parabolas. This is handled in (1P4T1) of §6.2.

5.4. **Hyperbola by 2 asymptotics and 3 tangents (2P23T).** Construct a conic passing through two points $[A], [B]$ and tangent to three lines $a, b, c$. This is a hyperbola with asymptotic directions $[A], [B]$ and tangent to three lines. Proceeding as in §5.1, we draw parallels to those directions from each vertex of the triangle $A_0B_0C_0$ with side-lines $a, b, c$. These parallels define on each side two points $A_1, B_1$ on $a, A_2, B_2$ on $b$ etc. The common harmonics ($A', A''$) of point pairs ($A_1, B_1$) and ($C_0, B_0$) and the corresponding common harmonics for the other sides, define the cevian triangle $A'B'C''$, its perspector $D$ and the corresponding harmonic associates $D_1, D_2, D_3$ (see Figure 41). Each of the requested hyperbolas is constructed as a conic tangent to the three lines $a, b, c$ plus the two lines joining $D_i$ to the points at infinity $A, B$, i.e. the parallels from $D_i$ to $[A]$ and $[B]$ (see Figure 42). Given the directions of lines $a, b, c$, there are four solutions if drawing parallels to these directions and to $[A], [B]$, later are not separated by the first. Otherwise there are no solutions.
6. One point and four tangents

6.1. Conic by 1 point and 4 tangents ($1P4T$). Construct a conic tangent to four given lines $a$, $b$, $c$, $d$ and passing through a given point $A$. One way to the construction is to reduce the problem to its dual ($4P1T$) of §3.1. In fact, if $IJK$ is the diagonal triangle of the complete quadrilateral whose sides are $a$, $b$, $c$, $d$ (see Figure 43), then the harmonic associates $A_1$, $A_2$, $A_3$ of $A$ with respect to $IJK$ are also points of the conic. Thus, one can apply the recipe of §3.1 by taking these four points and one of the four given lines.

Another way to define the conics is by using Desargues’ theorem in its dual form ([4, p. 57]) in order to locate a fifth tangent to the conic, namely the one passing through $A$. In fact, according to that theorem, the tangents from an arbitrary point $A$ to the conics of the one-parameter pencil of conics $\mathcal{D}$, which are tangent to four lines $a$, $b$, $c$, $d$, define an involution on the pencil $A^*$ of all lines passing
through that point. The tangents to the members of that pencil, which pass through $A$ are the fixed elements of this involution. By considering the intersection of each line through $A$ with line $a$, we represent this involution by a corresponding involution of points of $a$. The fixed elements of the involution in $A^*$ correspond to the fixed points of the corresponding involution on $a$. It is easy to see that two particular pairs of points in involution on $a$ are the pairs $(B, B')$ and $(E, E')$, where $B = (b, a), E = (d, a), B' = (AD, a), E' = (AC, a)$. The common harmonics $F_1, F_2$ of these two pairs define the two requested tangents, which in turn define the two conics (see Figure 44).

There are two solutions if point $A$ is in one of the five domains (1) – (5) shown in Figure 45. Otherwise there are no solutions. The reason for this is, as is visible in the figure, that the pencil $D$ of conics tangent to four given lines $a, b, c, d$ does not cover all connected domains defined by the four lines ([2, p. 200, II]). Noticeable in the figure is also the fact that for every point in these five domains there are two conics of the pencil passing through the point.
A third method to construct the requested conics is to use the dual of the **eleven points conic** of §3.1, which is the **eleven tangents conic**, defined by four lines $a, b, c, d$ and a point $A$ ([1, p. 97]). This conic is the envelope $k$ of the polars of $A$ with respect to all conics tangent to the four given lines. This conic is tangent to the 3 sides of the diagonal triangle $RST$ of the quadrilateral of the four lines. It is also tangent to the 6 polars of $A$ with respect to all line-pairs of the quadrilateral, and is also tangent to the two tangents $t_1, t_2$ to the requested conics at $A$ (see Figure 46).

Conic $k$ can be constructed by the methods of §7.1 and then $t_1, t_2$ can be found by drawing the tangents to $k$ from $A$. The two requested conics can be defined by applying again the methods of the next section and determining the conic tangent to five lines $a, b, c, d, t_i$, $(i = 1, 2)$.

### 6.2. Parabola by 1 point, 3 tangents ($1P4T_1$)

Construct a conic tangent to the line at infinity, i.e. a parabola, and also tangent to three lines $a, b, c$ and passing through a point $E$. Any of the methods of the previous section can be modified to produce the requested parabolas. For example, applying the second method, we draw first parallels to $b, c$ through $E$ intersecting $a$ at $E', E''$ (see Figure 47). The pairs of lines $(EE', EB)$ and $(EE'', EC)$ through $E$ are related with respect to the involution defined by Desargues’ theorem. They are tangents from $E$ to degenerate members of the pencil of parabolas tangent to four lines three of which
are \(a, b, c\). The common harmonics \(F_1, F_2\) of these pairs define the tangents at \(E\) of the requested parabolas passing through \(E\). There are two solutions if point \(E\) lies in one angular domain containing the triangle \(ABC\) but outside of the triangle. Otherwise there are no solutions.

![Figure 48. The pencil of parabolas tangent to \(a, b, c\)](image)

An alternative solution is obtained by using the *nine tangents conic* of the three lines \(a, b, c\), the line at infinity \(e\) and the point \(E\). This is the conic \(k\), defined as envelope of all polars of \(E\) with respect to the members of the pencil of parabolas tangent to the three lines \(a, b, c\) (see Figure 48). Conic \(k\) is tangent to \(a', a''b', b'', c', c''\), where \(a', a''\) are parallel to \(a\), respectively, from point \(A = (b, c)\) and the symmetric \(E_a\) of \(E\) with respect to \(a\), and the other lines are defined analogously. The tangents to the requested parabolas at \(E\) are the two tangents from \(E\) to \(k\).

6.3. *Conic, 1 infinity, 4 tangents \((1P_4T)\).* Construct a conic tangent to four lines \(a, b, c, d\) and passing through a point at infinity \([E]\). Again a solution results by adapting any of the methods of section §6.1. For example, to adapt the second method to the present configuration, define the points \(B', C'\) on \(b\), to be the projections of \(D, A\) parallel to \([E]\) (see Figure 49). Points \(F_1, F_2\) are the common harmonics of pairs \((B, B'), (C', C)\), and the parallels \(EF_1, EF_2\) define the tangents at \(E\) of the requested conics, which are constructed from five tangents. Fixing lines \(a, b, c, d\), there are two solutions when the parallel to \([E]\) from \(B\)
falls inside the angle $CAB$ or the parallel from $D$ falls inside the complement of $ADC$. Otherwise there are no solutions. The above construction assumes that the lines through $F_1, F_2$ are ordinary and as a consequence the conics are hyperbolas. If the conic is tangent to the line at infinity, thus a parabola, then $[E]$ is uniquely determined from the lines $a, b, c, d$. This is handled in $(0P5T_1)$ of §7.2.

7. Five tangents

7.1. Conic by five tangents $(0P5T)$. Construct a conic tangent to five lines $a, b, c, d, e$. A first solution is to reduce the construction to its dual of a conic through five points, as in §2.1. For this, use Brianchon’s theorem to find the contact points with the sides ([19, p. 225]). In fact, the diagonals $BD, CE$ of the pentagon of the given lines intersect at a point $O$ (see Figure 50), which lies also on the line joining the remaining vertex of the pentagon $A$ to the contact point $F$ of the opposite side. Thus $F$ is constructible from the data. Analogously are found the other contact points with the sides of the pentagon. Using again the theorem of Brianchon in its general form for hexagons circumscribed to a conic, one can construct arbitrary many other tangents to the conic. In fact, take a point $F$ on side $AB$ and define the intersection point $O$ of $FD$ and $AC$ (see Figure 51). By Brianchon’s theorem the diagonal $GE$ will pass also through $O$. Hence the position of $G$ can be found by intersecting $BC$ with $OE$. Thus, moving $F$ on line $AB$ and determining $G$ on $BC$ by the above procedure, we can find arbitrary many tangents $FG$ to the conic. There is always a unique solution.

An image of the pencil of conics tangent to four lines, related to this problem, is contained in §6.1.
7.2. *Parabola by four tangents* \((0P5T_1)\). Construct a conic tangent to the line at infinity, i.e. a parabola, tangent to four given lines \(a, b, c, d\). The case is projectively equivalent to the previous one and the method used there can be adapted to solve the problem. For this, apply Brianchon’s theorem to the pentagon \(ABCDE\), which now has points \(C, D\) at infinity (see Figure 52). The contact points of the sides opposite to the vertices are denoted correspondingly \(A', B', C', D', E'\). Point \(A'\) is at infinity, and determines the axis of the parabola. By Brianchon’s theorem, line \(AA'\) passes through the intersection \((EC, BD)\), which is constructible from the data. Analogously are constructible the intersections \((AD, EC), (BE, AD), (CA, BE), (DB, CA)\). Join these points correspondingly with the vertices \(B, C, D, E\) to find \(B', C', D', E'\) through their intersections with the opposite sides of the pentagon. To the four points on the parabola a fifth one \(G\) can be defined by taking the middle \(F\) of \(D'E'\) and the middle \(G\) of \(FB\). Thus the parabola can be constructed as a conic passing through the five points \(B', C', D', E', G\) ([2, II, p. 212]).

Another way to solve the problem, is through the properties of the created parabola related to the Miquel circles of the quadrilateral of the four given lines ([12, p. 83]). These are the circumcircles of the four triangles formed by the four given lines. A theorem of Miquel asserts that all four circles pass through the same point \(F\) (see Figure 53). A theorem of Steiner ([17, p. 161]) completes then the construction, by showing that this point \(F\) is the focus of the parabola, while the directrix carries all four orthocenters of the aforementioned triangles ([19, p. 70]). Thus, in order to construct the parabola, it suffices to take the circumcircles and the orthocenters of two such triangles and define \(F\) and their orthocenters \(H_1, H_2\) ([12, p.45], [14, p. 100, II]). The parabola then is constructed from its focus and the directrix \(H_1H_2\). There is always a unique solution.

8. *Four points one tangent one coincidence*

8.1. *Conic by three points and one tangent-at* \((4P1T)_{1}\). Construct a conic passing through four points \(A, B, C, D\) and tangent to a line \(e\) at \(D\). In this case it is easy to find a fifth point on the conic and reduce the construction to that of \((5P0T)\) in
§2.1. In fact, take $I = (BC, e), I' = I(B, C)$. Then line $I'D$ is the polar of $I$ (see Figure 54). If $J = (I'D, AI)$, then $A' = A(I, J)$ is a point on the conic. There is always a unique solution.

Noticeable in the figure is point $K = (DA', AI')$. It is a fixed point on line $AI'$, since the cross ratio $(A, K, I', I'') = (A, A', J, I) = -1$. Hence points $D, A',$ and through them, the various conics passing through $A, B, C$ and tangent to $e$, are defined by turning a line about $K$ and considering its intersections with the fixed lines $e$ and $IA$. Figure 55 shows the structure resulting by making the same construction with respect to the other sides of $ABC$. In this point $E$ is the tripolar of line $e$ and $K_A, K_B, K_C$ are its harmonic associates with respect to $ABC$. Each point $D \in e$ defines three other points of the conic passing through $A, B, C$ and tangent to $e$ at $D$. These points are $A' = (DK_A, K_BK_C), B' = (DK_B, K_CK_A), C' = (DK_C, K_AK_B)$. It is easily seen that points $K_A, K_B, K_C$ are the harmonic associates of $D$ with respect to $A'B'C'$.

8.2. Parabola by three points and axis-direction $(4P_11T_1)$. Construct a conic tangent to the line at infinity, hence a parabola, passing through four points $A, B, C, [D]$. The last point, at infinity, defines the direction of the axis of the parabola. The construction is carried out by locating two more points $B'', C''$ on the parabola and
Figure 55. Conic through \(A, B, C\) tangent to \(e\) at \(D\). Three additional points \(A', B', C'\)

Figure 56. Parabola through \(A, B, C\) and given axis-direction \([D]\) passing a conic through \(A, B, C, B'', C''\) (see Figure 56). The construction of the additional points is the same with that of the previous section.

Figure 57. The pencil of conics through \(A, B, C, [D]\)

The pencil of conics involved is a specialization of the one in \(\S 1.1\), resulting from it by sending \(D\) at infinity (see Figure 57). The construction shows that in this pencil, all members except one are hyperbolas with one asymptotic direction \([D]\). The one exceptional member is the requested parabola.

8.3. Conic by 2 points, 1 at infinity, 1 tangent-at \((4P_t1T)_1\). Construct a conic passing through four points \(A, B, C, [D]\) and tangent to a line \(e\) at \(C\). Additional points on the conic can be found as in \(\S 8.1\). In Figure 58 points \(L, K\) are two such
additional points. They are constructed as conjugates of $B$ and $D$ after constructing the polars $CF$ of $G = (e, AB)$ and $CH$ of $I = (e, AL)$. There is always one solution, which, in general, is a hyperbola. The pencil involved is the one of conics through $A, B, C, [D]$, seen in §8.2. Line $e$ can be considered to turn about point $C$, defining in each of the obtained locations a corresponding member of that pencil. There is a single line through $C$, for which the corresponding conic is a parabola with axis $[D]$. In all other cases the conic is a hyperbola with an asymptotic direction $[D]$.

Fixing $A, B, C, e$, and varying $[D]$ we obtain, in general, hyperbolas. There are, though, two special directions $[D]$, determined in terms of $A, B, C, e$, for which the resulting conic is a parabola. This is handled in $(3P2T1)_1$ of §9.2.

8.4. Hyperbola by 3 points, 1 asymptote $(4P11T)_i$. Construct a conic passing through four points $A, B, C, [D]$ and tangent to a line $e$ at $[D] \in e$. This is equivalent with the construction of a hyperbola passing through the points $A, B, C$ and having the asymptote line $e$. Adapting the method of §8.1, we can find two additional points $B', C'$ and construct the requested conic as a $(5PT0)$ conic. For example, to define $B'$, take successively $C_1 = (AC, e), C_2 = C_1(A, C)$. The parallel to $e$ from $C_2$ is the polar of $C_1$. Take then the intersection $B_2$ of that line with $BC_1$ and $B' = B(C_1, B_2)$, which is a point on the conic. Analogously is defined point $C'$. There is always one solution. The pencil of conics involved is the one shown in $(4P11T1)_i$ of 8.2. A line parallel to $[D]$ determines a unique member of the pencil having this line as an asymptote.
8.5. Hyperbola by 1 point 2 asymptotics 1 tangent at $A_1P_2T_1$. Construct a conic passing through four points $A, B_1, [C], [D]$ and tangent to a line $e$ at $B \in e$. The conic is a hyperbola with given asymptotic directions $[C], [D]$, passing through a point $A$ and tangent at a point $B$ to a given line $e$. In this case, as we did in §8.1,

![Figure 60. Hyperbola through $A, B, [C], [D]$ tangent to $e \ni B$](image)

we can find additional points on the conic. For this let $A_0$ be the intersection point with $e$ of the parallel to $[D]$ through $A$. The symmetric $A_1$ of $A_0$ with respect to $A$ defines line $BA_1$ which is the polar of $A_0$. The parallel to $[C]$ from $A_0$ intersects $A_1B$ at $A_2$ and the middle $A_3$ of $A_0A_2$ is on the conic. Repeating the construction with $A_3$ in place of $A$ and continuing this way, we can construct arbitrary many points on the conic. There is always one solution.

8.6. Hyperbola by 2 points 1 asymptote 1 asymptotic ($A_1P_2T_i$). Construct a conic passing through four points $A, B, [C], [D]$ and tangent to a line $e$ at $[D]$. The requested conic is a hyperbola passing through two points $A, B$ having one asymptotic direction $[C]$ and an asymptote $e \ni [D]$. By a well known property of the hyperbola ([4, p. 42]), the segments $AA', BB'$ intercepted by the asymptotes on the secant $AB$ are equal, hence, knowing $AA'$, we locate $B'$ on $AB$ and given the direction of the other asymptote $[C]$ we determine it completely and find its

![Figure 61. Hyperbola through $A, B, [C], [D]$ and asymptote $e \ni [D]$](image)

intersection point $O$ with the given asymptote $e$, which is the center of the hyperbola (see Figure 61). Two additional points $A'', B''$ are immediately constructed,
by taking the symmetrics of $A, B$ with respect to $O$. Arbitrarily many points on the conic can be then constructed by the method of the previous section. There is always a unique solution.

9. Three points two tangents one coincidence

9.1. Conic by 2 points, 1 tangent, 1 tangent-at $(3P2T)_1$. Construct a conic passing through three points $A, B, C$ and tangent to two lines $d \ni A, e$. Using again the power of Desargues’ theorem, we find first the contact points of the requested conics with line $e$. These are the common harmonics $A', A''$ of the point-pairs $(E, F), (G, H)$, where $F = (d, e), E = (e, BC), G = (e, AB), H = (e, AC)$ (see Figure 62). Note that these pairs are defined as intersections of $e$ with two degenerate members of the pencil $D$ of conics tangent to $d$ at $A$ and passing through $B, C$. The first pair is the intersection with the degenerate conic of two lines $d \cdot BC$ and the second with the degenerate conic of the two lines $AB \cdot AC$. After locating the contact point, a fifth point on the conic can be obtained by using the polar of $F$, which is $AA'$ or $AA''$ and taking the conjugate of $B$ or $C$. There are no solutions if only one of the lines $d, e$ separates points $B, C$. Otherwise there are two solutions. Figure 63 displays a pencil $D$ of conics tangent to line $d$ at a fixed point $A$ and passing through two points $B, C$. It is visible there that from every point of the plane passes a unique member of the pencil and that for every line of the plane not separating $B, C$ there are two members tangent to that line ([2, II, p. 193]).
Figure 64. The two conics through $A, B, C$, tangent to $e$ and also to $d$ at $A$

For an alternative method, as in §3.1, we consider the poles $P$ of line $e$ with respect to all members $e$ of $D$. Their locus is the eleven points conic $k$ intersecting the line $e$ at the contact points $A', A''$ of the requested conics (see Figure 64).

9.2. Parabola by 2 points, 1 tangent-at $(3P2T_1)_1$. Construct a conic passing through three points $A, B, C$, tangent to line $d$ at $A$ and tangent to the line at infinity $e$, thus a parabola. The involution on $e$, induced by its intersections with the members of the pencil $D$ of conics tangent to $d$ at $A$ and passing through $B, C$, induces an involution on the pencil $A^*$ of lines through $A$ and, through the intersections of these lines with $d$, induces also an involution on $d$. Two, related by this involution point-pairs on $d$ are $(B, C)$ and $(F, [BC])$, where $F = (BC, d)$ and $[BC]$ the point at infinity of $BC$. The common harmonics $A', A''$ of these two pairs define, by joining them with $A$, the directions of the axes of the parabolas (they pass from the contact point at infinity) (see Figure 65). Two additional points on each parabola can be defined by projecting $B, C$ parallel to $d$ on the parallel to the corresponding axis through $A$ and doubling the resulting segments. There are two solutions if the line $d$ does not intersect the interior of segment $BC$ and no solution if it does.

A different way to think about this problem is the following (see Figure 66). All conics $c$ tangent to line $d$ at $A$ and passing through $B, C$ have their centers on a conic $k$, passing through $A$ and $D = (d, BC)$ and also through the middles $P, Q, R$ of segments $AB, AC, BC$ ([19, p. 299]). This is the eleven points conic of $D$ with respect to the line at infinity. If this conic is a hyperbola, then its points at infinity are the centers of the two requested parabolas. To find the parabolas in this case draw from the middle $Q$ of $AC$ parallels $C'Q, C''Q$ to the asymptotes.
intersecting $d$ ad $C', C''$ respectively. One of the requested parabolas is tangent to lines $C'A, C'C$ at $A, C$ correspondingly and passes through $B$. This is the case $(3P2T)_{2}$ of §10.1. Analogously can be constructed the other parabola, starting with $C''$ instead of $C'$. Figure 67 shows the pencil of all conics passing through $B, C$ and tangent to $d$ at $A$, but having $B, C$ on both sides of $d$. All conics are hyperbolas and the conic $k$ of their centers is an ellipse. This is the reason of non-existence of solutions in this case.

9.3. Parabola by 2 points, 1 tangent, axis-direction $(3P1T_1)$. Construct a conic passing through two points $A, B, [C]$, tangent to line $d$ and also tangent to the line at infinity $e$, hence a parabola. The point at infinity $[C]$ determines the direction of
the parabola’s axis. By Desargues’ theorem, the pencil $D$ of all parabolas passing through $A, B, [C]$, i.e. passing through $A, B$ and having axis direction $[C]$, define through their intersection points $X, X'$ with line $d$ an involution. The fixed points $D, D'$ of this involution are contact points of the requested parabolas. There are two obvious, degenerate, parabolas passing through $A, B, [C]$ defining two pairs of points in involution. One pair consists of the intersections $(X, X')$ with $d$ of the two parallels to $[C]$ from $A$ and $B$ (see Figure 68). The other pair consists of $(E, [d])$, defined by line $AB$ and the line at infinity, where $E = (AB, d)$. The fixed points $D, D'$ of the involution are the common harmonics of these point-pairs. Once the contact points $D, D'$ are known, the requested parabolas are easily constructible by the method of §10.2. If $A, B$ are on the same side of $a$ then there are two solutions, otherwise there is no solution.

Another computational solution of the problem results by using the homothety relating the two parabolas. In fact, the intersection point $O$ of $d$ with the parallel $e$ to $[C]$ from the middle of $AB$ is the center of a homothety, mapping one of the parabolas to the other. The other common tangent $d'$ to the two parabolas from $O$, can be constructed from the given data, since it is the harmonic conjugate of $d$ with respect to the line pair $(c, e)$, where $e$ is the parallel to $AB$ from $O$. The computations are straightforward and I omit them. See the remark in (2P3T1) of §5.2, which relates that problem to the present one.
9.4. **Hyperbola by 2 points, 1 tangent, 1 asymptote** \((3P_22T)_1\). Construct a conic passing through points \(A, B, [C]\) and tangent to line \(d\) at \([C]\) and to line \(e\). Thus, \(d\) is an asymptote and the conic is a hyperbola. The hyperbolas touch line \(e\) at \(D, D'\), which are the common harmonics of pairs of points \((E, F), (G, H)\), with \(E = (e, AB), F = (d, e), G = (AC, e), H = (BC, e)\) (see Figure 70). Additional points can be found by considering conjugate points with respect to the polar of \(F\). There are two hyperbolas, if \(A, B\) are in one and the same angular domain out of the four defined by lines \(d, e\), or they lie on opposite angular domains. Otherwise there are no solutions. Figure 71 shows the pencil \(D\) of hyperbolas through \(A, B\) with one asymptote line \(d\). It shows also a line \(e\), for which there are no tangent members of the pencil. Conic \(k\) is the locus of poles of the line \(d\) with respect to the members of the pencil (the eleven points conic of \(D\) with respect to \(e\)). It is a hyperbola with one asymptote parallel to \(d\), passing through \(F = (d, AB)\).

9.5. **Hyperbola by 2 asymptotics, 1 tangent-at** \((3P_22T)_1\). Construct a conic passing through three points \(A, [B], [C]\), tangent to \(d\) at \(A\) and tangent also to line \(e\). The conic is a hyperbola with asymptotic directions \([B], [C]\). In analogy to the method of §9.1, project first \(A\) on \(e\) parallel to \([B], [C]\) to find respectively points \(G, H\). The contact points \(A', A''\) of the conics with line \(e\) are the common harmonics \(A', A''\) of the point-pairs \((G, H)\) and \((F, [e])\), where \(F = (d, e)\). Once the two contact points of lines \(d, e\) are known, the methods of \((4P_21T)_1\) in §8.5 can be used to complete the construction. There are two solutions if the parallels to \([B], [C]\) from \(F\) fall in the same angular domain of lines \((d, e)\). Otherwise there
are no solutions. Figure 73 shows the pencil \( \mathcal{D} \) of conics tangent to \( d \) at \( A \) and asymptotic directions \([B],[C]\). Shown is also the hyperbola \( k \), defined as the locus of poles of a fixed line \( e \) with respect to the members of the pencil (the eleven points conic of \( \mathcal{D} \) and \( e \)). The intersection points \( A', A'' \) with \( e \) are the contact points of requested conics with line \( e \).

9.6. Conic by 1 asymptote 1 asymptotic 1 point 1 tangent (3P2T)\(_i\). Construct a conic passing through three points \( A, [B], [C] \), tangent to \( d \supset [B] \) and tangent to \( e \). This is a hyperbola with an asymptote \( d \), an asymptotic direction \([C]\), passing through point \( A \) and tangent to line \( e \). The following construction method is a variation of the one given in §9.1. The pencil of conics \( \mathcal{D} \), used in the theorem of Desargues, consists now of all conics tangent to \( d \) at \([B]\) and passing through \([C]\). This is the pencil of hyperbolas having their centers on line \( d \), one asymptote \( d \), the other parallel to \([C]\) and passing through \( A \) (see Figure 74). Two degenerate members of this pencil consist of (a) the product of line \( AB \) and the line at infinity, (b) the product of lines \( d \cdot AC \). These two members define on \( e \) respectively the point-pairs \((G,[e]),(E,F)\), where \( G = (e,AB), E = (e,AC) \) and \( F = (d,e) \). The contact points \( D,D' \) of the requested hyperbolas with line \( e \) are the common harmonics of these two point-pairs. They lie on \( d \) symmetrically with respect to \( G \) (see Figure 75). Once the contact points with line \( e \) are known, the methods of (3P2T)\(_i\) in §10.1 can be applied to complete the construction of the conics. Fixing the positions of \( A, d, e \), there are two solutions if \([C]\) defines \( E = (AC,e) \), such that \( EF \) is not separated by \( G \). Otherwise there are no solutions. The points \( D,D' \) can be found also as intersections of line \( e \) with the conic \( k \) which is the locus of
poles of $e$ with respect to the members of the pencil (the eleven points conic). In this case $k$ is a hyperbola with one asymptote parallel to $d$.

10. Three points two tangents two coincidences

10.1. Conic by two tangents-at and a point $(3P2T)_2$. Construct a conic passing through three points $A, B, C$ and tangent to two lines $d, e$ at $A \in d$ and $B \in e$. In this case it is easy to find additional points and pass the conic through five points. Line $AB$ is the polar of $F = (d, e)$ and the conjugate $D = C(F, J)$, where $J = (FC, AB)$ is on the conic. The conjugate $I = J(A, B)$ is the pole of $FC$ and more points can be constructed as shown in Figure 75. The problem has always one solution.

Remark. In this case the simplicity of the analytic solution is worth noticing. Representing lines $d, e$ with two equations respectively $f = 0, g = 0$, and line $AB$ with $h = 0$, the general equation of the conic passing through $A, B$ and tangent there to lines $d, e$ is given by a quadratic equation ([18, p. 234])

$$j = \lambda \cdot (f \cdot g) + \mu \cdot h^2 = 0,$$

where $\lambda$ and $\mu$ are arbitrary constants. The requirement for the conic to pass through $C$, namely, $j(C) = 0$, determines, the constants $\lambda, \mu$ up to a multiplicative factor, and through these determines a unique conic. The conics resulting for variable $\lambda, \mu$ build the bitangent pencil ([2, II, p. 187]), used also in §4.1.
Figure 77 presents such a kind of (type IV) pencil. All conics of the pencil are tangent to the lines $d,e$ at their intersections with line $g$. The two conics $c_1 = d \cdot e$ and $c_2 = g^2$ are degenerate members of the pencil.

10.2. Parabola by 1 point, 1 tangent-at, axis-direction $(3P_12T_1)$. Construct a conic passing through three points $A,B,[C]$, tangent to line $d$ at $A$ and also tangent to the line at infinity $e$. Thus, the conic is a parabola with axis parallel to the direction $[C]$. Project $B$ parallel to $d$ on $AC$ to $M$ and extend $BM$ to the double to find $B'$, which is on the parabola (this map $B \mapsto B'$ is the affine reflection with axis $AC$ and conjugate direction $d$ ([5, p. 203])). Define $N$ to be the symmetric of $M$ with respect to $A$ (see Figure 78). Since $BN$ is tangent at $B$ to the parabola, we can construct arbitrary many points of the parabola by repeating this procedure. There is always one solution.

10.3. Hyperbola 1 point 1 tangent-at 1 asymptote $(3P_12T_1)_1$. Construct a conic passing through points $A,B,[C]$ and tangent to lines $d \ni [C], e \ni B$. This is a hyperbola with asymptote $d$. A slight variation of the solution in §10.1, leading to the determination of the other asymptote and the center of the hyperbola, is as follows. Let $F = (d,e)$. Then the symmetric $F_1$ of $F$ with respect to $B$ is on the other asymptote $d'$ of the hyperbola (see Figure 79). Draw also the parallel to $d$ from $A$ intersecting $e$ at $D$. The symmetric $D'$ of $D$ with respect to $A$ defines the polar $BD'$ of $D$. The intersection point $E = (BD',d)$ is the pol of line $AD$. Hence $AE$ is the tangent at $A$ and the symmetric $E_1$ of $E$ with respect to $A$ defines
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A

b

d

A'

FF'

D

D'

E

E1

O

d'

[127x739]Figure 79. Hyperbola through A, asymptote d and tangent to e at B

a point on the other asymptote $d'$ of the hyperbola. Thus, the other asymptote $d'$ can be constructed to pass from the two points $F_1$, $E_1$. The intersection point $O$ of the two asymptotes defines the center of the hyperbola and by the symmetry with respect to $O$ we can find more points on the conic. An additional point on the conic is also $A'$, constructed by first drawing the parallel from $B$ to $d$. This parallel is the polar of $F$ and if $F'$ is its intersection point with $AF$, then the harmonic conjugate of $A$ with respect to $F, F'$ is on the conic. There is always one solution.

10.4. Hyperbola from two asymptotes and a point $(3P2T)_1$. Construct of a conic passing through three points $A, [B], [C]$ and tangent to two lines $d \ni B, e \ni C$. This is a hyperbola with asymptotes the lines $d, e$ passing through a point $A$. This can be done by determining the successive symmetrics $D, D', A'$ of $A$ with respect to the axes and a fifth additional point $D''$ easily constructible from the data (see Figure 80). In fact, draw a parallel to the asymptote $d$ intersecting the other asymptote in $A_0$. The polar of $A_0$ is the line parallel to $e$, such that its intersection $A_1$ with $AA_0$ is the symmetric of $A_0$ with respect to $A$. Consider the intersection $D_1$ of that polar with line $A_0D'$. The harmonic conjugate $D''$ of $D'$ with respect to $(A_0, D_1)$ is on the conic and coincides with the middle of $D_1D'$. Note the $A_0$ divides $D'D''$ in ratio $(2 : 1)$. There is always a unique solution.

11. Two points three tangents one coincidence

11.1. Conic by 1 point, 1 tangent-at, 2 tangents $(2P3T)_1$. Construct a conic passing through two points $D, E$ and tangent to three lines $a \ni D, b, c$. The solution can be given by applying a special case of the dual of Desargues’ theorem, referred
to also as Plücker’s theorem ([4, p. 25], [2, p. 202, II]). This case concerns the one-parameter pencil $D$ of all conics, which are tangent to line $a$ at $D$ and also tangent to two lines $b, c$ (see Figure 84 below in this section). If $E$ is another, arbi-

Figure 81. Conics tangent $a, b, c$, passing through $D \in a, E$

trary, but fixed point, not lying on any of $a, b, c$, Desargues’ theorem asserts, that the pairs of tangents to these conics from $E$ define an involution on the pencil $E^*$ of lines through $E$. By intersecting the rays of this pencil with a line, such as $a$, we can represent this involution through one which permutes the points of that line. Thus, the tangents from $E$ to an arbitrary conic of that pencil intersect line $a$ at a pair of points $(E_1, E_2)$, related by this involution (see Figure 81). The requested conics are those, which pass through $E$ and their tangents at $E$ pass through the fixed points $X, X'$ of this involution. In order to construct these points it suffices to find two easily constructible pairs of points in involution on $a$. One such pair consists of the points $C = (a, b), B = (a, c)$. Another pair is found by drawing

Figure 82. A particular conic tangent to $a, b, c$, passing through $D \in a$

the parallel $a'$ to $a$ through $E$, intersecting $c$ at $F$ (see Figure 82). The second tangent from $E$ to the conic inscribed in the quadrilateral with sides $a', b, a, c$ and passing through $D$, can be found by applying Brianchon’s theorem to the pentagon $CBFEG$. This theorem guarantees that lines $DE, CF, BG$ pass through a common point $P$. Thus, $P$ is constructed by intersecting $DE$ with $CF$ and $G$ is found as the intersection $G = (PB, b)$. In this case the two tangents from $E$ are $EG, EF$ and consequently $H$ corresponds by the involution to the point at infinity of line $a$. It follows that the fixed points $X, X'$ of the involution are common harmonics of pairs $(H, [a])$ and $(B, C)$. Once the tangents at $E$ are found, each one of the two conics can be constructed by locating one more point on it and applying the recipe of §10.1 using Brianchon’s theorem. There are two solutions if points $D, E$
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Figure 83. The two conics tangent to \(a, b, c\), passing through \(D \in a, E\)

are in the same angular domain defined by lines \(b, c\) or they are in opposite angular domains. In all other cases there are no solutions. This is visible also in Figure 84,

Figure 84. A pencil of conics tangent to \(a, b, c\) and passing through \(D \in a\)

which displays a pencil of conics tangent to \(a, b, c\) and passing through \(D \in a\). When \(D\) is on the exterior of segment \(BC\), then the conics are all located in the angular domain of \(b, c\) containing \(D\) and its opposite (\([2, II, p. 201]\)).

An alternative solution of the problem is the following. Consider the triangle with sides \(a, b, c\) and the conic \(k\) passing through its vertices and tangent to \(EB, EC\) at \(B, C\) respectively, which is a construction of the type \((3P2T)_2\) of

Figure 85. The locus of \(F = (B'C, C'B)\) for \(B'C'\) passing through \(E\)

\(\S 10.1\) (see Figure 85). It is easy to see, using Maclaurin’s theorem (\([1, p. 77], [18, p. 230]\)), that this conic is the geometric locus of points \(F\), which are intersections of diagonals of quadrilaterals \(B'BCC'\) with \(B'C'\) passing through \(E\). From Brianchon’s theorem follows that if \(B'C'\) were the tangent at \(E\) to our requested conic,
then the diagonals $B'C, C'B$ would intersect on line $DE$. Thus, their intersection point $F$ must coincide with the intersection points $F_1, F_2$ of line $DE$ with the conic $k$. Having these two points, the construction of the two tangents at $E$ is immediate and the rest, of the construction of conics, goes as before.

11.2. Parabola by 1 point, 1 tangent, 1 tangent-at $(2P3T_1)_1$. Construct a conic passing through two points $A, B$, tangent to two lines $c \ni A, d$ and tangent also to the line at infinity, thus a parabola. By Desargues’ theorem, applied as in §11.1, the tangents at $B$ to the members of the pencil $D$ (see Figure 86) of all conics tangent to $c$ at $A$, passing through $B$ and also tangent to $d$ and the line at infinity (thus parabolas), define an involution on the pencil $B^*$ of all lines through $B$. The tangents at $B$ to the requested parabolas are the fixed elements of this involution. We can represent this involution through points on the line $d$, by corresponding to each ray through $B$ its intersection point with $d$. There are two particular degenerate parabolas of this pencil, coinciding with the lines parallel to $c, d$ through $B$. The parallel to $d$ defines the pair of corresponding points $(B_2, [d])$, where $B_2 = (AB, d)$. The parallel to $c$ defines the pair of corresponding points $(B_1, D)$, where $B_1 = (B[c], d)$ and $D = (c, d)$. The rays through $B$ representing the fixed elements of the involution in $B^*$ are $BC_1, BC_2$, where $C_1, C_2$ are the common harmonics of the pairs $(B_1, D), (B_2, [d])$. Once the tangents through $B$ are located, the parabolas are constructed as in the next section. There are two solutions if points $A, B$ are not separated by line $d$ and no solution if they are.
11.3. **Parabola by 1 point, 2 tangents, axis-direction** \((2P_13T_1)\). Construct a conic passing through two points \(A, [B]\), tangent to two lines \(c, d\) and also tangent to the line at infinity, thus a parabola. Point \([B]\) determines the direction of the axis of the parabola. The pencil of parabolas tangent to \(c, d\) with axis direction \([B]\) can be easily constructed by taking the harmonic conjugate \(e\) of \(CB\) with respect to \(c, d\). This is namely the direction of chords bisected by \(CB\), where \(C = (c, d)\). Having that direction, we can define a second point \(A'\) on the requested parabola. This is the result of the affine reflexion on \(CB\) parallel to \(e\) (see Figure 88). The rest of the construction is thus reducible to that of §9.3, which gives either two solutions or none, if point \(A\) is outside of the angular domains determined by \(c, d\), which contain the parallel \(CB\) to the given direction \([B]\).

Another, computational, method to locate the two parabolas could be the one using the equation of the parabola with respect to the axes \(e, CB\) (see Figure 89).

In these axes the equation of the parabola has the form
\[
y = \alpha x^2 + \beta,
\]
and constants \(\alpha, \beta\) are easily determined by the data. In fact, the given point \(A\) has known coordinates \((x_1, y_1)\) with respect to these axes and the coordinates \((x_2, y_2)\) of the contact point \(K\) with \(c\) satisfy \(y_2 = 2\beta\) and \(\frac{y_2}{x_2^2} = \lambda\), later being a constant determined by the data. It turns out that \(\alpha, \beta\) satisfy the two equations
\[
y_1 = \alpha x_1^2 + \beta,\quad \alpha\beta = \frac{\lambda^2}{4},
\]
which determine the same solutions under the same conditions as before.
11.4. Hyperbola by 1 point, 1 asymptote, 2 tangents \((2P_13T)\). Construct a conic passing through two points \(A, [B]\), tangent to three lines \(c, d, e \ni B\). This is a hyperbola with an asymptote \(e\). By the method of §11.1, the tangents at \(A\) of the requested conics are determined by the common harmonics \(B_1, B_2\) of the two pairs of points \((C, E)\) and \((F, [B])\), where \(C = (e, d), E = (e, c), D = (c, d), F = (AD, e)\) (see Figure 90). If \(O\) is a diagonal point of the quadrilateral formed by the three tangents \(c, d, e, AB_1\), then, by Brianchon’s theorem, the intersection point \(A'\) of \(d\) with the parallel to \(e\) through \(O\) will be the contact point of \(d\) with the conic. The problem reduces then to the construction of the conic tangent at \(A' \in d, A \in AB_1\) and passing through \(A\), which is \((3P2T)_2\) of §10.1. Analogous properties hold for the other conic with tangent at \(A\) the line \(AB_2\). Fixing the lines \(c, d\), there are two solutions if \(A, [B]\) are in the same or opposite angular domains defined by \(c, d\). Otherwise there are no solutions.

11.5. Hyperbola 1 asymptote 1 asymptotic 2 tangents \((2P_23T)\). Construct a conic passing through two points \([A], [B]\) and tangent to three lines: \(a\) at \([A]\), \(b, c\). This is a hyperbola with one asymptote \(a\), the other asymptotic direction \([B]\) and two other tangents \(b, c\). The problem is projectively equivalent to \((2P3T)_1\) of §11.1. Here

![Figure 90. The two hyperbolas through \(A\) with asymptote \(e\) and tangent to \(c, d\)](image)

![Figure 91. The two hyperbolas with asymptote \(a\), asymptotic \([B]\) and tangents \(b, c\)](image)
again the recipe is essentially the one of §11.1, with some simplifications allowing
for a faster determination of the conics. In Figure 91, displaying the conics, \( A_0 = (a, EB) \) and points \( A_1, A_2 \) are the common harmonics of pairs \((C, D), (A_0, [A])\). Parallels to \([B]\) define the two quadrilaterals \(DA_2C_2E, DA_1C_1E\). Each quadrilateral determines a conic tangent to its sides. In this case the contact points of the conics with the sides of the quadrilateral are easily determined. In fact, by the well known property of segments intercepted between asymptotes follows, that the contact points \(X_1, X_2\) are respectively the middles of \(DF_2, CC_1\) and the contact points \(Y_1, Y_2\) are the middles of \(DF_1, CC_2\). There are two solutions if, drawing parallels from a point to \(b, c\) and to \([B]\), later does not fall between the two first. Otherwise there are no solutions.

12. Two points three tangents two coincidences

12.1. Conic by two tangents-at and a tangent \((2P3T)_2\). Construct a conic tangent to three lines \(a, b, c\), passing through two points \(A, B\) with \(A \in a\) and \(B \in b\). In this case the contact point \(C\) of the requested conic with the third line is easily constructed, since all lines joining the vertices of the triangle formed by the three lines to the opposite contact point pass through the same point \(P\), the perspector of the conic with respect to that triangle \(A'B'C'\) (see Figure 92). There is always a unique solution.

12.2. Parabola by 2 tangents-at \((2P3T_1)_2\). Construct a conic passing through two points \(A, B\), tangent to two lines \(c \ni A, d \ni B\) and also tangent to the line at infinity, thus a parabola. If point \(C = (a, b)\), taking the middle \(D\) of \(AB\) and the middle \(M\) of \(CD\) we construct a new point on the parabola. Analogously
are obtained new points \( K, L \) from the middles of \( MA, MB \) respectively. The parabola is led as a conic through the five points \( A, B, M, K, L \). There is always one solution identified with one first-kind Artzt parabola of triangle \( ABC \) ([13, p. 518]). Triangle \( ABC \) is referred by times as an Archimedes triangle ([8, p. 239]).

12.3. Parabola I tangent I tangent-at, axis-direction \((2P; 3T_1)_1\). Construct a conic passing through two points \( A, [B] \), tangent to two lines \( c \ni A, d \) and tangent also to the line at infinity, thus, a parabola with axis parallel to \( [B] \). One solution is to construct, as in the previous section, the direction \( GH \) of chords of the parabola, which are bisected by \( CB \). Then, find the point \( A' \) on the parabola, such that \( AA' \) is parallel to \( GH \) and bisected by \( CB \). Point \( A' \) is the contact point of the parabola with \( d \) and the construction reduces to that of \((2P3T_1)_2\) of §12.2. There is always a unique solution.

12.4. Hyperbola I asymptote I tangent I tangent-at \((2P; 3T)_3\). Construct a conic passing through two points \([A], B\) and tangent to three lines \( c \ni A, d \ni B, e \). This is a hyperbola with one asymptote \( c \), tangent to \( d \) at \( B \) and also tangent to \( e \). The triangle \( CDE \) with sides the tangents \( c, d, e \) is known and the perspector \( P \) of the conic, tangent to the sides of this triangle, can be found (see Figure 95). In fact, draw from \( C = (d, e) \) parallel to the asymptote \( c \) and find its intersection \( P \) with \( BD \), where \( D = (c, e) \). If \( E = (c, d) \), then line \( PE \) passes through the contact point \( F \) of \( e \) with the conic. The case, as the one of §12.1, has always a unique solution.
12.5. **Hyperbola 2 asymptotes 1 tangent (2P₃T)₂.** Construct a conic passing through two points \([A], [B]\) and tangent to three lines \(a \ni A, b \ni B, c\). This is a hyperbola with given asymptotes \(a, b\) and a tangent \(c\). This is an easy case, since the contact point of the tangent \(c\) is the middle \(F\) of \(DE\), where \(D = (a, c), E = (b, c)\). The parallel to \(a\) from \(F\) is the polar of \(D\) and the parallel to the other asymptote \(b\) from \(D\) intersects the first parallel at \(G\). The middle \(H\) of \(DG\) is a point of the hyperbola. An analogous point can be constructed starting with \(E\). Taking the symmetrics with respect to the center \(O = (a, b)\) of the hyperbola we have enough points to define the conic through five points. There is always a unique solution.

### 13. One point four tangents one coincidence

13.1. **Conic by one tangent-at and three tangents (1P₄T)₁.** Construct a conic tangent to a given line \(a\) at a given point \(D\) and also tangent to three other lines \(b, c, e\). The basic underlying structure results from Brianchon’s theorem. In fact, consider the intersection point \(O\) of the diagonals of the quadrilateral \(BCGF\) defined by the lines \(a, b, c, e\). The problem is thus reducible to \((4P₁T)₁\) of §8.1 and has one solution. The pencil involved here is the one of conics tangent to \(a\) at \(D\) and also tangent to \(b, c\), appearing also in \((2P₃T)₁\) of §11.1.
Remark. Besides quadrangle $BCGF$, the complete quadrilateral, defined by lines $a, b, c, e$, contains also the quadrangles $AGMB$, whose diagonals intersect at $L$ and $CMFA$, whose diagonals intersect at $K$. It is also easily seen, that the contact points $N, E, M$ are the harmonic associates of $D$ with respect to the diagonal triangle $OLK$ of the complete quadrilateral. Thus the definition of $N, E, M$ from $D$, does not depend on which one of the three quadrangles (and corresponding intersection of diagonals $O, K$ or $L$) we select to work with.

13.2. Parabola by 1 tangent-at, 2 tangents $(1PT_1)$. Construct a conic tangent to the line at infinity, i.e. a parabola, tangent to line $a$ at $E \in a$ and tangent to two lines $b, c$. Here the construction is somewhat simpler than that of the previous case, because of the nice properties of parabolas. In fact, let $O$ be the intersection of the parallels from $B$ to $b$ and from $C$ to $c$ (see Figure 98). Then the line $EO$ is parallel to the axis of the parabola. Having the direction of the axis, we can construct more points on the parabola using the method of $(2P_13T_1)$ of §11.3. Using this we can find the direction of the chords bisected by the parallel to the axis from $B$ and determine the contact point $F$ with line $c$. Analogously we can find the contact point $G$ of $b$, and from these points by similar methods find other arbitrary many points on the parabola. Alternatively, we can use the fact that points $\{F, G, O\}$ are collinear and the line $d$, carrying them, intersects line $BC$ at the harmonic conjugate $E' = E(B, C)$. There is always a unique solution.

13.3. Parabola by axis-direction, 3 tangents $(1P_4A_4T_1)$. Construct a conic tangent to the line at infinity, i.e. a parabola, tangent to three lines $a, b, c$ and passing through $[D]$, i.e. with given axis-direction. This is a case similar to the previous one. Again we construct the intersection point $O$ of the parallels to $b, c$ from the
opposite vertices. The line through $O$, parallel to the given axis-direction, determines now on $a$ the contact point $E$ with the parabola. From there the construction of the other contact points $F, G$ with sides $c, b$ and the completion of the parabola construction is the one described in the previous section. There is always a unique solution.

13.4. **Hyperbola, 1 asymptote, 3 tangents** $(1P_14T)$. Construct a conic tangent to line $e$ at its point at infinity $[E]$, i.e. a hyperbola with an asymptote $e$, and tangent to three lines $a, b, c$. In analogy to §13.1 we can find the contact points of the three tangents with the hyperbola. In fact, consider the intersection point $P$ of the diagonals of quadrilateral $FGHI$, whose all sides are given and are tangents to the hyperbola. By Brianchon’s theorem, the line $e'$ parallel to the asymptote $e$ from point $P$ will intersect the side $a$ of the quadrilateral at its contact point $A$ with the hyperbola (see Figure 100). Consider now an arbitrary line $h$ and its intersection points $A' = (e', h)$, $I' = (IG, h)$, $F' = (FH, h)$. The other chord of contact points $BC$ will intersect line $h$ at the harmonic conjugate $B'$ of $A'$ with respect to $(I', F')$. Thus, $B'$ is constructible from the given data, and drawing $PB'$ we determine the positions $B, C$ of the contact points on the tangents $b, c$, respectively. Having one asymptote and the contact points on the tangents, we can determine the other asymptote, the center, and, by symmetry to that center, three more points on the conic. The method is described already in $(3P_12T)_1$, of §10.3. There is always one solution.

**References**


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On Two Triads of Triangles Associated With the Perpendicular Bisectors of the Sides of a Triangle

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Abstract. We discover some properties of the triangle centers related to the two triads of triangles associated with the perpendicular bisectors of the sides of a triangle.

1. Introduction

Given a triangle \( T := ABC \), let the perpendicular bisector of the side \( BC \) intersect the sidelines \( AC \) and \( AB \) at \( B_c \) and \( C_b \) respectively. Define \( C_a, A_c, A_b, B_a \) similarly. In this paper we study the two triads of triangles (i) \( T_a := AA_bA_c, T_b := B_aBB_c, T_c := C_aCcC \) (see Figure 1a) and (ii) \( T'_a := AB_aC_a, T'_b := A_bBC_b, T'_c := A_cB_cC \) (see Figure 1b).

Homogeneous barycentric coordinates are used throughout this work. With the usual notations in triangle geometry, \( a, b, c \) for the lengths of the sides \( BC, CA, AB \), and
\[
S_A = \frac{b^2 + c^2 - a^2}{2}, \quad S_B = \frac{c^2 + a^2 - b^2}{2}, \quad S_C = \frac{a^2 + b^2 - c^2}{2},
\]
the points on the perpendicular bisectors are
\[
A_b = (0 : -S_A + S_B : S_A + S_B), \quad A_c = (0 : S_C + S_A : S_C - S_A); \\
B_c = (S_B + S_C : 0 : -S_B + S_C), \quad B_a = (S_A - S_B : 0 : S_A + S_B); \\
C_a = (-S_C + S_A : S_C + S_A : 0), \quad C_b = (S_B + S_C : S_B - S_C : 0).
\]

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Denote by $T'$ the orthic triangle of $ABC$. Its vertices are

$$A' = (0 : S_C : S_B), \quad B' = (S_C : 0 : S_A), \quad C' = (S_B : S_A : 0).$$

The sidelengths $a', b', c'$ of $B'C', C'A', A'B'$ of the orthic triangle are given by

$$a'^2 = \frac{a^2S_{AA}}{b^2c^2}, \quad b'^2 = \frac{b^2S_{BB}}{c^2a^2}, \quad c'^2 = \frac{c^2S_{CC}}{a^2b^2}.$$

From these,

$$a'^2 : b'^2 : c'^2 = a^4S_{AA} : b^4S_{BB} : c^4S_{CC}.$$

Corresponding to $S_A, S_B, S_C$, we have

$$S'_A := \frac{1}{2}(b'^2 + c'^2 - a'^2) = \frac{S_{ABC}}{a^2b^2c^2} \frac{S^2 - S_{AA}}{S_A},$$

$$S'_B := \frac{1}{2}(c'^2 + a'^2 - b'^2) = \frac{S_{ABC}}{a^2b^2c^2} \frac{S^2 - S_{BB}}{S_B},$$

$$S'_C := \frac{1}{2}(a'^2 + b'^2 - c'^2) = \frac{S_{ABC}}{a^2b^2c^2} \frac{S^2 - S_{CC}}{S_C},$$

and

$$S'_A : S'_B : S'_C = S_{BC}(S^2 - S_{AA}) : S_{CA}(S^2 - S_{BB}) : S_{AB}(S^2 - S_{CC}).$$

**Lemma 1.** (a) The triangles $AA_bA_c$, $B_aBB_c$, and $C_aC_bC$ are all similar to $T'$.
(b) The triangles $AB_aC_a$, $A_bBC_b$, and $A_cB_cC$ are all similar to $T$.

2. A common point of circumcircles

**Theorem 2.** The circumcircles of triangles in the two triads $(T_a, T_b, T_c)$ and $(T'_a, T'_b, T'_c)$ all contain the Euler reflection point

$$E = \left( \frac{a^2}{b^2 - c^2} : \frac{b^2}{c^2 - a^2} : \frac{c^2}{a^2 - b^2} \right)$$

of the circumcircle of $T$.

**Proof.** We compute the coordinates of $E$ with respect to these triangles, and show from these coordinates that $E$ lies on the circumcircle of each. With respect to $T$,

$$E = ((S_B + S_C)(S_C - S_A)(S_A - S_B) : (S_B - S_C)(S_C + S_A)(S_A - S_B) : (S_B - S_C)(S_C - S_A)(S_A + S_B))$$

with coordinate sum $\sigma = a^2S_{AA} + b^2S_{BB} + c^2S_{CC} - 6S_{ABC}$.

The coordinates of $E$ in triangle $T_a = AA_bA_c$ are
Two triads of triangles

\[ \text{Area}(E_A E_a) : \text{area}(A E_A E) \]

\[ = \frac{1}{4\sigma_{BC}} \begin{vmatrix} (S_B + S_C)(S_C - S_A)(S_A - S_B) & (S_B - S_C)(S_C + S_A)(S_A - S_B) & (S_B - S_C)(S_C - S_A)(S_A + S_B) \\ 0 & -(S_A - S_B) & S_A + S_B \\ 0 & S_C + S_A & S_C - S_A \end{vmatrix} \]

\[ = \frac{1}{2\sigma_C} \begin{vmatrix} (S_B + S_C)(S_C - S_A)(S_A - S_B) & (S_B - S_C)(S_C + S_A)(S_A - S_B) & (S_B - S_C)(S_C - S_A)(S_A + S_B) \\ 1 & 0 & 0 \\ 0 & S_C + S_A & S_C - S_A \end{vmatrix} \]

\[ = \frac{1}{2\sigma_B} \begin{vmatrix} (S_B + S_C)(S_C - S_A)(S_A - S_B) & (S_B - S_C)(S_C + S_A)(S_A - S_B) & (S_B - S_C)(S_C - S_A)(S_A + S_B) \\ 0 & -(S_A - S_B) & S_A + S_B \\ 0 & S_C - S_A & 0 \end{vmatrix} \]

\[ = \frac{S_A(S_B + S_C)^2}{2(S_B - S_C)} : \frac{S_B(S_C + S_A) + S_C(S_A + S_B)}{S_A - S_B} : \frac{S_C(S_A + S_B)(S_B - S_C)}{S_C - S_A} \]

\[ = \frac{S_A(S_B + S_C)^2}{2S_A(S_B - S_C)} : \frac{S_B(S_C + S_A)^2}{S_C + S_A(S_A - S_B)} : \frac{S_C(S_A + S_B)^2}{S_A + S_B(S_C - S_A)} \]

This is the isogonal conjugate (in triangle $T_a = AA_b A_c$) of the point

\[ (2S_A(S_B - S_C) : (S_C + S_A)(S_A - S_B) : (S_A + S_B)(S_C - S_A)), \]

which is an infinite point since the coordinate sum

\[ 2S_A(S_B - S_C) + (S_C + S_A)(S_A - S_B) + (S_A + S_B)(S_C - S_A) = 0. \]
This shows that $E$ is on the circumcircle of triangle $T_\alpha$. Similarly, $E$ is also on the circumcircles of the triangles $T_b$ and $T_c$, with coordinates

\[
\frac{a'^2}{(S_B + S_C)(S_A - S_B)} : \frac{b'^2}{2S_B(S_C - S_A)} : \frac{c'^2}{(S_A + S_B)(S_B - S_C)}
\]

and

\[
\frac{a'^2}{(S_B + S_C)(S_C - S_A)} : \frac{b'^2}{(S_C + S_A)(S_B - S_C)} : \frac{c'^2}{2S_C(S_A - S_B)}
\]

respectively. A similar calculation shows that $E$ is also on the circumcircles of triangles $T'_a$, $T'_b$, $T'_c$, with coordinates

\[
\frac{a'^2}{2S_A(S_B - S_C)} : \frac{b'^2}{(S_C + S_A)(S_A - S_B)} : \frac{c'^2}{(S_A + S_B)(S_C - S_A)}
\]

\[
\frac{a'^2}{(S_B + S_C)(S_A - S_B)} : \frac{b'^2}{2S_B(S_C - S_A)} : \frac{c'^2}{(S_A + S_B)(S_B - S_C)}
\]

\[
\frac{a'^2}{(S_B + S_C)(S_C - S_A)} : \frac{b'^2}{(S_C + S_A)(S_B - S_C)} : \frac{c'^2}{2S_C(S_A - S_B)}
\]

respectively in these triangles.

\[\square\]

3. **Counterparts of a point in the triad $T_\alpha, T_b, T_c$**

Let $P$ be a point with homogeneous barycentric coordinates $(x : y : z)$ with respect to triangle $T = ABC$. The counterparts of $P$ in the triangles $T_\alpha, T_b, T_c$ are the points $A_P, B_P, C_P$ which have the same coordinates $(x : y : z)$ in these triangles. In homogeneous barycentric coordinates,

\[
A_P = (2S_{BC}x : S_C(-S_A + S_B)y + S_B(S_C + S_A)z : S_C(S_A + S_B)y + S_B(S_C - S_A)z),
\]

\[
B_P = (S_A(S_B + S_C)z + S_C(S_A - S_B)x : 2S_{CA}y : S_A(-S_B + S_C)z + S_C(S_A + S_B)x),
\]

\[
C_P = (S_B(-S_C + S_A)x + S_A(S_B + S_C)y : S_B(S_C + S_A)x + S_A(S_B - S_C)y : 2S_{AB}z)
\]

Denote by $T(P)$ the triangle $A_PB_PC_P$. Basic properties of $T(P)$ can be found in [1].

**Theorem 3 (Bui).** The triangle $T(P)$ is

(a) oppositely similar to $T$,

(b) orthologic to $T$,

(c) perspective with $T$ if and only if $P$ lies on the Jerabek hyperbola. In this case, the perspector traverses the Euler line.

If $P = (x : y : z)$, the perpendiculars from $A$ to $BPC_P$, $B$ to $C_PA_P$, $C$ to $A_PB_P$ concur at

\[
Q = \left(\frac{1}{-b^2c^2S_{BC}x + c^2S_AS_{CC}y + b^2S_AS_{BB}z} : \cdots : \cdots\right),
\]

which lies on the circumcircle of $T$ (see Figure 3).
On the other hand, the perpendiculars from $A_P$ to $BC$, $B_P$ to $CA$, $C_P$ to $AB$ concur at

$$Q' = (S_{BC}(a^2(S_{AA} - S_{BC}) - S_A(S_{BB} + S_{CC})x + c^2a^2S_Ay + a^2b^2S_AS_{BB})z : \cdots : \cdots).$$

According to [1], the coordinates of $Q'$ with respect to $T(P)$ are the same as those of $Q$ with respect to $T$. It follows that $Q'$ is a point on the circumcircle of $T(P)$.

**Proposition 4.** The triangle $T(P)$ has orthocenter $O$.

**Proof.** The sum of the coordinates of $A_P$ given at the beginning of the present section is $S_{BC}(x + y + z)$; similarly for $B_P$ and $C_P$. The orthocenter of $T(P)$ has coordinates

$$(2S_{BC}x, S_C(-S_A + S_B)y + S_B(S_C + S_A)z, S_C(S_A + S_C)z + S_C(S_A + S_B)x, 2S_{CA}y, S_A(-S_B + S_C)z + S_C(S_A + S_B)x, \cdots).$$

This is the circumcenter $O$ of $T$. □

**Theorem 5.** The points $A_P$, $B_P$, $C_P$ and $P$ are concyclic.
Proof. The coordinates of \( P \) with respect to \( T(P) = A_PB_PC_P \) are
\[
\text{Area}(PBP_C) : \text{area}(A_PC_PC) : \text{area}(A_PB_P) = \frac{(b^2S_Bx - a^2S_{AB})(a^2S_Az - c^2SCx)}{4S_{ABC}S_A(x+y+z)^2} : \frac{(c^2SCy - b^2S_{Bz})(b^2S_Bx - a^2S_{AB})}{4S_{ABC}S_B(x+y+z)^2} : \frac{(a^2S_Az - c^2SCx)(c^2SCy - b^2S_{Bz})}{4S_{ABC}S_C(x+y+z)^2}
\]
\[
= \frac{a^2}{b^2} : \frac{b^2}{c^2} : \frac{c^2}{a^2}
\]
This shows that \( P \) is the isogonal conjugate (in triangle \( T(P) \)) of an infinite point. It is a point on the circumcircle of \( T(P) \). \( \square \)

Since \( T(P) \) is similar to \( T \), its circumcenter is the point
\[
O_P := \frac{a^2S_A \cdot A_P + b^2S_B \cdot B_P + c^2S_C \cdot C_P}{2S^2}
\]
in absolute barycentric coordinates. In homogeneous barycentric coordinates with respect to \( T \),
\[
O_P = (S_{BC}(4S^2 \cdot S_A - a^2b^2c^2)x + c^2a^2S_{CCAY} + a^2b^2S_{ABB}z : b^2c^2S_{BCC}x + S_{CA}(4S^2 \cdot S_B - a^2b^2c^2)y + a^2b^2S_{AAB}z : b^2c^2S_{BCC}x + c^2a^2S_{CAAY} + S_{AB}(4S^2 \cdot S_C - a^2b^2c^2)z).
\]

4. Counterparts of a point in the triad \( T'_a, T'_b, T'_c \)

For \( P = (x : y : z) \) with respect to \( T \), we also consider its counterparts \( A'_P, B'_P, C'_P \) in the triangles \( T'_a, T'_b, T'_c \). These are the points
\[
A'_P = (2S_Ax - (a^2 - b^2)y - (a^2 - c^2)z : b^2z : c^2y),
\]
\[
B'_P = (a^2z : 2S_By - (b^2 - c^2)z - (b^2 - a^2)x : c^2x),
\]
\[
C'_P = (a^2y : b^2x : 2S_Cz - (c^2 - a^2)x - (c^2 - b^2)y).
\]

Denote by \( T'(P) \) the triangle \( A'_PB'_PC'_P \).

Proposition 6. For \( P = (x : y : z) \), the triangle \( T'(P) \) is
(a) oppositely similar to the orthic triangle \( T' \),
(b) perspective with \( T \) at the isogonal conjugate of \( P \) in \( T \), namely,
\[
P^* = \left( \frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z} \right).
\]
(c) orthologic to \( T \) if and only if \( P \) lies on the Euler line. In this case,
(i) the perpendiculars from \( A, B, C \) to \( B'_PC'_P, C'_PA'_P, A'_PB'_P \) are concurrent at
\[
X(265) = \left( \frac{S_A}{S^2 - 3S_{AA}} : \frac{S_B}{S^2 - 3S_{BB}} : \frac{S_C}{S^2 - 3S_{CC}} \right),
\]
(ii) if $OP : PH = t : 1 - t$, the perpendiculars from $A', B', C'$ to $BC, CA, AB$ are concurrent at $Q$ where $OQ : QH = 1 - t : t$.

**Remark.** $X(265)$ is the reflection conjugate of $O$. Equivalently, it is the reflection of $O$ in the Jerabek center $X(125)$.

**Proposition 7.** The points $A', B', C'$, $P$ are concyclic if and only if $P$ lies on the circumconic which is the isogonal conjugate (in $T$) of the perpendicular bisector of the segment $OH$.

**Proof.** With respect to $T'(P)$, the point $P$ has coordinates

$$x' : y' : z' = \frac{\text{Area}(PB'C') : \text{area}(A'PC') : \text{area}(A'B'P)}{2a^2S_Ax^2 + a^4yz - (S^2 + 2SBC - SCC)zx - (S^2 + 2SBC - SBB)xy}$$

$$= \frac{2b^2S_by^2 + b^4zx - (S^2 + 2SCA - SAA)xy - (S^2 + 2SCA - SCC)yz}{4SBC(x + y + z)^2}$$

$$= \frac{2c^2S_cz^2 + c^4xy - (S^2 + 2SAB - SBB)yz - (S^2 + 2SAB - SAA)zx}{4SAB(x + y + z)^2}$$

$$= S_A(2a^2S_Ax^2 + a^4yz - (S^2 + 2SBC - SCC)zx - (S^2 + 2SBC - SBB)xy)$$

$$: S_B(2b^2S_by^2 + b^4zx - (S^2 + 2SCA - SAA)xy - (S^2 + 2SCA - SCC)yz)$$

$$: S_C(2c^2S_cz^2 + c^4xy - (S^2 + 2SAB - SBB)yz - (S^2 + 2SAB - SAA)zx)$$
The point \( P \) lies on the circumcircle of \( T'(P) \) (which is similar to \( T' \)) if and only if

\[
0 = a'^2 y' z' + b'^2 z' x' + c'^2 x' y'
= a'^4 S_{AA} y' z' + b'^4 S_{BB} z' x' + c'^4 S_{CC} x' y'
= 2S_{ABC} : S^2(x + y + z)^2|HP|^2 \left( \sum_{\text{cyclic}} a^4(S^2 - 3S_{AA})yz \right).
\]

There are two possibilities.
(i) \(|HP| = 0 \implies P = H\). In this case, \( A'_H = B'_H = C'_H = O\).
(ii) \( P \) lies on the circumconic \( \sum_{\text{cyclic}} a^4(S^2 - 3S_{AA})yz = 0 \), which is the isogonal conjugate (in \( T \)) of the perpendicular bisector of the segment \( OH \). \( \square \)

5. Triangle centers of \( T_a, T_b, T_c \) and \( T'_a, T'_b, T'_c \)

Consider the circumcenter \( O = (a^2S_A : b^2S_B : c^2S_C)\). Note that \( A_O, B_O, C_O \) are not the circumcenters of the triangles \( T_a, T_b, T_c \) respectively. Indeed, these three points coincide with \( O; A_O = B_O = C_O = O \). Instead, the circumcenters of triangles \( T_a, T_b, T_c \) are the points

\[
O_a = (a^4(S^2 - S_{AA}) : b^2(2S^2 \cdot S_B - S_C(S^2 - S_{BB})) : c^2(2S^2 \cdot S_C - S_B(S^2 - S_{CC}))),
O_b = (a^2(2S^2 \cdot S_A - S_C(S^2 - S_{AA})) : b^4(S^2 - S_{BB}) : c^2(2S^2 \cdot S_C - S_A(S^2 - S_{CC}))),
O_c = (a^2(2S^2 \cdot S_A - S_B(S^2 - S_{AA})) : b^2(2S^2 \cdot S_B - S_A(S^2 - S_{BB})) : c^4(S^2 - S_{CC})).
\]

These form the vertices of \( T(P) \) for

\[
P = X(1147) = (a^4S_A(S^2 - S_{AA}) : b^4S_B(S^2 - S_{BB}) : c^4S_C(S^2 - S_{CC})),
\]

which, according to the ENCYCLOPEDIA OF TRIANGLE CENTERS [2], is the midpoint of \( O \) and \( X(155) \), the orthocenter of the tangential triangle (see Figure 5). The three circumcenters are concyclic with \( X(1147) \) (see Theorem 5). The center of the circle containing them is \( X(156) \), the nine-point center of the tangential triangle.

More generally, let \( P \) be a triangle center of \( ABC \), with coordinates expressed in terms of \( a, b, c \). The same triangle center \( P_a \) of \( T_a = AA_bA_c \) is the point with coordinates in which \( a, b, c \) are replaced by \( a', b', c' \) respectively (likewise \( S_A, S_B, \))
Two triads of triangles

Figure 5.

$S_C$ by $S'_A, S'_B, S'_C$ respectively). For example, for orthocenters,

$$H_a = \left( \frac{1}{S'_A} : \frac{1}{S'_B} : \frac{1}{S'_C} \right)$$

$$= \left( \frac{S_A}{S^2 - S_{AA}} : \frac{S_B}{S^2 - S_{BB}} : \frac{S_C}{S^2 - S_{CC}} \right)$$

with respect to $T_a$

$$= \frac{S_A}{S^2 - S_{AA}} \cdot (1, 0, 0) + \frac{S_B}{S^2 - S_{BB}} \cdot \left( 0, -S_A + S_B, S_A + S_B \right)$$

$$+ \frac{S_C}{S^2 - S_{CC}} \cdot \left( 0, S_C + S_A, S_C - S_A \right)$$

in absolute barycentric coordinates

$$= (S^2 - S_{BB})(S^2 - S_{CC}) : a^2S_C(S^2 - S_{AA}) : a^2S_B(S^2 - S_{AA})$$

with respect to $T$.

The orthocenters $H_a, H_b, H_c$ are the vertices of $T(Q)$ for

$$Q = X(68) = \left( \frac{S_A}{S^2 - S_{AA}} : \frac{S_B}{S^2 - S_{BB}} : \frac{S_C}{S^2 - S_{CC}} \right).$$

The triangle center $X(68)$ is the superior of $X(1147)$. It lies on the circle containing the three orthocenters (see Figure 5 and Theorem 5). The circle $H_aH_bH_c$ also contains the orthocenter $H$. 
The perpendiculars from triangle center

\[ \begin{align*}
G_a &= (2S_{BC} : 2S_{BC} + S_A(S_B - S_C) : 2S_{BC} - S_A(S_B - S_C)), \\
G_b &= (2S_{CA} - S_B(S_C - S_A) : 2S_{CA} : 2S_{CA} + S_B(S_C - S_A)), \\
G_c &= (2S_{AB} + S_C(S_A - S_B) : 2S_{AB} - S_C(S_A - S_B) : 2S_{AB}).
\end{align*} \]

In this case, \( G_aG_bG_c = T(G) \).

On the other hand, since the triangles \( T'_a = AB_aC_a, T'_b = A_bBC_b, T'_c = C_aC_bC \) are similar to \( ABC \), we have

\[ P'_a = A'_p, \quad P'_b = B'_p, \quad P'_c = C'_p. \]

For example, the circumcenters are

\[ \begin{align*}
O'_a &= A'_O = (3a^2S_{AA} - S_A(S_B - S_C)^2 - a^2S_{BC} : b^2c^2S_C : b^2c^2S_B), \\
O'_b &= B'_O = (c^2a^2S_C : 3b^2S_{BB} - S_B(S_C - S_A)^2 - b^2S_{CA} : c^2a^2S_A), \\
O'_c &= C'_O = (a^2b^2S_B : a^2b^2S_A : 3c^2S_{CC} - S_C(S_A - S_B)^2 - c^2S_{AB}).
\end{align*} \]

\[ \begin{align*}
G'_a &= (2b^2 + 2c^2 - 3a^2 : b^2 : c^2), \\
G'_b &= (a^2 : 2c^2 + 2a^2 - 3b^2 : c^2), \\
G'_c &= (a^2 : b^2 : 2a^2 + 2b^2 - 3c^2).
\end{align*} \]

\[ H'_a = H'_b = H'_c = O. \]

6. Orthology with \( T \) and pedal triangles

6.1. The triangle \( O_aO_bO_c \).

**Proposition 8.** The triangle \( O_aO_bO_c \) is orthologic to \( ABC \).

(a) The perpendiculars from \( O_a \) to \( BC \), \( O_b \) to \( CA \), and \( O_c \) to \( AB \) concur at the triangle center\(^1\)

\[ Y_{(1)} := (a^2(S_{AA}(S_{BB} + S_{BC} + S_{CC}) - S_{BB}S_{CC}) : \cdots : \cdots). \]

(b) The perpendiculars from \( A \) to \( O_bO_c \), \( B \) to \( O_cO_a \), and \( C \) to \( O_aO_b \) concur at the triangle center \( X(74) \) on the circumcircle of \( T \).

**Remark.** The orthology center \( Y_{(1)} \) lies on the circle \( O_aO_bO_c \).

**Proposition 9.** The triangle \( O_aO_bO_c \) is orthologic with the pedal triangle of \( P \) if and only if \( P \) lies on the Euler line. If \( P \) lies on this line,

(a) the perpendiculars from \( O_a, O_b, O_c \) to the corresponding sides of the pedal triangle of \( P \) are concurrent at a point on the conic

\[ \begin{align*}
16S^2 \sum_{\text{cyclic}} a^2S_A(S_B - S_C)yz - (x + y + z)(f_a x + f_b y + f_c z) &= 0,
\end{align*} \]

\(^1\)The triangle center \( Y_{(1)} \) does not appear in the current edition of [2]. It has (6-9-13)-search number \(-5.64011769173\cdots\).
where

\[ f_a = b^2 c^2 (S_B - S_C)(a^4 S_A^3 + a^2 S_A A (S_{BB} - 3S_{BC} + S_{CC}) - (8S_A + S_B + S_C)S_{BB} S_{CC}), \]

and \( f_b, f_c \) are defined cyclically,

(b) the perpendiculars from the vertices of the pedal triangle of \( P \) to the corresponding sides of \( O_a O_b O_c \) are concurrent at a point on the line

\[ \sum_{ \text{cyclic} } \frac{S_A (S_B - S_C)}{a^2 S_{AA} - S_A (S_B - S_C)^2 - a^2 S_{BC}} x = 0. \]

6.2. The triangle \( G_a G_b G_c \).

**Proposition 10.** The triangle \( G_a G_b G_c \) is orthologic to \( ABC \).

(a) The perpendiculars from \( G_a \) to \( BC \), \( G_b \) to \( CA \), and \( G_c \) to \( AB \) concur at the triangle center \(^2\)

\[ Y_{(2)} := (a^2 (S_{BB} + S_{BC} + S_{CC}) S_{AA} + S_{BB} S_{CC} (2S_A - S_B - S_C) : \cdots : \cdots). \]

\(^2\)The triangle center \( Y_{(2)} \) does not appear in the current edition of [2]. It has (6-9-13)-search number \( -5.65228493146 \cdots \).
(b) The perpendiculars from $A$ to $G_b G_c$, $B$ to $G_c G_a$, and $C$ to $G_a G_b$ concur at the triangle center $X(1294) = \left( \frac{1}{S_{AA} (S_{BB} - S_{BC} + S_{CC}) - S_{BB} S_{CC}} : \ldots : \ldots \right)$ on the circumcircle.

Remark. The circle $G_a G_b G_c$ contains the centroid $G$ and the orthology center $Y_2$.

Proposition 11. The triangle $G_a G_b G_c$ is orthologic with the pedal triangle of $P$ if and only if $P$ lies on the line

$$\sum_{cyclic} b^2 c^2 (S_B - S_C) (a^2 S_A - S_{BC}) x = 0,$$

which passes through $O$ and $X(64) = \left( \frac{a^2}{a^2 S_A - S_{BC}} : \frac{b^2}{b^2 S_B - S_{CA}} : \frac{c^2}{c^2 S_C - S_{AB}} \right)$.

If $P$ lies on this line,

(a) the perpendiculars from $G_a$, $G_b$, $G_c$ to the corresponding sides of the pedal triangle of $P$ are concurrent at a point on the conic

$$6 S_{ABC} \sum_{cyclic} (S_B - S_C) (a^2 S_A - S_{BC}) y z$$

$$- (x + y + z) \left( \sum_{cyclic} S_A (S_B - S_C) (S_{AA} (S_{BB} - 3S_{BC} + S_{CC}) - S_{BB} S_{CC}) x \right) = 0.$$
(b) the perpendiculars from the vertices of the pedal triangle of $P$ to the corresponding sides of $G_aG_bG_c$ are concurrent at a point on the line

$$\sum_{\text{cyclic}} \frac{S_A(S_B - S_C)(a^2S_A - S_{BC})}{a^2S_A - 2S_{BC}} = 0.$$ 

6.3. The triangle $H_aH_bH_c$.

**Proposition 12.** The triangle $H_aH_bH_c$ is orthologic to $ABC$.

(a) The perpendiculars from $H_a$ to $BC$, $H_b$ to $CA$, and $H_c$ to $AB$ concur at the orthocenter $H$ of $T$.

(b) The perpendiculars from $A$ to $H_bH_c$, $B$ to $H_cH_a$, and $C$ to $H_aH_b$ concur at

$$X(1300) = \left( \frac{1}{S_A(a^2(S_{AA} - S_{BC}) - S_A(S_B - S_C)^2)} : \cdots : \cdots \right)$$

on the circumcircle of $T$.

*Remark.* $X(1300)$ is the second intersection of the circumcircle with the line $EH$. 
Proposition 13. The triangle $H_a H_b H_c$ is orthologic with the pedal triangle of $P$ if and only if $P$ lies on the line

$$\frac{S_B - S_C}{a^2S_A}x + \frac{S_C - S_A}{b^2S_B}y + \frac{S_A - S_B}{c^2S_C}z = 0$$

joining the circumcenter $O$ to $X(394) = (a^2S_{AA} : b^2S_{BB} : c^2S_{CC})$.

If $P$ lies on this line,
(a) the perpendiculars from $H_a$, $H_b$, $H_c$ to the corresponding sides of the pedal triangle of $P$ are concurrent at a point on the conic

$$4S^2 \sum_{\text{cyclic}} S_{BC}(S_B - S_C)yz + (x + y + z) \left( \sum_{\text{cyclic}} a^2S_A(S_B - S_C)(S^2 - S_{AA})x \right) = 0,$$

with center at the nine-point center $N$;
(b) the perpendiculars from the vertices of the pedal triangle of $P$ to the corresponding sides of $H_a H_b H_c$ are concurrent at a point on the line

$$\sum_{\text{cyclic}} \frac{S_B - S_C}{S_A(a^2S_{AA} - a^2S_A(S_{BB} + S_{CC}) - S_{BC}(S_B - S_C)^2)}x = 0.$$

6.4. The triangle $O'_a O'_b O'_c$.

Proposition 14. The triangle $O'_a O'_b O'_c$ is orthologic to $ABC$.
(a) The perpendiculars from $O'_a$ to $BC$, $O'_b$ to $CA$, and $O'_c$ to $AB$ concur at the orthocenter $H$ of $T$.
(b) The perpendiculars from $A$ to $O'_b O'_c$, $B$ to $O'_c O'_a$, and $C$ to $O'_a O'_b$ concur at the triangle center

$$X(265) = \left( \frac{S_A}{S^2 - 3S_{AA}} : \frac{S_B}{S^2 - 3S_{BB}} : \frac{S_C}{S^2 - 3S_{CC}} \right)$$

(see Proposition 6).

Proposition 15. The triangle $O'_a O'_b O'_c$ is orthologic with the pedal triangle of $P$ if and only if $P$ lies on the Euler line. If $P$ lies on this line,
(a) the perpendiculars from $O'_a$, $O'_b$, $O'_c$ to the corresponding sides of the pedal triangle of $P$ are concurrent at a point on the conic

$$4S^2 \sum_{\text{cyclic}} a^2S_A(S_B - S_C)yz$$

$$+ a^2b^2c^2(x + y + z)(S_A(S_B - S_C)x + S_B(S_C - S_A)y + S_C(S_A - S_B)z) = 0,$$

with center $N$;
(b) the perpendiculars from the vertices of the pedal triangle of $P$ to the corresponding sides of $O'_a O'_b O'_c$ are concurrent at a point on the line

$$\sum_{\text{cyclic}} \frac{S_B - S_C}{a^2S_A - 2S_{BC}}x = 0.$$
6.5. The triangle $G'_aG'_bG'_c$.

**Proposition 16.** The triangle $G'_aG'_bG'_c$ is orthologic to $ABC$.

(a) The perpendiculars from $G'_a$ to $BC$, $G'_b$ to $CA$, and $G'_c$ to $AB$ concur at the triangle center

$$X(381) = \left( a^2 S_A + 4S_{BC} : b^2 S_B + 4S_{CA} : c^2 S_C + 4S_{AB} \right)$$
on the Euler line.

(b) The perpendiculars from $A$ to $G'_bG'_c$, $B$ to $G'_cG'_a$, and $C$ to $G'_aG'_b$ concur at the triangle center

$$X(265) = \left( \frac{S_A}{S^2 - 3S_{AA}} : \frac{S_B}{S^2 - 3S_{BB}} : \frac{S_C}{S^2 - 3S_{CC}} \right)$$

(see Proposition 6).

**Proposition 17.** The triangle $G'_aG'_bG'_c$ is orthologic with the pedal triangle of $P$ if and only if $P$ lies on the Euler line. If $P$ lies on this line,

(a) the perpendiculars from $G'_a$, $G'_b$, $G'_c$ to the corresponding sides of the pedal triangle of $P$ are concurrent at a point on the conic

$$6 \sum_{cyclic} a^2 S_A (S_B - S_C)yz + (x + y + z) \left( \sum_{cyclic} b^2 c^2 (S_B - S_C)x \right) = 0$$

with center at the centroid $G$,

(b) the perpendiculars from the vertices of the pedal triangle of $P$ to the corresponding sides of $G'_aG'_bG'_c$ are concurrent at a point on the line

$$\sum_{cyclic} \frac{S_B - S_C}{a^2 S_A - 2S_{BC}} x = 0.$$
7. The triangles $T(P)$ and $T(Q)$

**Proposition 18.** The triangles $T(P)$ and $T(Q)$ are perspective if and only if the line $PQ$ contains the circumcenter $O$. In this case, the triangles are homothetic at $O$.

**Proof.** Let $P = (x : y : z)$ and $Q = (u : v : w)$. The line containing $A_P$ and $A_Q$ has equation

$$a^2S_A(yv - wz)X + (c^2S_C(vx - uy) - S_B(S_C - S_A)(uz - wx))Y$$

$$+ (b^2S_B(uz - wx) + S_C(S_A - S_B)(vx - uy))Z = 0.$$ 

Similarly, we have the equations of the lines $B_PB_Q$ and $C_PC_Q$. The three lines are concurrent if and only if $f \cdot g = 0$, where

$$f : = (c^2S_Cv - b^2S_Bw)x + (a^2S_Aw - c^2S_Cu)y + (b^2S_Bu - a^2S_Av)z,$$

$$g : = (u + v + w)^2 \left( \sum_{\text{cyclic}} a^4S_{AAY}z \right)$$

$$- (x + y + z) \left( \sum_{\text{cyclic}} ((c^2S_Cv + b^2S_Bw)^2 - 4S_{ABC} \cdot S_{Auvw})x \right).$$

Note that

(i) the equation $f = 0$ represents the line $OQ$;

(ii) the equation $g = 0$ represents a conic with center $Q$. Since the conic also contains $Q$ and intersects the sidelines at imaginary points, its represents only the point $Q$.

From these we conclude that the triangles $T(P)$ and $T(Q)$ are perspective if and only if $O, P, Q$ are collinear. Each of the lines $A_PA_Q, B_PB_Q, C_PC_Q$ contains $O$. The perspector is $O$. \qed
Theorem 19. The triangle bounded by the lines $A_P A_Q$, $B_P B_Q$, $C_P C_Q$ has circumcenter $O$.

Proof. Since $O$ is the orthocenter of the triangle $T(P)$, the circles $OB_P C_P$, $OC_P A_P$, and $OA_P B_P$ have equal radii. Note that $O$ is also the incenter of each of triangles $T_a$, $T_b$, $T_c$, and these triangles are all similar to $T'$. Therefore, $\angle O A_P A_Q = \angle O B_P B_Q = \angle O C_P C_Q$. Let $A''B''C''$ be the triangle bounded by the lines $A_P A_Q$, $B_P B_Q$ and $C_P C_Q$. Applying the law of sines to triangles $OA''C_P$, $OB''A_P$, $OC''B_P$, we conclude that $OA'' = OB'' = OC''$.

Proposition 20. The triangles $T'(P)$ and $T'(Q)$ are perspective if and only if $Q$ lies on the line $HP$. If the condition is satisfied, the triangles are homothetic at $O$.

8. The triangles $T(P)$ and $T'(P)$

Theorem 21. The triangles $T(P)$ and $T'(P)$ are perspective if and only if $P$ lies on

(a) the Jerabek hyperbola or

(b) the line $\sum x (S_B - S_C) = 0$.

Remarks. (1) If $P$ lies on the Jerabek hyperbola, the perspector is the circumcenter $O$.

(2) The line in (b) contains the Jerabek center $X(125)$ and $X(122)$ (also $X(684)$, $X(1650)$, $X(2972)$). In this case the lines $A_P A'_P$, $B_P B'_P$, $C_P C'_P$ are parallel.

Theorem 22. Let $P$ be a point with coordinates $(x : y : z)$ in $T$.

(a) The circumcenter of $T(P)$ has coordinates $(x : y : z)$ in triangle $O'_a O'_b O'_c = T'(O) = A'_O B'_O C'_O$.

(b) The circumcenter of $T'(P)$ has coordinates $(x : y : z)$ in $O_a O_b O_c$, which is $T(Q)$ for

$$Q = X(1147) = (a^4 S_A (S^2 - S_{AA}) : b^4 S_B (S^2 - S_{BB}) : c^4 S_C (S^2 - S_{CC}))$$

Proof. (a) The point with coordinates $(x : y : z)$ with respect to $O'_a O'_b O'_c$ is

$$\overline{O} = (S_{BC}(4S^2 \cdot S_A - a^2 b^2 c^2)x + c^2 a^2 S_{CC} ay + a^2 b^2 S_{ABB} z$$

$$\quad : b^2 c^2 S_{BCC} x + S_{CA}(4S^2 \cdot S_B - a^2 b^2 c^2)y + a^2 b^2 S_{ABB} z$$

$$\quad : b^2 c^2 S_{BCC} x + c^2 a^2 S_{CA} ay + S_{AB}(4S^2 \cdot S_C - a^2 b^2 c^2)z).$$

Its square distance from $A_P$ is

$$\frac{a^2 b^2 c^2 Q(x, y, z)}{16 S^2 \cdot (S_{ABC})^2 (x + y + z)^2},$$

where

$$Q(x, y, z) = b^2 c^2 S_{BB} S_{CC} x^2 + c^2 a^2 S_{CC} S_{AA} y^2 + a^2 b^2 S_{AA} S_{BB} z^2$$

$$- 2 S_{ABC}(a^2 S_{AA} y z + b^2 S_{BB} x z + c^2 S_{CC} x y).$$

The symmetry of $Q$ in $S_A$, $S_B$, $S_C$ and $x$, $y$, $z$ shows that this is the same if $A_P$ is replaced by $B_P$ or $C_P$. The point $\overline{O}$ therefore is the circumcenter of $A_P B_P C_P$. 

\qed
Theorem 23. (a) Triangles $O_aO_bO_c$ and $O'_aO'_bO'_c$ are perspective at the circumcenter $O$ of triangle $ABC$.
(b) The six circumcenters $O_a, O_b, O_c, O'_a, O'_b, O'_c$ are concyclic. The center of the circle containing them is $X(156)$, the nine-point center of the tangential triangle.

Proof. With respect to $O_aO_bO_c$,

$$O'_a = (-a^2S_{BC} : S_C(a^2S_A + 2S_{BC}) : S_B(a^2S_A + 2S_{BC})) = \left(\frac{-a^2}{a^2S_A + 2S_{BC}} : \frac{b^2}{b^2S_B} : \frac{c^2}{c^2S_C}\right);$$

$$O'_b = (S_C(b^2S_B + 2S_{CA}) : -b^2S_{BC} : S_A(b^2S_B + 2S_{CA})) = \left(\frac{a^2}{a^2S_A} : \frac{-b^2}{b^2S_B + 2S_{CA}} : \frac{c^2}{c^2S_C}\right);$$

$$O'_c = (S_B(c^2S_C + 2S_{AB}) : S_A(c^2S_C + 2S_{AB}) : -c^2S_{BC}) = \left(\frac{a^2}{a^2S_A} : \frac{b^2}{b^2S_B} : \frac{-c^2}{c^2S_C + 2S_{AB}}\right).$$

These expressions show that
(a) triangles $O_aO_bO_c$ and $O'_aO'_bO'_c$ are perspective at the orthocenter of $O_aO_bO_c$,
(b) $O'_a, O'_b, O'_c$ are on the the circumcircle of $O_aO_bO_c$. □

The orthocenter of $O_aO_bO_c$ is the circumcenter $O$. The circumcenter is $X(156)$, which is the nine-point center of the tangential triangle.

Proposition 24. The triangle $O_aO_bO_c$ is perspective with the inferior triangle of the tangential triangle at a point on its circumcircle.

Proof. With respect to $O_aO_bO_c$, the midpoint of the $A$-side of the tangential triangle has coordinates

$$(a^2S_{AA}(2S_{BB} - 7S_{BC} + 2S_{CC}) + S_{ABC}(3S_{BB} - 2S_{BC} + 3S_{CC}) + a^2S_{BB}S_{CC}) : a^2b^2S_A(e^2S_C - 2S_{AB}) : c^2a^2S_A(b^2S_B - 2S_{CA});$$
similarly for the midpoints of the other two sides of the tangential triangle. From these coordinates, it is clear that the medial triangle of the tangential triangle is perspective to \(O_aO_bO_c\) at the point with coordinates
\[
\left( \frac{a^2}{a^2S_A - 2S_{BC}} : \frac{b^2}{b^2S_B - 2S_{CA}} : \frac{c^2}{c^2S_C - 2S_{AB}} \right).
\]
Since triangle \(O_aO_bO_c\) is similar to \(T\), this perspector is a point on the circumcircle of \(O_aO_bO_c\). It is the isogonal conjugate (with respect to \(O_aO_bO_c\)) of the infinite point of the Euler line of the triangle.

**Remark.** With respect to \(T\), this perspector has coordinates
\[
(a^2(S_{AA}(S_{BB} + S_{BC} + S_{CC}) - S_{BB}S_{CC}) : \cdots : \cdots).
\]

See Proposition 8.

**Proposition 25.** The triangle \(O_a'O_b'O_c'\) is perspective with the tangential triangle at
\[
X(195) = (a^2(-3a^2S_A^3 + (5S_{BB} + 6S_{BC} + 5S_{CC})S_{AA} + 9a^2S_{ABC} + 4S_{BB}S_{CC}) : \cdots : \cdots).
\]

**Remark.** \(X(195)\) is the circumcenter of the triangle of reflections (see [3]).

9. A family of circumcircles of \(T(P)\) containing the Euler reflection point

**Proposition 26.** Let \(P = (x : y : z)\). The circumcircle of \(T(P) = A_PPBC_P\) contains the Euler reflection point \(E\) if and only if \(P\) lies on the conic
\[
a^4S_{AAY}z + b^4S_{BB}zx + c^4S_{CC}xy = 0.
\]

**Proof.** The coordinates of \(A_P, B_P, C_P\) are given at the beginning of §3. Computing the coordinates of \(E\) with respect to triangle \(A_PPBC_P\), we have
\[
x' : y' : z' = \frac{S_{BC}(\mu \left( \sum_{\text{cyclic}} a^4S_{AAY}z \right) + \tau(x + y + z) \left( \frac{S_{BC}x}{S_B - S_C} + \frac{2S_{AACY}}{b^2(S_A - S_B)} + \frac{2S_{ABB}z}{c^2(S_C - S_A)} \right))}{S_{CA}(\mu \left( \sum_{\text{cyclic}} a^4S_{AAY}z \right) + \tau(x + y + z) \left( \frac{2S_{BB}BCx}{\mu (S_A - S_B)} + \frac{S_{CA}y}{S_C - S_A} + \frac{2S_{ABB}z}{c^2(S_B - S_C)} \right))}
\]
\[
: \frac{S_{AB}(\mu \left( \sum_{\text{cyclic}} a^4S_{AAY}z \right) + \tau(x + y + z) \left( \frac{2S_{BC}CCx}{\mu (S_C - S_A)} + \frac{2S_{CC}AY}{b^2(S_B - S_C)} + \frac{S_{AB}z}{S_A - S_B} \right))}{S_{CA}(\mu \left( \sum_{\text{cyclic}} a^4S_{AAY}z \right) + \tau(x + y + z) \left( \frac{2S_{BB}BCx}{\mu (S_A - S_B)} + \frac{S_{CA}y}{S_C - S_A} + \frac{2S_{ABB}z}{c^2(S_B - S_C)} \right))}
\]

where
\[
\mu = a^2S_{BC} + b^2S_{CA} + c^2S_{AB} - 6S_{ABC},
\]
\[
\tau = (S_{BB} - S_{CC})(S_{CC} - S_{AA})(S_{AA} - S_{BB}).
\]

This is a point on the circumcircle of triangle \(A_PPBC_P\) (which is similar to \(ABC\)) if and only if
\[
(S_B + S_C)y'z' + (S_C + S_A)z'x' + (S_A + S_B)x'y' = 0.
\]
This reduces to
\[
\mu^2 \left( \sum_{\text{cyclic}} a^4 S_{AAyz} \right) \cdot G = 0,
\]
where
\[
G = \sum_{\text{cyclic}} (b^2 c^2 S_{BB} S_{CC} x^2 - 2S_{ABC} \cdot a^2 S_{AAyz})
= S^2 \left( \sum_{\text{cyclic}} S_{BC}^2 x^2 - 2S_{AB} S_{AB} yz \right) + S_{ABC}^2 (x + y + z)^2.
\]

\(G = 0\) is the equation of a conic with center \(O\). Since \(G(O)=0\), we conclude that \(G = 0\) defines only the point \(O\). Therefore, the circle \(A_P B_P C_P\) contains \(E\) if and only if \(P\) lies on the circumconic \(\sum_{\text{cyclic}} a^4 S_{AAyz}\). \(\Box\)

**Theorem 27.** Let \(Q\) be a triangle center on the circumcircle. The circle \(Q_a Q_b Q_c\) contains the Euler reflection point \(E\).

**Proof.** Let \(Q = \left( \frac{a^2}{(b^2-c^2)(a^2+t)} : \frac{b^2}{(c^2-a^2)(b^2+t)} : \frac{c^2}{(a^2-b^2)(c^2+t)} \right)\) be a triangle center on the circumcircle. For
\[
P = \left( \frac{a^2}{(b^2-c^2)(a^2+t)} : \frac{b^2}{(c^2-a^2)(b^2+t)} : \frac{c^2}{(a^2-b^2)(c^2+t)} \right)
\]
we have \(Q_a = A_P\), \(Q_b = B_P\), \(Q_c = C_P\). The circle \(Q_a Q_b Q_c\) is the same as \(A_P B_P C_P\). With respect to triangle \(O'_a O'_b O'_c\), the center of \(A_P B_P C_P\) has coordinates given above. Therefore the locus of the center of \(Q_a Q_b Q_c\) is the circle \(O'_a O'_b O'_c\), which is the nine-point circle of the tangential triangle of triangle \(ABC\).

Note that \(P\) lies on the conic \(\sum_{\text{cyclic}} a^4 S_{AAyz} = 0\). By Proposition 26, the circle \(A_P B_P C_P\) contains the Euler reflection point \(E\). \(\Box\)

**Theorem 28.** The circumcircle of \(T'(P)\) contains the Euler reflection point if and only if \(P\) lies on the circumcircle.

**References**


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Archimedean Circles Related to the Schoch Line

Hiroshi Okumura

Abstract. We give several Archimedean circles of the arbelos related to the Schoch line.

Let us consider an arbelos consisting of three semicircles $\alpha$, $\beta$ and $\gamma$ with diameters $AO$, $BO$ and $AB$, respectively, where $O$ is a point on the segment $AB$. Thomas Schoch has considered the circles $\alpha'$ and $\beta'$ with centers $A$ and $B$ and passing through the point $O$. He has found that the circle touching the two circles externally and the semicircle $\gamma$ internally is Archimedean [1]. The perpendicular to $AB$ from the center of this circle is called the Schoch line (see Figure 1). In this note we give several Archimedean circles touching this line or one of the two circles.

Let $a$ and $b$ be the radii of $\alpha$ and $\beta$, respectively. The radius of Archimedean circles is $\frac{ab}{a+b}$, which is denoted by $r_A$. We use a rectangular coordinate system with origin $O$ such that the points $A$ and $B$ have coordinates $(2a, 0)$ and $(-2b, 0)$, respectively, where we assume that all the semicircles are constructed in the region $y > 0$. Let $s = r_A \cdot \frac{b-a}{b+a}$. The Schoch line is expressed by the equation $x = s$ (see [2]). If $b > a$, then $s > 0$, and the Schoch line intersects the semicircle $\alpha$.

Let $O_\alpha$ and $O_\beta$ be the centers of $\alpha$ and $\beta$ respectively, and $\mu$ the circle with $O_\alpha O_\beta$ as diameter (see Figure 2).

Theorem 1. (1) The two circles each touching the circle $\mu$ and $\alpha$ (respectively $\beta$), all externally, and the Schoch line from the side opposite to $B$ (respectively $A$) are Archimedean.

(2) If $b > a$, then the circle touching the circle $\mu$ internally, $\alpha$ externally, and the Schoch line from the side opposite to $A$ is Archimedean.
Proof. Let $x$ be the radius of the circle touching the circle $\mu$ and $\alpha$ externally, and the Schoch line from side opposite to $B$ in (1). By the Pythagorean theorem,

$$(a + x)^2 - (s + x - a)^2 = \left(\frac{a + b}{2} + x\right)^2 - \left(s + x - \frac{a - b}{2}\right)^2.$$ 

Solving the equation for $x$, we get $x = r_A$, i.e., the circle is Archimedean. The rest of the theorem is proved similarly.  \hfill \Box

Let $L_\alpha$ be the perpendicular to $AB$ from the point of intersection of $\gamma$ and $\alpha'$. The line $L_\beta$ is defined similarly. Each of the two lines touches one of the twin circles of Archimedes [1] (see Figure 3). The proof of the following theorem is similar to that of Theorem 1, and is omitted.

**Theorem 2.** Each of the circles touching $\alpha$ externally, $\alpha'$ internally, and $L_\alpha$ from the side opposite to $O$ (respectively $\beta$, $\beta'$ and $L_\beta$) is Archimedean.

**References**


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Optimal Packings of Two Ellipses in a Square

Thierry Gensane and Pascal Honvault

Abstract. For each real number $E$ in $[0, 1]$, we describe the densest packing $P_E$ of two non-overlapping congruent ellipses of aspect ratio $E$ in a square. We find three different patterns as $E$ belongs to $[0, 1/2]$, $[1/2, E_0]$ where $E_0 = \sqrt{(6\sqrt{3} - 3)/11}$, and $[E_0, 1]$. The technique of unavoidable sets – used by Friedman for proving the optimality of square packings – allows to prove the optimality of each packing $P_E$.

1. Introduction

We consider the following generalization of the classical disk packing problem in a compact convex domain $K$: Let $n \in \mathbb{N}$ and $E \in [0, 1]$, what is the densest packing of $n$ non-overlapping congruent ellipses of aspect ratio $E$ in $K$?

In this paper, we describe for each aspect ratio $E$ in $[0, 1]$, the densest packing $P_E$ of two congruent unit ellipses of aspect ratio $E$ in the square $K = [0, 1]^2$ and we prove the optimality of these packings. In Figure 1, we display six representative optimal packings of two congruent ellipses. For $E = 1$, the optimal packing $P_1$ is composed of two disks lying in opposite corners, see [4] for a large list of dense packings of congruent disks in the square. An introductory bibliography on disk packing problems can be found in [1, 3]. When $E$ decreases from 1 to $E_0 = \sqrt{(6\sqrt{3} - 3)/11} \approx 0.8198$, the ellipses of optimal packings $P_E$ flatten by keeping a constant tilted angle equals to $-\pi/4$. For $E \in [1/2, E_0]$, the angle of the two ellipses of $P_E$ decreases and when $E = 1/2$ the ellipses reach a third side of the square. When $E$ decreases from 1/2 to 0, the ellipses slide along the sides and move towards the diagonal.

In all the following we consider only unit ellipses that is, ellipses whose equation is $x^2 - y^2/E^2 = 1$ when their major and minor axes coincide with the cartesian axes. We can reformulate our problem: What is the side length $s_2(E)$ of the smallest square which contains two non-overlapping unit ellipses of aspect ratio $E$?

In order to prove the optimality of square packings, Friedman [2] used sets of unavoidable points. We adapt his definition to the case of ellipse packings: Let $E \in [0, 1]$ and let $P$ be a set of $n - 1$ points in the square $K_s = [0, s]^2$ with $s > 0$. We say that $P$ is a set of unavoidable points in $K_s$ if any unit ellipse of aspect ratio $E$ in $K_s$ contains an element of $P$ (possibly on its boundary). If $P$ is a set of unavoidable points in $K_s$, then $s_n(E) \geq s$. For the convenience of the reader, we recall the proof given by Friedman: Shrinking the square $K_s$ by a factor of $1 - \varepsilon/s$ gives a set $P'$ of $n - 1$ points in a square $K_{s-\varepsilon}$ so that any unit ellipse in $K_{s-\varepsilon}$ contains an element of $P'$ in its interior. Therefore no more than
$n - 1$ non-overlapping unit ellipses can be packed into a square of side $s - \varepsilon$, and $s_n(E) > s - \varepsilon$. Since this is true for all $\varepsilon > 0$, we must have $s_n(E) \geq s$. The upper bound $s_n(E) \leq s$ is obtained by constructing a packing of $n$ non-overlapping unit ellipses in $K_s$.

In the case of $n = 2$ ellipses and in order to get $s_2(E) \geq s$, it suffices to show that the center $\Omega$ of $K_s = [0, s]^2$ belongs to each unit ellipse $e \subset K_s$ of aspect ratio $E$. In fact, we will consider only unit ellipses $e_{\alpha} = e_{(\lambda, \mu), \alpha, E}$ centered at $(\lambda, \mu)$ with $\lambda > 0, \mu > 0$, tilted at an angle $\alpha \in [-\pi/2, -\pi/4]$ and which are tangent to the axes $x = 0$ and $y = 0$:

**Fact 1.** Let $K_s = [0, s]^2$ be a square of side length $s$ and $\Omega = (s/2, s/2)$ its center. If for all $\alpha \in [-\pi/2, -\pi/4]$, the ellipse $e_\alpha$ contains the point $\Omega$, then all unit ellipses $e$ included in $K_s$ contain $\Omega$.

This fact is trivially obtained by contraposition (if a unit ellipse $e \subset K_s$ does not contain the point $\Omega$, we apply a translation followed by a reflection or a rotation and we get an ellipse $e_\alpha$ with $\alpha \in [-\pi/2, -\pi/4]$ which does not contain the point $\Omega$). As we want to find the minimal value of $s$ such that each ellipse $e_\alpha \subset K_s$ contains the center $\Omega$, we consider the intersection points $I = (x_I(\alpha), x_I(\alpha))$ and $J$ of the diagonal $y = x$ and the ellipse $e_\alpha$, the abcissa of $I$ being larger than the one of $J$. In Section 2 and 3 we will prove that:

- If $0 < E \leq 1/2$, there exists a unique $\alpha_0 \in [-\pi/2, -\pi/4]$ such that $x_I(\alpha_0) = \mu(\alpha_0) = \mu_0$. The center $\Omega = (\mu_0, \mu_0)$ is an unavoidable point in $K_{2\mu_0}$ and then $s_2(E) \geq 2\mu_0$. The square $K_{2\mu_0}$ is displayed on the right hand side of Figure 2.
- If $1/2 < E < E_0$, the abcissa $x_I(\alpha)$ has a minimum value for a unique $\alpha_0 \in [-\pi/2, -\pi/4]$. The center $\Omega = (x_I(\alpha_0), x_I(\alpha_0))$ is an unavoidable point.
Optimal packings of two ellipses in a square

[Diagrams showing three ellipses in different configurations]

\[ \alpha = -\frac{\pi}{2}, \quad 0 < \alpha < \alpha_0, \quad \alpha = \alpha_0 \]

Figure 2. Three ellipses \( e_\alpha \subset K_{2\mu} \) in the case \( E \leq \frac{1}{2} \) in \( K_{2xI(\alpha_0)} \) and then \( s_2(E) \geq 2xI(\alpha_0) \). The three squares in Figure 3 represent \( K_{2xI(\alpha_0)} \).

\[ \alpha = -\frac{\pi}{2} < \alpha_0, \quad \alpha = \alpha_0, \quad \alpha = -\frac{\pi}{4} > \alpha_0 \]

Figure 3. For \( \frac{1}{2} < E < E_0 \) and if \( \alpha \neq \alpha_0 \), the center of \( K_{2xI(\alpha_0)} \) belongs to the interior of \( e_\alpha \).

- If \( E_0 \leq E \leq 1 \), the abscissa \( x_I(\alpha) \) is decreasing for \( \alpha \in [-\pi/2, -\pi/4] \). The center \( \Omega = (x_I(-\pi/4), x_I(-\pi/4)) \) is an unavoidable point in \( K_{2xI(-\pi/4)} \) and then \( s_2(E) \geq 2xI(-\pi/4) \), see Figure 4 in Section 3.

We finish the paper by remarking that among all the optimal packings \( \mathcal{P}_E \), the densest optimal packings of two congruent ellipses in the square is \( \mathcal{P}_{1/2} \).

2. Technical lemmas.

First we precise the coordinates of the center of the ellipse \( e_\alpha \) and the parametrization of \( e_\alpha \) used in Lemma 3.

**Lemma 1.** (a) The coordinates of the center of the ellipse \( e_\alpha \) are equal to

\[ \lambda = \sqrt{\cos^2 \alpha + E^2 \sin^2 \alpha} \quad \text{and} \quad \mu = \sqrt{\sin^2 \alpha + E^2 \cos^2 \alpha}. \]  

(1)
The function $\mu$ is decreasing for $\alpha \in [-\pi/2, -\pi/4]$ and we have $\mu(-\pi/2) = 1$ and $\mu(-\pi/4) = \sqrt{(1+E^2)/2}$.

(b) We have

$$
(2\lambda\mu)^2 = 4E^2 + (1 - E^2)\sin^2 2\alpha. 
$$

(c) The ellipse $e_\alpha$ can be parameterized by

$$
e_\alpha(t) = \left( \begin{array}{ll} 2\lambda\cos^2 \left( \frac{t+\phi}{2} \right) \\ 2\mu\cos^2 \left( \frac{t+\psi}{2} \right) \end{array} \right),
$$

where the angles $\phi \in [-\pi/2, 0]$ and $\psi \in [-\pi, -\pi/2]$ are the respective arguments of the complex numbers $\cos \alpha + iE \sin \alpha$ and $\sin \alpha - iE \cos \alpha$.

Proof. (a) Let us consider the parametrization of the ellipse $e_\alpha$

$$
e_\alpha(t) = \left( \begin{array}{ll} x(t) \\ y(t) \end{array} \right) = \left( \begin{array}{ll} \lambda \\ \mu \end{array} \right) + \left( \begin{array}{ll} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{array} \right) \left( \begin{array}{ll} \cos t \\ E\sin t \end{array} \right). 
$$

Let us recall that the orthoptic curve of $e_\alpha$, i.e the locus of all points where the curve’s tangents meets at right angles, is the circle centered at $\omega = (\lambda, \mu)$ with radius $\sqrt{1+\lambda^2}$. Since the axes $x = 0$ and $y = 0$ are orthogonal, the origin $(0, 0)$ belongs to this circle and we have

$$
\lambda^2 + \mu^2 = 1 + E^2. 
$$

The ellipse $e_\alpha$ touches tangentially the axe $x = 0$. Therefore for some $t$ we have $x(t) = x'(t) = 0$, which gives

$$
\lambda = -\cos \alpha \cos t + E \sin \alpha \sin t, 
$$

$$
0 = -\cos \alpha \sin t - E \sin \alpha \cos t. 
$$

By adding the squares of equations (6-7), we find $\lambda^2 = \cos^2 \alpha + E^2 \sin^2 \alpha$. The value of $\mu$ comes from (5).

(b) We have

$$
(\lambda\mu)^2 = (\cos^2 \alpha + E^2 \sin^2 \alpha)(\sin^2 \alpha + E^2 \cos^2 \alpha) = E^2(\cos^4 \alpha + \sin^4 \alpha) + (1 + E^4) \cos^2 \alpha \sin^2 \alpha = E^2(1 - 2\sin^2 \alpha \cos^2 \alpha) + (1 + E^4) \cos^2 \alpha \sin^2 \alpha,
$$

which gives the result.

(c) By definition of $\phi$, we have $\lambda(\cos \phi + i \sin \phi) = \cos \alpha + iE \sin \alpha$. We get

$$
x(t) = \lambda + \cos \alpha \cos t - \sin \alpha E \sin t = \lambda + \lambda \cos(t + \phi) = 2\lambda \cos \left( \frac{t + \phi}{2} \right).
$$

The expression of $y(t)$ is obtained similarly. \quad \square

In the proof of Lemma 3, we change the variable $\alpha$ to the variable $T$ that we now define:

Lemma 2. Let us consider the angles $\phi$ and $\psi$ associated to the ellipse $e_\alpha$ and defined in Lemma 1. We set $\delta = \frac{\psi - \phi}{2} \in [-\pi/2, 0]$ and $T = -\cot \delta$. Then the real number $T$ decreases monotonically from 1 at $\alpha = -\pi/2$ to $E$ at $\alpha = -\pi/4$. 

Hence, for $\alpha$ (b) we find

$\frac{\mu e^{i(\psi - \varphi)}}{\lambda} = \frac{\sin(\alpha) - iE \cos(\alpha)}{\cos \alpha + iE \sin \alpha}$

$= \frac{1}{\lambda^2} \left( \sin \alpha \cos(1 - E^2) - iE \right)$.

By (8) and the formula $\tan u = (1 - \cos 2u) / \sin 2u$, we find

$\tan \left( \frac{\psi - \varphi}{2} \right) = \frac{1 - \sin 2\alpha(1 - E^2)}{2\lambda \mu - \sin 2\alpha(1 - E^2)} = \frac{2\lambda \mu - \sin 2\alpha(1 - E^2)}{-2E}$.

With Lemma 1 (b) we find

$\tan \left( \frac{\psi - \varphi}{2} \right) = \frac{1}{2E} \left( -\sqrt{4E^2 + (1 - E^2)^2 \sin^2 2\alpha + (1 - E^2) \sin 2\alpha} \right)$.

As $\sin 2\alpha$ decreases monotonically from 0 to -1 on the interval $[-\pi/2, -\pi/4]$, we find that $\tan ((\psi - \varphi)/2)$ decreases from $-1$ to $1/\tan((\psi - \varphi)/2)$ decreases from 1 to $E$.

Let us recall that $I = (x_I, x_I)$ is the intersection point of the diagonal $y = x$ and the ellipse $e_\alpha$ with a maximal $x_I$.

**Lemma 3.** Let us consider $E_0 = \sqrt{(6\sqrt{3} - 3)/11} \approx 0.8198$.

(a) If $E \in [E_0, 1]$, $x_I$ is decreasing for $\alpha \in [-\pi/2, -\pi/4]$.

(b) If $E \in [1/2, E_0]$, the function $x_I$ reaches a unique minimal value for a unique real number $\alpha_0 \in [-\pi/2, -\pi/4]$.

(c) If $E \in [0, 1/2]$, the function $x_I$ is increasing for $\alpha \in [-\pi/2, -\pi/4]$.

**Proof.** First we prove that

$$x_I(\alpha) = \frac{E}{\sqrt{T} \left( \sqrt{ET^2 + (1 + E^2)T - E} \right)}$$

where $T = -\cot \delta$ has been defined in Lemma 2. We use the parametrization of the ellipse $e_\alpha$ given by (3) and we look for some $t \in [0, 2\pi]$ such that

$$\sqrt{\lambda} \cos \left( \frac{t + \varphi}{2} \right) = \varepsilon \sqrt{\mu} \cos \left( \frac{t + \psi}{2} \right)$$

where $\varepsilon = \pm 1$. Since the point $I$ occurs for some $t \in [0, \pi]$ (and $J$ for some $t \in [\pi, 2\pi]$), we have $(t + \varphi)/2 \in [-\pi/4, \pi/2]$, $(t + \psi)/2 \in [-\pi/2, \pi/4]$ and we get $\varepsilon = +1$ because the two cosines in (11) are positive. In this equality we expand $\cos((t + \psi)/2) = \cos((t + \varphi)/2 + \delta)$ and we find

$$\sqrt{\lambda} \cos \left( \frac{t + \varphi}{2} \right) = \sqrt{\mu} \left( \cos \left( \frac{t + \varphi}{2} \right) \cos \delta - \sin \left( \frac{t + \varphi}{2} \right) \sin \delta \right)$$
which gives
\[
\tan \left( \frac{t + \varphi}{2} \right) = \frac{\sqrt{\mu} \cos \delta - \sqrt{\lambda}}{\sqrt{\mu} \sin \delta}
\]
and
\[
x_I = 2\lambda \cos^2 \left( \frac{t + \varphi}{2} \right) = \frac{2\lambda}{1 + \tan^2 \left( \frac{t + \varphi}{2} \right)} = \frac{2\lambda \mu \sin^2 \delta}{\lambda + \mu - 2\sqrt{\lambda \mu} \cos \delta}.
\] (12)

By (8) and Lemma 1 (b) and since \( \sin \) and \( \cos \) are increasing with respect to \( P \), the discriminant of \( 0 \) and \( P \) is \( E \). The previous equality gives \( 2\lambda \mu = -2E/\sin 2\delta \) and by (5) we have \( \lambda + \mu = (1 + E^2 + 2\lambda \mu)^{1/2} \). Substituting these values in (12) we find
\[
x_I = \frac{-E \sin \delta}{\cos \delta \left( \sqrt{1 + E^2 - \frac{E}{\sin \delta} \cos \delta} - \sqrt{-2E \cos \delta} \right)}
\]
\[
= \frac{E}{\sqrt{T} \left( \sqrt{(1 + E^2)T + \frac{E}{\sin^2 \delta}} - \sqrt{T \sqrt{2E}} \right)}.
\]

It remains to use \( 1/\sin^2 \delta = 1 + T^2 \) and we get (10). Let us denote by \( f(T) \) the denominator of the right hand side of (10). We find
\[
f'(T) = \frac{g(T) - h(T)}{2\sqrt{T} \sqrt{ET^2 + (1 + E^2)T + E}},
\]
where \( h(T) = 3\sqrt{2E} T \sqrt{ET^2 + (1 + E^2)T + E} \) and \( g(T) = 3ET^2 + 2(1 + E^2)T + E \). Since the functions \( g(T) \) and \( h(T) \) are positive, the sign of \( f'(T) \) is equal to the one of the polynomial \( P(T) = g^2(T) - h^2(T) \), that is
\[
P(T) = -9E^2T^4 - 6E(1 + E^2)T^3 + 4(1 + E^2)T^2 + 4E(1 + E^2)T + E^2.
\] (13)

We get \( P'(T) = -36E^2T^3 - 18E(1 + E^2)T^2 + 8(1 + E^2 + 1)T + 4E(1 + E^2) \).
The discriminant of \( P''(T) = -4(27E^2T^2 + 9E(1 + E^2)T - 2(E^4 - E^2 + 1)) \) is \( \Delta = 16 \cdot 27E^2(11E^4 - 2E^2 + 11) > 0 \). We remark that \( P''(0) > 0 \) and \( \lim_{T \to \infty} P''(T) = -\infty \), then \( P''(T) \) has a unique positive root \( T_2 \). Since \( P'(0) > 0 \) and \( \lim_{T \to \infty} P'(T) = -\infty \), the polynomial \( P'(T) \) has a unique root \( T_1 > T_2 \) and \( P'(T) \geq 0 \) for all \( T \in [0, T_1] \). Finally, \( P(0) = E^2 > 0 \) implies that the polynomial \( P(T) \) has a unique positive root \( T_0 \).

Moreover, \( P(E) = -E^2(11E^4 + 6E^2 - 9) \) vanishes at a unique positive value \( E_0 = \sqrt{(-3 + 6\sqrt{3})/11} \). We remark that \( T_0 = E_0 \) if and only if \( E = E_0 \). In the three following cases we conclude with Lemma 2:

(a) If \( E \in [E_0, 1] \), we have \( P(E) \leq 0 \). Then \( T_0 \leq E \leq 1 \) which implies that \( P(T) \leq 0 \) for all \( T \in [E, 1] \). So \( f(T) \) is decreasing on \([E, 1]\) and then \( x_I \) is increasing with respect to \( T \).
3. Calculation of $s_2(E)$

Now we can describe the various optimal packings of two ellipses in the square and the corresponding side lengths $s_2(E)$.

**Theorem 4.** If $E \leq 1/2$, then

$$s_2(E) = \sqrt{(1 + E)^2 + \sqrt{(1 + E)^4 - 8E^2}}.$$  \hspace{1cm} (14)

The minimum value $s_2(E)$ is obtained for two parallel ellipses $e_1 = e_{\alpha_0}$ and $e_2$ with

$$\alpha_0 = -\arccos \frac{1}{2} \sqrt{\frac{4 - s_2^2(E)}{1 - E^2}},$$  \hspace{1cm} (15)

and where $e_2$ is the reflection of $e_1$ through the center of the square.

**Proof.** If $\alpha = -\pi/2$, the center $\Omega = (\mu, \mu)$ of $K_{2\mu}$ does not belong to $e_\alpha$ (except for $E = 1/2$). If $\alpha = -\pi/4$, the center $\Omega$ is also the center of the ellipse $e_\alpha$.

We know by Lemma 1 (a) and Lemma 3 (c) that $x_I$ is increasing for $\alpha \in [-\pi/2, -\pi/4]$. Then there exists a unique angle $\alpha_0 \in [-\pi/2, -\pi/4]$ such that $I = \Omega_0 = (\mu_0, \mu_0)$ with $\mu_0 = \mu(\alpha_0)$. We note that the center $\Omega_0$ belongs to the boundary of the ellipse $e_{\alpha_0}$, see Figure 2. Let us show that in the square $K_{2\mu_0}$, the center $\Omega_0$ is an unavoidable point. By Fact 1, it suffices to show that any ellipses $e_\alpha$ included in $K_{2\mu_0}$ contain $\Omega_0$:

- If $\alpha < \alpha_0$, the ellipse $e_\alpha$ is not contained in $K_{2\mu_0}$ because it intersects the upper side $y = 2\mu_0$ (except for $\alpha \in [-\pi/2, -\pi/4]$).
- If $\alpha > \alpha_0$, the point $\Omega_0$ belongs to the interior of the ellipse $e_\alpha$ because $x_I$ is increasing for $\alpha \in [\alpha_0, -\pi/4]$.

It remains to calculate $\alpha_0$ and $\mu_0$. First, we show that

$$\mu_0 - \lambda_0 = \frac{E}{\mu_0}.$$  \hspace{1cm} (16)

We have by (4) that $e_\alpha(t) = (\mu, \mu)$ if and only if

$$\mu - \lambda = \cos \alpha \cos t - E \sin \alpha \sin t,$$

$$0 = \sin \alpha \cos t + E \cos \alpha \sin t.$$

Substituting $-E \cos \alpha \sin t/ \sin \alpha$ for $\cos t$ in the first equality, we find

$$-E \sin(t) = (\mu - \lambda) \sin \alpha,$$

$$\cos t = (\mu - \lambda) \cos \alpha,$$
which implies $E^2 = (\mu - \lambda)^2(\sin^2 \alpha + E^2 \cos^2 \alpha) = (\mu - \lambda)^2 \mu^2$, and then (16) since $\mu \geq \lambda$ for $\alpha \in [-\pi/2, -\pi/4]$. By (5) we get for $\alpha = \alpha_0$,

$$1 + E^2 + 2\lambda_0 \mu_0 = (\lambda_0 \mu_0)^2 = (\lambda_0 - \mu_0 + 2\mu_0)^2 = \left(2\mu_0 - \frac{E}{\mu_0}\right)^2.$$  

Since (16) implies $\mu_0^2 - E = \lambda_0 \mu_0$, we have

$$2(\mu_0^2 - E) = 4\mu_0^2 + \frac{E^2}{\mu_0^2} - 4E - (1 + E^2).$$

This equation in $\mu_0^2$ leads to

$$4\mu_0^2 = (1 + E)^2 + \varepsilon \sqrt{(1 + E)^4 - 8E^2},$$

where $\varepsilon = \pm 1$. The case $\varepsilon = -1$ leads to $4\lambda_0 \mu_0 = 4\mu_0^2 - 4E = (1 - E)^2 - \sqrt{(1 + E)^4 - 8E^2} \leq 0$ what is impossible.

Then $s_2(E) \geq 2\mu_0 = \sqrt{(1 + E)^2 + \sqrt{(1 + E)^4 - 8E^2}}$. We can pack in $K_{2\mu_0}$ the reflection of $e_{\alpha_0}$ through $\Omega_0$ and we get the equality (14). We finally obtain the angle (15) by considering $\mu_0^2 = \sin^2 \alpha_0 + E^2 \cos^2 \alpha_0$ and $s_2(E) = 2\mu_0$.

**Theorem 5.** If $1/2 < E < E_0$, then

$$s_2(E) = \frac{2E}{\sqrt{T_0 \left(\sqrt{ET_0^2 + (1 + E^2)T_0} + E - T_0 \sqrt{2E}\right)},}$$

where $T_0$ is the unique positive root of (13). The minimum value $s_2(E)$ is obtained for two parallel ellipses $e_1 = e_{\alpha_0}$ and $e_2$ with

$$\alpha_0 = -\frac{1}{2} \left(\pi + \arcsin \frac{E(T_0^2 - 1)}{T_0(1 - E^2)}\right),$$

and where the ellipse $e_2$ is the reflection of $e_1$ through the center of the square.

**Proof.** We denote by $\alpha_0$ the unique angle $\alpha$ such that $T_0 = T(\alpha)$ and by $I_0$ the intersection of $e_{\alpha_0}$ with $y = x$. Since the continuous function $x_I$ is decreasing for $\alpha \in [-\pi/2, \alpha_0]$ and increasing for $\alpha \in [\alpha_0, -\pi/4]$, the point $I_0$ belongs to the interior of $e_\alpha$ if $\alpha \neq \alpha_0$. Then any ellipse $e_\alpha$ in $K_{2x_I}$ contains $I_0$. As Fact 1 gives that $I_0$ is an unavoidable point, we have $s_2(E) \geq 2x_{I_0}$. We can pack the reflection of the ellipse $e_{\alpha_0}$ through $I_0$ and we get $s_2(E) \leq 2x_{I_0}$. The value of $x_{I_0}$ is given by (10).

By (2), (9) and since $-1/T = \tan(\psi - \varphi)/2$, we have

$$(2\lambda \mu)^2 = \left(\frac{2E}{T} + (1 - E^2) \sin 2\alpha\right)^2 = 4E^2 + (1 - E^2)^2 \sin^2 2\alpha,$$

which gives $\sin 2\alpha = E(T^2 - 1)/((1 - E^2)T)$ and (18) for $2\alpha \in [-\pi, -\pi/2]$. □

**Theorem 6.** If $E_0 \leq E \leq 1$, then

$$s_2(E) = \sqrt{2} \left(\sqrt{1 + E^2} + E\right).$$ (19)
The minimum value $s_2(E)$ is obtained for two parallel ellipses $e_1 = e_\alpha$ and $e_2$ with $\alpha = -\pi/4$ and where $e_2$ is the reflection of $e_1$ through the center of the square.

Proof. We consider again the intersection point $I_0$ of the ellipse $e_{-\pi/4}$ and the diagonal $y = x$. Since $x_I$ is decreasing for $\alpha \in [-\pi/2, -\pi/4]$, any ellipse $e_\alpha$ in $K_{2x_{I_0}}$ contains $I_0$. As Fact 1 gives that $I_0$ is an unavoidable point, we get $s_2(E) \geq 2x_{I_0} = 2x_{(-\pi/4)} = \sqrt{2}(\sqrt{1+E^2} + E)$. As in the two previous cases, we can pack the reflection of the ellipse $e_{-\pi/4}$ through $I_0$ and we get the equality (19). □

It is not surprising that among all the optimal packings $\mathcal{P}_E$, the densest one is $\mathcal{P}_{1/2}$, see Figure 1. We denote by $d(E)$ the density of $\mathcal{P}_E$ and we have for all $E$ in $[0, 1]$,

$$d(E) = \frac{2\pi E}{s_2^2(E)}.$$

The formulas (14), (17), (19) for $s_2(E)$ give that $d(E)$ equals to

$$d_1(E) = \frac{2\pi E}{(1+E)^2+\sqrt{(1+E)^4-8E^2}} \text{ if } E \in [0, \frac{1}{2}],$$

$$d_2(E) = \frac{\pi E}{2E} T_0 \left( \sqrt{ET_0^2 + (1+E^2)T_0^2} + E - T_0 \sqrt{2E} \right)^2 \text{ if } E \in \left[ \frac{1}{2}, E_0 \right[,\)$$

$$d_3(E) = \frac{\pi E}{\left( \sqrt{1+E^2} + E \right)^2} \text{ if } E \in [E_0, 1].$$

The optimality of $\mathcal{P}_E$ for all $E$ on each interval $]0, 1/2], ]1/2, E_0[, [E, 1]$ implies the continuity of $d(E)$ on $]0, 1[$. It is easy to verify that $d_1(1/2) = d_2(1/2) = $
\[ d_2(E_0) = \frac{\pi E_0}{\left(\sqrt{1 + E_0^2} + E_0\right)^2}. \] We show that \( d(E) \) is increasing on \([0, 1/2]\) and decreasing on \([1/2, 1]\). For \( E \in [0, \frac{1}{2}] \), we get

\[ d'_1(E) = \frac{2\pi (1 - E^2)}{\sqrt{(E + 1)^4 - 8E^2 \left(1 + 2E + E^2 + \sqrt{(E + 1)^4 - 8E^2}\right)}} > 0 \]

and for \( E \in [E_0, 1] \),

\[ d'_3(E) = \frac{\pi \left(\sqrt{1 + E^2} - 2E\right)}{\sqrt{1 + E^2 \left(\sqrt{1 + E^2} + E\right)^2}} < 0. \]

In the case of Theorem 5, we have \( s_2(E) = 2E/f(T_0) \) and \( d_2(E) = (\pi/(2E))f^2(T_0) \).

Since \( T_0 = T_0(E) \) is a single root of (13), \( T_0(E) \) is differentiable at \( E \in [1/2, E_0[ \]
and we get

\[ d'_2(E) = \frac{\pi}{2E^2} \left(2Ef(T_0)f'(T_0)\frac{dT_0}{dE} - f^2(T_0)\right). \]

As \( f'(T_0) = 0 \), we obtain \( d'_2(E) < 0 \).

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The Diagonal Point Triangle Revisited

Martin Josefsson

Abstract. We derive a formula for the area of the diagonal point triangle belonging to a tangential quadrilateral in terms of the four tangent lengths, and prove a characterization for a tangential trapezoid.

1. Introduction

Consider a convex quadrilateral with no pair of opposite parallel sides. Let the two diagonals intersect at $E$ and the extensions of opposite sides intersect at $F$ and $G$. Then the triangle $EFG$ is called the diagonal point triangle or sometimes just the diagonal triangle (see Figure 1).

The significance of the diagonal point triangle is most evident in projective geometry, where it is studied in connection with the complete quadrilateral. It is for instance a well known property that the diagonal point triangle associated with a cyclic quadrilateral is self-conjugate.

In [5] we derived a formula for the area of the diagonal point triangle belonging to a cyclic quadrilateral in terms of the four sides. In this note we shall derive a formula for this triangle area in connection with a tangential quadrilateral (a quadrilateral with an incircle), but here it will be in terms of the tangent lengths instead. The tangent lengths $e$, $f$, $g$, $h$ in a tangential quadrilateral are defined to be the distances from the vertices to the points where the incircle is tangent to the sides (see Figure 2).

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2. More on the area of the diagonal point triangle

We will use Richard Guy’s version of Hugh ApSimon’s formula to derive a formula for the area of the diagonal point triangle belonging to a tangential quadrilateral. According to it (see [2]), the diagonal point triangle belonging to a convex quadrilateral $ABCD$ has the area

$$ T = \frac{2T_1 T_2 T_3 T_4}{K(T_1 T_2 - T_3 T_4)} \tag{1} $$

where $T_1, T_2, T_3, T_4$ are the areas of triangles $ABC, ACD, ABD, BCD$ respectively, and $K$ is the area of the quadrilateral.

**Theorem 1.** If $e, f, g, h$ are the tangent lengths in a tangential quadrilateral with no pair of opposite parallel sides, then the associated diagonal point triangle has the area

$$ T = \frac{2efghK}{|ef - gh||eh - fg|} $$

where

$$ K = \sqrt{(e + f + g + h)(efg + fgh + ghe + hef)} $$

is the area of the quadrilateral.

**Proof.** In a tangential quadrilateral, triangle $ABD$ has the area (see Figure 2)

$$ T_3 = \frac{1}{2}(e + f)(e + h) \sin A = (e + f)(e + h) \sin \frac{A}{2} \cos \frac{A}{2}. $$

According to Theorem 8 in [4], we have that

$$ \sin^2 \frac{A}{2} = \frac{efg + fgh + ghe + hef}{(e + f)(e + g)(e + h)}. $$
Using the trigonometric Pythagorean theorem yields
\[
\cos^2 \frac{A}{2} = 1 - \sin^2 \frac{A}{2} = \frac{(e + f)(e + g)(e + h) - (efg + fgh + ghe + hef)}{(e + f)(e + g)(e + h)} = \frac{e^2(e + f + g + h)}{(e + f)(e + g)(e + h)},
\]
Thus we get the subtriangle area
\[
T_3 = \frac{(e + f)(e + h)e\sqrt{(efg + fgh + ghe + hef)}(e + f + g + h)}{(e + f)(e + g)(e + h)} = \frac{eK}{e + g}.
\]
The last equality is due to formula (2) in [4] which gives the area of a tangential quadrilateral in terms of the tangent lengths. By symmetry we also have
\[
T_1 = \frac{fK}{f + h}, \quad T_2 = \frac{hK}{f + h}, \quad T_4 = \frac{gK}{e + g}.
\]
Combining the last four formulas gives
\[
T_1T_2 - T_3T_4 = \frac{fhK^2}{(f + h)^2} - \frac{egK^2}{(e + g)^2} = K^2 \left( \frac{(e + g)^2fh - eg(f + h)^2}{(e + g)^2(f + h)^2} \right).
\]
Expanding the numerator, canceling the two double products and factoring it yields
\[
(e + g)^2 fh - eg(f + h)^2 = e^2 f h + f g^2 h - e f^2 g - e g h^2 = (ef - gh)(eh - f g).
\]
Now by inserting the expressions for the triangle areas $T_1, T_2, T_3, T_4$ into (1), we get the area of the diagonal point triangle belonging to a tangential quadrilateral. Hence this is
\[
T = \frac{2efghK^4}{K(e + g)^2(f + h)^2} \cdot \frac{(e + g)^2(f + h)^2}{K^2(ef - gh)(eh - f g)}
\]
and the formula in the theorem follows by simplification and adding an absolute value to the denominator to cover all cases. \[\square\]

Except in projective geometry, where the notion of area is irrelevant, we have only found one source where the diagonal point triangle associated with a tangential quadrilateral is treated. This is in the old extensive paper [1] on quadrilateral geometry by Dostor. He derives a formula for this triangle area,\[^1\] but that formula is wrong. It states incorrectly (using other notations) that the area is given by
\[
T = \frac{4efghK}{(e^2 - g^2)(f^2 - h^2)}
\]
where $e, f, g, h$ are the tangent lengths and $K$ is the area of the quadrilateral. In [5] we concluded that Dostor’s formula for the area of the diagonal point triangle belonging to a cyclic quadrilateral is also wrong, and then derived the correct formula.

An interesting observation is that the correct formula in [5] for a cyclic quadrilateral has the exact same form as Dostor’s incorrect formula for a tangential quadrilateral.

\[^1\]Formula CCXVII on page 308 in [1]. We used $e, f, g, h$ in the citation of his formula to easily be able to compare it to Theorem 1 in this note.
quadrilateral (except for a factor of 2). But there is one big difference. The letters
used in Dostor’s formula stands for the tangent lengths in a tangential quadrilateral,
whereas in Theorem 1 in [5], we used $a$, $b$, $c$, $d$ which stands for the side lengths in
a cyclic quadrilateral.

3. A characterization of tangential trapezoids

If two opposite sides in the quadrilateral are parallel, then one of the points $F$
or $G$ becomes a point at infinity. Then the area of the diagonal point triangle is
infinite. This is equivalent to having a denominator in Theorem 1 that is zero, so
we get a necessary and sufficient condition for parallel opposite sides this way.
Hence opposite sides are parallel if and only if $ef = gh$ or $eh = fg$.

Now we give a second proof of these characterizations of a tangential trapezoid
(a trapezoid with an incircle; see Figure 3) where it is easier to determine which
pair of opposite sides that are parallel in each case.

**Theorem 2.** The opposite sides $AB$ and $CD$ in a tangential quadrilateral $ABCD$
with tangent lengths $e$, $f$, $g$, $h$ are parallel if and only if

$$eh = fg.$$  

The opposite sides $AD$ and $BC$ are parallel if and only if

$$ef = gh.$$  

**Proof.** According to Theorem 3 in [6], the opposite sides $AB$ and $CD$ in a convex
quadrilateral are parallel if and only if

$$\tan \frac{A}{2} \cdot \tan \frac{D}{2} = \tan \frac{B}{2} \cdot \tan \frac{C}{2}.$$  

Since $\tan \frac{A}{2} = \frac{r}{e}$, $\tan \frac{B}{2} = \frac{r}{f}$, $\tan \frac{C}{2} = \frac{r}{g}$ and $\tan \frac{D}{2} = \frac{r}{h}$ in a tangential quadrilateral with inradius $r$ (see Figure 2), we have that $AB$ and $CD$ are parallel if and only if

$$\frac{r}{e} \cdot \frac{r}{h} = \frac{r}{f} \cdot \frac{r}{g} \quad \Leftrightarrow \quad eh = fg.$$  

The second condition is proved in the same way using the angle characterization

$$\tan \frac{A}{2} \cdot \tan \frac{B}{2} = \tan \frac{C}{2} \cdot \tan \frac{D}{2}$$

for when $AD$ and $BC$ are parallel in a convex quadrilateral $ABCD$. □

In [4, p.129] we concluded that the inradius in a tangential trapezoid with tan-

gent lengths $e$, $f$, $g$, $h$ is given by

$$r = \sqrt[4]{efgh}.$$  

Combining this with Theorem 2 yields that the formula for the inradius in a tan-
gential trapezoid $ABCD$ with bases $AB$ and $CD$ can be simplified to

$$r = \sqrt{eh} = \sqrt{fg}.$$
These formulas can also be derived without the use of trigonometry. We give two other short proofs. Let the incircle be tangent to \( AB \) and \( CD \) at \( W \) and \( Y \) respectively, and \( I \) be the incenter (see Figure 3). Then triangles \( AWI \) and \( IYD \) are similar (AAA), so \( \frac{r}{h} = \frac{e}{r} \). Whence \( r^2 = eh \) and the second formula follows in a similar way. Another derivation starts by noting that the angle \( AID \) is a right angle when \( AB \parallel CD \). Using the Pythagorean theorem in the three triangles \( AWI \), \( DYT \) and \( AID \) yields \( AI^2 = e^2 + r^2 \), \( DI^2 = h^2 + r^2 \) and \( AI^2 + DI^2 = (e + h)^2 \). Combining these, we have \( r^2 + e^2 + r^2 + h^2 = (e + h)^2 \), and thus \( r^2 = eh \).

![Figure 3. A tangential trapezoid](image)

When the bases of the tangential trapezoid instead are \( AD \) and \( BC \), the corresponding formulas are

\[
r = \sqrt{ef} = \sqrt{gh}.
\]

They can be derived in the same way by any of the three methods used above.

As a final remark, we note that the related equality \( ef = fh \) gives another necessary and sufficient condition in tangential quadrilaterals. Two different proofs (both using other notations) were given in [7] and [3, p.104] that this is a characterization for when a tangential quadrilateral is also cyclic.

References

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A Simple Construction of an Inconic

Francisco Javier García Capitán

Abstract. We give a simple construction of an inscribed inconic with given perspector and its traces by constructing two extra points on the conic, without explicitly constructing the center of the conic.

Let $ABC$ be a given triangle. Consider a point $P$ with its cevian triangle $XYZ$. It is well known that there is an inscribed conic tangent to the sidelines at $X$, $Y$, $Z$. The center $Q$ of the conic is the inferior of the isotomic conjugate of $P$ (see [1, p.128]). The reflections of $X$, $Y$, $Z$ in $Q$ are three extra points on the conic. We give below a construction of two extra points $Y'$ and $Z'$ on the conic, without first locating the center $Q$.

**Proposition 1.** Given a triangle $ABC$ and a point $P$ with cevian triangle $XYZ$, let $X'$, $Y'$, $Z'$ be the second intersections of cevians $AX$, $BY$, $CZ$ and the inconic with perspector $P$. The cross ratios $(AX, PX')$, $(BY, PY')$, and $(CZ, PZ')$ are all equal to $4 : 1$ (see Figure 1a).

![Figure 1a](image1.png)

**Figure 1a**

**Figure 1b**

**Proof.** It is enough to prove this for the incircle of the equilateral triangle, regarded as the Steiner inellipse with perspector at the centroid (see Figure 1b). Then a projective transformation that maps our triangle $ABC$ and $P$ into the equilateral triangle with its centroid preserves these cross ratios. Now, for the equilateral triangle, clearly,

$$(AX, PX') = \frac{AP}{PX} : \frac{AX'}{X'X} = \frac{2}{1} : \frac{1}{2} = 4 : 1.$$
We simply use the usual method to construct a point with a given cross ratio. Let $D$ be the point that divides the segment $AP$ in the ratio $3:1$. We construct the intersection points $M = DZ \cap CA$ and $N = DY \cap AB$, draw parallels to $AX$ through $N$ and $M$ that intersect $BY$ and $CZ$ at $Y'$ and $Z'$ respectively (see Figure 2). If $J$ is the infinite point of the line $AP$, then

$$(BY,PY') = (AD,PJ) = \frac{AP}{PD} : \frac{AJ}{JD} = -4 : -1 = 4 : 1.$$ 

Therefore, $Y'$ lies on the inconic with perspector $P$. The same reasoning shows that $Z'$ lies on the same conic.

![Figure 2](image_url)

Reference


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On a Circle Containing the Incenters of Tangential Quadrilaterals

Albrecht Hess

Abstract. When we fix one side and draw different tangential quadrilaterals having the same side lengths but different angles we observe that their incenters lie on a circle. Based on a known formula expressing the incircle radius of a tangential quadrilateral by its tangent lengths, some older results will be presented in a new light and the equation of the before mentioned circle will appear. This circle encodes information about tangential and bicentric quadrilaterals that leads to an apparently new characterization of tangential quadrilaterals. Curiously enough, no trigonometric formulae are needed.

1. Introduction

Figure 1 shows a tangential quadrilateral $ABCD$, its incircle with incentre $I$ and radius $r$. Let $W, X, Y, Z$, be the tangency points and denote the tangent lengths $AW$ etc. by $e, f, g$ and $h$. While the tangent lengths determine the sides of a tangential quadrilateral $AB = a = e + f, BC = b = f + g, CD = c = g + h, DA = d = h + e$, the tangent lengths cannot be derived unambiguously from the sides - in contrast to the triangle, where the tangent lengths are $e = s - a, f = s - b, g = s - c$ with semiperimeter $s$.

The reason is that the condition $a + c = b + d$ for the sides in a tangential quadrilateral produces one degree of freedom in the solutions of the equations for the tangent lengths. This dichotomy between ambiguous and unambiguous tangent lengths in quadrilaterals and triangles continues to hold in polygons of an even and odd number of vertices.

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2. Inradius and distances of incenter from vertices of the quadrilateral

Given the tangent lengths \(e, f, g, h\), the radius \(r\) of the inscribed circle of the tangential quadrilateral is determined according to the formula

\[
r^2 = \frac{fgh + egf + efh + efg}{e + f + g + h}
\]  

(1)

cf. [8], [9, Lemma 2], [13, (1)], [16]. In [8] this equation is derived from

\[
\text{Im}((r + ei)(r + fi)(r + gi)(r + hi)) = \sigma_1 r^3 - \sigma_3 r = 0,
\]

(2)

wherein \(\sigma_k\) are the \(k\)-th degree elementary symmetric functions of \(e, f, g, h\). Formula (2) says that the four right-angled triangles with the legs \(r\) and \(e\), \(r\) and \(f\), \(r\) and \(g\), and \(r\) and \(h\) can be put together to form an angle of \(180^\circ\). From one pair of each of these triangles one can form a tangential quadrilateral as shown in Figure 2.

The generalization of formula (2) for arbitrary polygons can be resolved unambiguously for \(r > 0\) only in the case of the triangle and the tangential quadrilateral and leads for the triangle to the formula

\[
r^2 = \frac{efg}{e + f + g} = \frac{(s - a)(s - b)(s - c)}{s},
\]

which expresses the radius \(r\) of its incircle by the tangent lengths or the side lengths. This also gives a short proof of Heron’s triangle area formula [18] in the spirit of ”proofs without words”. From (1) we obtain

\[
AI^2 = r^2 + e^2 = \frac{fgh + egf + efh + efg}{e + f + g + h} + e^2 = \frac{(e + f)(e + g)(e + h)}{e + f + g + h},
\]

(3)

compare with [13, (5)]. Either multiply out the right hand side, or observe that \(\sigma_1(r^2 + e^2) = \sigma_3 + e^2\sigma_1\) is a monic third degree polynomial in \(e\) vanishing for \(e = -f, e = -g\) and \(e = -h\). Similar formulae hold for the other distances between the vertices and the incenter \(I\).

For later use, we note

\[
\frac{AI \cdot CI}{BI \cdot DI} = \frac{e + g}{f + h},
\]

(4)

which is an immediate consequence of (3). Similar formulae are in [7, Theorem 8].
3. Construction of the bicentric quadrilateral from its side lengths

With given side lengths $AB = a$, $BC = b$, $CD = c$, $DA = d$, that fulfill the condition $a + c = b + d$, we can construct in general different incongruent tangential quadrilaterals. Among them, the tangential quadrilateral with the greatest area, and therefore with the greatest radius of its incircle, is the bicentric quadrilateral with sides $a$, $b$, $c$, $d$, (cf. [3, p.135, (5)], [4, p. 238, (35)], [20]). The conditions for its constructability $a < b + c + d$ etc. as a cyclic quadrilateral from its side lengths are obviously satisfied and the construction can be carried out by use of the circle of Apollonius as in [11, pp. 82-83], or by joining two similar cyclic quadrilaterals such that they form a trapezoid with the same angles, but in a different order (see Figure 3). It seems that Bretschneider must have had this gluing method of similar geometric figures in mind when he produced this impressive series of formulas for triangles and quadrilaterals in [3] and [4]. Recently, Varverakis used this trick in his article [20] to have a short and visual proof of the maximum area property for cyclic quadrilaterals.

![Figure 3](image)

The triangle inequalities for the shaded triangle in Figure 3, i.e. the constructability conditions for the trapezoid and hence for the bicentric quadrilateral are satisfied. Consider, for example, this one

$$\left( d + \frac{bc}{a} \right) + \left( d \right) > \left( b + \frac{dc}{a} \right).$$

It is equivalent to $(c - a)(a + b + c - d) > 0$ and is satisfied if the dilated quadrilateral is glued to the larger of the sides $a$ and $c$. In the same way the other triangle inequalities are proved. For $c = a$, there is no need to paste two similar quadrilaterals together since cyclic quadrilaterals with a pair of equal opposite sides are trapezoids.

The area $K$ of a bicentric quadrilateral is $K = \sqrt{(s - a)(s - b)(s - c)(s - d)} = \sqrt{abcd}$ by Brahmagupta’s formula (cf. [3, p.135, (5)], [4, p. 238, (35)], [5, Th.).
3.22], [6, pp.62-63], [10], [11, Th. 109], [15]), and the radius of its incircle

\[ r_{\text{max}} = \frac{K}{s} = \frac{\sqrt{abcd}}{a+c} = \frac{\sqrt{abcd}}{b+d} \]  

is the largest among the incircle radii of all tangential quadrilaterals with sides \( a, b, c, d \).

4. Construction of a tangential quadrilateral from given tangent lengths and conclusions

If we want to construct a tangential quadrilateral with given tangent lengths \( e, f, g, h \), we could determine its inradius \( r \) with some intricate segment multiplications and divisions based on formula (1) and then assemble the right-angled triangles with the legs \( r \) and \( e \), \( r \) and \( f \), \( r \) and \( g \), and \( r \) and \( h \) as in Figure 2. Another way that takes us further and leads to the main formula (7), starts from the bicentric quadrilateral with side lengths \( a = e + f \), \( b = f + g \), \( c = g + h \), \( d = h + e \).

We get the same bicentric quadrilateral with sides \( a, b, c, d \) from different tangent lengths, the degree of freedom being the choice of one of them, e.g. \( e \). Then the other tangent lengths are

\[ BW = BX = f = a - e, \]
\[ CX = CY = g = b - a + e, \]
\[ DY = DZ = b = d - e; \]  

see Figure 4.

![Figure 4](image-url)

The right hand sides are positive, if \( a \) is the smallest side and \( e < a \). The green zones indicate where the endpoints of the segments of lengths \( e, f, g, h \) are
On a circle containing the incenters of tangential quadrilaterals

situated, the red zones are excluded. According to (1), the incircle radius for a
tangential quadrilateral with the tangent lengths from (6) is

\[ r^2 = \frac{(f + g)eh + (h + e)fg}{(f + g) + (h + e)} \]

\[ = \frac{be(d - c) + d(a - e)(b - a + e)}{b + d} \]

\[ = - \left( e - \frac{ad}{b + d} \right)^2 + \frac{abcd}{(b + d)^2} \]

\[ = - \left( e - \frac{ad}{s} \right)^2 + r^2_{\text{max}}. \]

(7)

This formula has the following consequences.

**Theorem 1.** A tangential quadrilateral \(ABCD\) with sides \(a, b, c, d\), and semiperimeter \(s = a + c = b + d\) is cyclic if and only if its tangent lengths are

\[ e = \frac{ad}{s}, \quad f = \frac{ab}{s}, \quad g = \frac{bc}{s}, \quad h = \frac{cd}{s}. \]

(8)

With these formulae, it is easy to deduce the characterization of bicentrics in
Problem 10804 in the MONTHLY [19] or in Hajja’s article [9, Lemma 1].

**Corollary 2.** A tangential quadrilateral is bicentric if and only if \(eg = fh\).

One direction is obvious from (8). If, on the other hand, \(eg = fh\), then \(eb = e(f + g) = f(h + e) = fd\), and together with \(e + f = a\), we find \(e = \frac{ad}{b + d}\),
\(f = \frac{ab}{b + d}\), i.e. (8).

From \(\frac{AC}{BD} = \frac{ad + bc}{ab + cd}\), a companion formula of Ptolemy’s theorem, valid for all
cyclic quadrilaterals (cf. [1, p.130], [2, Lemma 2], [11, p. 85]) we get immediately
from (8) the second criterion of [9].

**Corollary 3.** A tangential quadrilateral \(ABCD\) is bicentric only if

\[ \frac{AC}{BD} = \frac{e + g}{f + h}. \]

(9)

To prove the converse of Corollary 3, we invert the points \(A, B, C, D\), with
respect to the incircle and insert (4) into the formulae

\[ AC = A'C' \cdot \frac{AI \cdot CI}{r^2}, \quad BD = B'D' \cdot \frac{BI \cdot DI}{r^2} \]

for the distances of the images \(A', B', C', D'\), and get

\[ \frac{AC}{BD} = \frac{e + g}{f + h} \cdot \frac{A'C'}{B'D'}. \]

(10)

Formula (10) says that under condition (9) the diagonals of the parallelogram \(A'B'C'D'\) are of equal lengths, so that \(A'B'C'D'\) is a rectangle, hence cyclic,
and the pre-images \(A, B, C, D\), are cyclic too.

Writing (7) as \(r^2 + \left( e - \frac{ad}{s} \right)^2 = r^2_{\text{max}}\), we obtain the following Theorem 4, or
more visually, Theorem 4’ as announced in the abstract.
**Theorem 4.** For different choices of the tangent lengths $e$, $f$, $g$, $h$, having the same sums $a = e + f$, $b = f + g$, $c = g + h$, $d = h + e$, we make the side $AB$ of the bicentric quadrilateral $ABCD$ with sides $a$, $b$, $c$, $d$ to the $x$-axis of a coordinate system with origin $A$. The points with coordinates $(e, r)$ move on a circle around the point $W$ of tangency of the incircle with $AB$, which goes through the incenter $I$ of $ABCD$.

![Diagram](image)

**Theorem 4'.** When we fix one side and draw different tangential quadrilaterals having the same side lengths by changing the angles, the incenters will move on a circle.

This theorem allows us to construct the inradius $r = \sqrt{\frac{L_{ge+egh+efh+efg}}{e+f+g+h}}$ of a tangential quadrilateral with the tangent lengths $e$, $f$, $g$, $h$ from the bicentric quadrilateral $ABCD$ with the sides $e + f$, $f + g$, $g + h$, $h + e$. Having once determined $r$, we can easily construct the tangential quadrilateral with given tangent lengths. To perform this construction of $r$ draw the perpendicular at the point $W$ with $AW = e$ on the side $AB$ of the bicentric quadrilateral $ABCD$. It intersects the above mentioned circle of the incenters around $W_{\text{max}}$ at a point whose distance to $AB$ is $r$. This construction can be carried out without any restriction if one starts with a tangency point within the green ranges of Figure 4. If, for the sake of simplicity, $AB$ is the shortest side of the quadrilateral, the inequalities

$$AW_{\text{max}} = \frac{ad}{s} \leq r_{\text{max}} = \frac{\sqrt{abcd}}{s},$$

$$BW_{\text{max}} = \frac{ab}{s} \leq r_{\text{max}} = \frac{\sqrt{abcd}}{s},$$

guarantee that the segment $AB$ lies within the circle with the center $W_{\text{max}}$ and the radius $r_{\text{max}}$, see Figure 5.
Another way to understand the formula (7) is the following characterization of tangential quadrilaterals, complementing the investigations of [14] and [17].

**Theorem 5.** The points $K, L, M, N$ are situated on the sides $a, b, c, d$, of a quadrilateral $ABCD$ such that they divide the respective sides in the ratio of the adjacent sides:

$$\frac{AK}{KB} = \frac{d}{b}, \frac{BL}{LC} = \frac{a}{c}, \frac{CM}{MD} = \frac{b}{d}, \frac{DN}{NA} = \frac{c}{a}. \quad (11)$$

Then $ABCD$ is a tangential quadrilateral if and only if $KLMN$ is cyclic.

**Proof.** For a tangential quadrilateral we infer from (7) that the points $K, L, M, N$, lie on a circle around $I$ with radius $r_{\text{max}} = \sqrt{abcd/a+c}$.

In order to prove the converse, let $KLMN$ be a cyclic quadrilateral and suppose that $a + c > b + d$. Then $AK = \frac{ad}{b+c} > \frac{ad}{a+c} = AN$ implies that $\angle AKN < \angle KNA$. Similar inequalities hold for the other angles between the sides of $KLMN$ and the sides of $ABCD$. Therefore the sum of the red angles at points $N$ and $L$, i.e. $\angle KNA + \angle DNM + \angle MLC + \angle BLK$ is greater than the sum of the corresponding blue angles at points $K$ and $M$. From this we get a contradiction because for a cyclic quadrilateral $KLMN$ both sums must be $180^\circ$. Hence we conclude that the quadrilateral $ABCD$ is tangential.

It is noteworthy that the radii $r_{\text{max}} = \frac{K}{r} = \frac{\sqrt{abcd}}{a+c}$ of the circles through points $K, L, M$ and $N$, for incongruent tangential quadrilaterals with equal side lengths but different angles do not depend on the latter. This permits another construction of the bicentric quadrilateral with side lengths $a, b, c, d, a + c = b + d$. This construction is as follows: Draw any tangential quadrilateral $ABCD$ with the side lengths $a, b, c, d$. Divide its sides in the ratio of the adjacent sides to get the points $K, L, M$ and $N$ (cf. (11)). Measure the distance of the incenter $I$ from one of the points $K, L, M$ and $N$ and construct a circle with this distance as radius, tangent to $AB$ in $K$. Then complete the construction by drawing tangents from $A$ and $B$.
to this circle whereon the segments $AD = d$ and $BC = b$ are given the prescribed lengths.

References


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Congruent Contiguous Excircles

Mihály Bencze and Ovidiu T. Pop

Abstract. In this paper we present some interesting lines in a triangle and we give some of their properties.

1. The e-property

For a given triangle $ABC$, and a point $X$ on the side $BC$, consider the contiguous subtriangles $ABX$ and $AXC$, with their excircles on the sides $BX$ and $XC$. We shall say that $X$ has the $e$-property if these two excircles are congruent (see Figure 1).

![Figure 1](image-url)

Figure 1

Denote by $a$, $b$, $c$ the lengths of the sides $BC$, $CA$, $AB$. The radii of the incircle and excircle on $BC$ are $r = \frac{\Delta}{s}$ and $r_a = \frac{\Delta}{s-a}$, where $s$ and $\Delta$ are the semiperimeter and area of triangle $ABC$. It is known that $BT_a = s - c$ and $T_aC = s - b$.

In the following, we shall denote by $s_T$ and $\Delta(T)$ the semiperimeter and area of a triangle $T$.

Let $I_1$, $I_2$, and $I_a$ be the centers of the excircles of triangles $ABX$, $AXC$, and $ABC$ on the sides $BX$, $XC$, and $BC$ respectively, with points of tangency indicated in Figure 1. We denote by $\rho_a$ the common exradius of the congruent contiguous excircles of $ABX$ and $ACX$. 

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Theorem 1. Let $X$ be the point on $BC$ with the $e$-property.

(a) $AX = \sqrt{s(s-a)}$.

(b) $\rho_a = \frac{\Delta}{(\sqrt{s}+\sqrt{s-a})\sqrt{s-a}}$.

(c) $BX = \frac{a(c+\sqrt{s(s-a)})}{(\sqrt{s}+\sqrt{s-a})}$, and $CX = \frac{a(b+\sqrt{s(s-a)})}{(\sqrt{s}+\sqrt{s-a})}$.

Proof. (a) In triangles $ABX$ and $AXC$ we have

$$\rho_a = \frac{\Delta(ABX)}{s_{ABX} - BX} = \frac{\Delta(AXC)}{s_{AXC} - XC} = \frac{\Delta(ABX) + \Delta(AXC)}{s_{ABX} - BX + s_{AXC} - XC} = \frac{\Delta}{AX + s - a}. \quad (1)$$

Since $I_1I_2$ is parallel to $BC$,

$$\frac{r_a - \rho_a}{r_a} = \frac{I_1I_2}{a} = \frac{T_bX + XT_c}{a} = \frac{(s_{ABX} - AX) + (s_{AXC} - AX)}{a} = \frac{s - AX}{a}.$$

From this, $\frac{r_a - \rho_a}{r_a} = \frac{s - AX}{a}$, and

$$AX - (s-a) = \frac{a\rho_a}{r_a}. \quad (2)$$

From (1) and (2), we have $AX^2 - (s-a)^2 = \frac{a\Delta}{r_a} = a(s-a)$. Therefore, $AX^2 = (s-a)^2 + a(s-a) = s(s-a)$. This proves (a).

(b) follows from (1) and (a).

(c) Let $BX = x$. We have

$$\rho_a = \frac{\Delta(ABX)}{s_{ABX} - BX} = \frac{\Delta(AXC)}{s_{AXC} - XC} \implies \frac{x}{c + AX - x} = \frac{a - x}{b + AX - (a - x)}.$$

This reduces to $x(b + AX - a + x) = (a - x)(c + AX - x)$, and $x = \frac{a(c + AX)}{b + c + 2AX}$. Making use of (a), we obtain the expression for $BX$ given in (b); similarly for $CX$.

Analogous to the congruent contiguous excircles, the problem of congruent contiguous incircles has been studied by P. Yiu ([3] and [4]). It was shown [3, §9.1.5] that for a point $X$ on $BC$, the incircles of triangles $ABX$ and $AXC$ are congruent if and only if their excircles on the sides $BX$ and $XC$ are congruent, i.e., $X$ has the $e$-property.

Theorem 2. For the points of tangency of the excircles indicated in Figure 1, we have

(a) $\frac{BT_b}{CT_c} = \frac{s-c}{s-b}$.

(b) $\frac{AT'_a}{CT_c} = \frac{BX}{CX} = \frac{c+\sqrt{s(s-a)}}{b+\sqrt{s(s-a)}}$. 

Proof. (a) From the excircle of triangle $ABX$ on the side $BX$, we have, making use of Theorem 1 (a) and (c),

$$BT_b = s_{ABX} - c = \frac{AX + BX - c}{2} = \frac{\sqrt{s(s-a) + \frac{a(c+\sqrt{s(s-a)})}{\sqrt{s+a}}} - c}{2} = \frac{(s-c)\sqrt{s-a}}{\sqrt{s} + \sqrt{s-a}}$$  \hspace{1cm} (3)

after simplification. Similarly,

$$CT_c = \frac{(s-b)\sqrt{s-a}}{\sqrt{s} + \sqrt{s-a}}.$$  \hspace{1cm} (4)

From (3) and (4), $\frac{BT_b}{CT_c} = \frac{s-c}{s-b}$. This proves (a).

(b) Note that

$$AT'_b = AB + BT'_b = AB + BT_b = c + \frac{(s-c)\sqrt{s-a}}{\sqrt{s} + \sqrt{s-a}} = \frac{\sqrt{s}\left(c + \sqrt{s(s-a)}\right)}{\sqrt{s} + \sqrt{s-a}}.$$  \hspace{1cm} (5)

Similarly, $AT'_c = \frac{\sqrt{s}\left(b + \sqrt{s(s-a)}\right)}{\sqrt{s} + \sqrt{s-a}}$. From these, (b) follows. □

**Theorem 3.** For the points of tangency of the excircles indicated in Figure 1, we have

(a) $T_bT_c = I_1I_2 = \frac{a\sqrt{s}}{\sqrt{s} + \sqrt{s-a}}$.

(b) $T_1T_2 = \frac{|b-c|\sqrt{s}}{\sqrt{s} + \sqrt{s-a}}$.

Proof. (a) Since the excircles are congruent, $I_1I_2T_cT_b$ is a rectangle. Using (3) and (4), we obtain

$$T_bT_c = BC - BT_b - CT_c = a - \frac{(s-c)\sqrt{s-a}}{\sqrt{s} + \sqrt{s-a}} - \frac{(s-b)\sqrt{s-a}}{\sqrt{s} + \sqrt{s-a}} = \frac{a\left(\sqrt{s} + \sqrt{s-a}\right) - a\sqrt{s-a}}{\sqrt{s} + \sqrt{s-a}} = \frac{a\sqrt{s}}{\sqrt{s} + \sqrt{s-a}}.$$
(b) If the excircles of triangles $ABX$ and $AXC$ touch the line $AX$ at $T_1$ and $T_2$ respectively, then, making use of (a) above and Theorem 1(b), we have

$$T_1T_2^2 = I_1I_2^2 - (\rho_a + \rho_a)^2$$

$$= \left(\frac{a\sqrt{s}}{s + \sqrt{s - a}}\right)^2 - \frac{4\Delta^2}{(\sqrt{s + \sqrt{s - a}})^2 (s - a)}$$

$$= \frac{a^2s - 4s(s - b)(s - c)}{(s + \sqrt{s - a})^2}$$

$$= \frac{s(b - c)^2}{(s + \sqrt{s - a})^2}.$$

\[\square\]

**Theorem 4.** If $X$ has the $e$-property, the line $AX$ bisects $I_1I_2$.

**Proof.** Let the bisectors $AI_1$ and $AI_2$ of angles $BAX$ and $XAC$ intersect $BC$ at $D$ and $E$ respectively. In triangle $ABX$ with bisector $AD$, we have, by Theorem 1(a) and (c),

$$DX = \frac{BX \cdot AX}{AB + AX} = \frac{a\sqrt{s(s - a)}}{(s + \sqrt{s - a})^2}.$$  

Similarly, $EX = \frac{a\sqrt{s(s - a)}}{(\sqrt{s + \sqrt{s - a}})^2} = DX$. This shows that $X$ is the midpoint of $DE$. Since $I_1I_2$ is parallel to $DE$, the line $AX$ bisects $I_1I_2$.  

\[\square\]

2. Some identities and inequalities involving points with the $e$-property

Analogous to the point $X$ on $BC$ with the $e$-property, there are also points $Y$ on $CA$, and $Z$ on $AB$ with the $e$-property, i.e., the contiguous triangles $BCY$, $BYA$ have congruent excircles on $CY$, $YA$, and the pair $CAZ$, $CZB$ also with congruent excircles on $AZ$, $ZB$ (see Figure 2). In general, the cevians $AX$, $BY$, and $CZ$ are not concurrent. If triangle $ABC$ is isosceles, they do, because of obvious symmetry.

**Proposition 5.** The area of triangle $XYZ$ is

$$\frac{(a + \sqrt{s(s - c))(b + \sqrt{s(s - a))(c + \sqrt{s(s - b))}}}{(\sqrt{s + \sqrt{s - a})^2(\sqrt{s + \sqrt{s - b}})(\sqrt{s + \sqrt{s - c}})^2}. \Delta.$$

We establish some identities and inequalities involving the cevian lines through the points with the $e$-property. Denote by $R$ the circumradius of triangle $ABC$.

**Theorem 6.** (a) $AX \cdot BY \cdot CZ = s\Delta$.

(b) $AX^2 + BY^2 + CZ^2 = s^2$.

(c) $\frac{1}{AX^2} + \frac{1}{BY^2} + \frac{1}{CZ^2} = \frac{4R + r}{s^2}$.

(d) $\frac{a}{AX^2} + \frac{b}{BY^2} + \frac{c}{CZ^2} = \frac{2(2R - r)}{\Delta}$. 


(e) $AX^4 + BY^4 + CZ^4 = s^2(s^2 - 2r^2 - 8Rr)$.
(f) $AX^6 + BY^6 + CZ^6 = s^4(s^2 - 12Rr)$.

**Proof.** These follow from Theorem 1(a), (c), and the basic relations

\[
ab + bc + ca = s^2 + r^2 + 4Rr,
\]
\[
a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr),
\]
\[
a^3 + b^3 + c^3 = 2s(s^2 - 3r^2 - 6Rr);
\]

see [1].

**Theorem 7.** Let $m_a$ and $w_a$ be the lengths of the median on $BC$ and angle bisector of angle $A$. If $X$ is the point on $BC$ with the e-property, then $w_a \leq AX \leq m_a$.

**Proof.** For the bisector of angle $A$,

\[
w_a = \frac{2bc}{b+c} \cos \frac{A}{2} = \frac{2\sqrt{bc}}{b+c} \cdot \frac{s(s-a)}{s} = \frac{2\sqrt{bc}}{b+c} \cdot AX \leq AX,
\]

since $2\sqrt{bc} \leq b+c$.

On the other hand, for the median on $BC$, the inequality $AX \leq m_a$ is equivalent to $s(s-a) \leq \frac{2(b^2+c^2)-a^2}{4}$. After some rearrangement, this reduces to $(b+c)^2 \leq 2(b^2+c^2)$, which is clearly valid.

**Theorem 8.** (a) $AX \cdot BY + BY \cdot CZ + CZ \cdot AX \leq s^2$.
(b) $3\sqrt{s\Delta} \leq AX + BY + CZ \leq \sqrt{3}s$. 

Figure 2
Proof. These follow from applying the well known inequalities 
\[(x+y+z)^2 \leq 3(x^2+y^2+z^2)\] and \[xy+yz+zx \leq x^2+y^2+z^2,\] applied to \(x = AX, y = BY,\) and \(z = CZ,\) and making use of Theorem 6 (a), (b).

\[\square\]

Remarks. (1) By using Schwarz’s Inequality, we have
\[(AX^2 + BY^2 + CZ^2) \left(\frac{1}{AX^2} + \frac{1}{BY^2} + \frac{1}{CZ^2}\right) \geq 9.\]

By using the formulae in Theorem 6(b), (c), we have \(s^2 \cdot \frac{4R+r}{rs^2} \geq 9.\) From this the well known Euler’s inequality \(R \geq 2r\) follows.

(2) Again, it is easy to establish
\[AX^2 \cdot BY^2 + BY^2 \cdot CZ^2 + CZ^2 \cdot AX^2 = (4R+r)rs^2.\]

From the inequality
\[AX^2 \cdot BY^2 + BY^2 \cdot CZ^2 + CZ^2 \cdot AX^2 \leq AX^4 + BY^4 + CZ^4,\]
and the formula in Theorem 6(e), we have \((4R+r)rs^2 \leq s^2(s^2 - 2r^2 - 8Rr).\) This leads to another known inequality
\[3(4R+r)r \leq s^2;\]
see [1, 2].

References


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Some Circles Associated with the Feuerbach Points

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Abstract. Consider a triangle with its nine-point circle tangent to the incircle and excircles at the Feuerbach point. We show that the four circles each through the circumcenter, nine-point, and Feuerbach point contain the nine-point center of the intouch triangle or the corresponding extouch triangle. Furthermore, the lines joining these Feuerbach points to the corresponding nine-point centers are concurrent on the nine-point circle of the given triangle.

1. Four coaxial circles through the Feuerbach points

The starting point of this note is the famous Feuerbach theorem, that for a given triangle, the nine-point circle is tangent internally to the incircle and externally to each of the excircles. Given triangle $ABC$, with incenter $I$, excenters $I_a, I_b, I_c$, and nine-point $N$, the points of tangency of the nine-point circle with these circles are the Feuerbach points $F_e, F_a, F_b, F_c$ on the lines $NI, NI_a, NI_b, NI_c$ with ratios of division

$$NF_e : F_e I = \frac{R}{2} : -r,$$

$$NF_a : F_a I_a = \frac{R}{2} : r_a,$$

$$NF_b : F_b I_b = \frac{R}{2} : r_b,$$

$$NF_c : F_c I_c = \frac{R}{2} : r_c,$$

where $R, r, r_a, r_b, r_c$ are the circumradius, inradius, and exradii.

Proposition 1. Let $O$ be the circumcenter of triangle $ABC$.

(a) $OJ_a^2 = R(R - 2r)$.

(b) $OJ_a^2 = R(R + 2r_a)$.

(c) $II_a^2 = 4R(r_a - r)$.

(d) The excentral triangle $I_aI_bI_c$ has circumcenter $I'$ at the reflection of $I$ in $O$, and circumradius $2R$.

For (a-c), see [1, Theorems 152-154]. For (d), see [4, §4.6.1].

Consider the circle through $O, N,$ and $F_e$. Since $I$ is an interior point of the segment $NF_e$, it is an interior point of the circle. Our first result relates the endpoint of the chord through $O$ and $I$ with the intouch triangle, whose vertices are the points of tangency of the incircle with the sidelines.

Theorem 2. The line $OI$ intersects the circle $ONF_e$ again at the nine-point center of the intouch triangle.

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Proof. Let $N_i$ be the second intersection of the line $OI$ with the circle $ONF_e$. By the intersecting chords theorem, $OI \cdot IN_i = NI \cdot IF_e$. Therefore,

$$\frac{IN_i}{OI} = \frac{NI \cdot IF_e}{OI^2} = \frac{(\frac{R}{2} - r)r}{R(R - 2r)} = \frac{r}{2R}.$$

Note that the intouch triangle is homothetic to the excentral triangle. Since the excentral triangle has circumcenter $I'$ and nine-point center $O$, its Euler line is the line $OI$. Since the intouch triangle has circumcenter $I$, its Euler line is also the line $OI$. From

$$\frac{IN_i}{PO} = \frac{IN_i}{OI} = \frac{r}{2R},$$

the homothetic ratio of the intouch and excentral triangles, we conclude that $N_i$ is nine-point center of the intouch triangle. \hfill \square

Analogous results hold if we replace the Feuerbach point $F_e$ by the other Feuerbach points, say, $F_a$. If the circle through $O$, $N$, $F_a$ intersects the line $OI_a$ at $N_a$, then $I_aN_a/O = \frac{r_a}{2R}$. Now, triangle $II_bI_c$ is homothetic to the $A$-extouch triangle formed by the points of tangency of the $A$-excircle with the sidelines of
Some circles associated with the Feuerbach points triangle $ABC$, with homothetic ratio $-\frac{r^2}{2R}$, since the circumradius of $II_bI_c$ is also $2R$. In fact, the circumcenter of $II_bI_c$ is the reflection of $I_a$ in $O$. It follows that $\frac{II_bN_a}{II_cO} = -\frac{II_bN_a}{II_cO} = -\frac{r}{2R}$, and $N_a$ is the nine-point center of the $A$-extouch triangle (see Figure 2).

**Theorem 3.** The points $O$, $N$, $F_a$ and $N_a$ are concyclic; so are $O$, $N$, $F_b$, $N_b$, and $O$, $N$, $F_c$, $N_c$.

**2. Concurrency of four lines on the nine-point circle**

In preparation for the proof of our next main result (Theorem 7 below), we establish an interesting relation between the centers $O$, $N$, $I$, $I_a$ given in Proposition 5. The reformulation as Proposition 6 in terms of directed angles ([2, §§16-19]) makes the proof of Theorem 7 independent of the relative position of $O$ and $N$ with respect to the bisector of angle $A$.

**Lemma 4.** Let $N$ be the nine-point center of triangle $ABC$.

(a) If $A > 60^\circ$, then $N$ and $O$ lie on the same side of the bisector of angle $A$.

(b) If $A = 60^\circ$, then $N$ lies on the bisector of angle $A$.

(c) If $A < 60^\circ$, then $N$ and $O$ lie on opposite sides of the bisector of angle $A$.

**Proof.** First consider the case when $A$ is an obtuse angle (see Figure 3A). Construct the perpendicular from $O$ to $BC$, to intersect the circumcircle at $M$ on the opposite side of $A$. The line $AM$ is clearly the bisector of angle $A$. If $X$ is the orthogonal projection of $A$ on $BC$, then the orthocenter $H$ and $X$ are on opposite sides of $A$. It follows that $O$ and $H$, and their midpoint $N$, all are on the same side of the bisector $AM$.

The same conclusion holds if $A = 90^\circ$, since the orthocenter $H$ coincides with $A$.

Now we assume $A$ an acute angle (see Figure 3B). It is known that $AH = 2R \cos A$, and that the bisector of angle $A$ also bisects the angle $OAH$. It divides
OH in the ratio $R:2R \cos A = 1:2 \cos A$. Therefore, $O$ and $N$ are on the same side of the bisector if and only if $2 \cos A < 1$, i.e., $A > 60^\circ$.

If $A = 60^\circ$, then $AH = AO$, and $N$ lies on the bisector. This completes the proof of the theorem.

\(\square\)

**Proposition 5.** (a) The angle $\angle IOI_a$ is acute, right, or obtuse according as $A$ is less than, equal to, or greater than $60^\circ$.

(b) $\angleINI_a = \begin{cases} 2\angle IOI_a, & \text{if } A \leq 60^\circ, \\ 360^\circ - 2\angle IOI_a, & \text{if } A > 60^\circ. \end{cases}$

**Proof.** (a) Applying the law of cosines to triangle $IOI_a$, and using the expressions for the lengths given in Proposition 2, we have

$$\cos IOI_a = \frac{O^2 + OI_a^2 - II_a^2}{2 \cdot OI \cdot OI_a} = \frac{R(R - 2r) + R(R + 2r_a) - 4R(r_a - r)}{2 \cdot OI \cdot OI_a} = \frac{R(R - r_a + r)}{OI \cdot OI_a}. \tag{1}$$

The angle $\angle IOI_a$ is acute, right, or obtuse according as $r_a - r$ is less than, equal to or greater than $R$. Since $\frac{r_a - r}{R} = 4 \sin^2 \frac{A}{2}$, these are the cases according as $A$ is less than, equal to, greater than $60^\circ$.

(b) From (1), we have

$$\cos 2 \cdot IOI_a = 2 \cos^2 IOI_a - 1 = \frac{(O^2 + OI_a^2 - II_a^2)^2 - 2 \cdot OI^2 \cdot OI_a^2}{2 \cdot OI^2 \cdot OI_a^2} = \frac{(2R(R - r_a + r))^2 - 2R(R - 2r) \cdot R(R + 2r_a)}{2R(R - 2r) \cdot R(R + 2r_a)} = \frac{R^2 - 6R(r_a - r) + 2(r_a^2 + r^2)}{(R - 2r)(R + 2r_a)}. \tag{2}$$

On the other hand,

$$\cos INI_a = \frac{NI^2 + NI_a^2 - II_a^2}{2NI \cdot NI_a} = \frac{(\frac{R}{2} - r)^2 + (\frac{R}{2} + r_a)^2 - 4R(r_a - a)}{2(\frac{R}{2} - r)(\frac{R}{2} + r_a)} = \frac{R^2 - 6R(r_a - r) + 2(r_a^2 + r^2)}{(R - 2r)(R + 2r_a)}. \tag{3}$$

Comparison of (2) and (3) shows that $\cos INI_a = \cos 2 \cdot IOI_a$. Therefore, $\angleINI_a = 2\angle IOI_a$ or $360^\circ - 2\angle IOI_a$. Taking (a) into account, we have (b). \(\square\)

**Remark.** When $A = 60^\circ$, $\angle IOI_a = 90^\circ$, and $\angleINI_a = 180^\circ$ (see Lemma 4(b)).

In terms of directed angles, we reformulate Proposition 5 below.
**Proposition 6.** \((NI, NI_a) = -2(OI, OI_a)\).

**Theorem 7.** The four lines \(F_eN_i, F_aN_a, F_bN_b, F_cN_c\) are concurrent at a point on the nine-point circle.

**Proof.** It is enough to prove this for the lines \(F_eN_i\) and \(F_aN_a\) (see Figure 4). Let \(P\) be the intersection of the line \(F_eN_i\) with the nine-point circle. We show that it also lies on the line \(F_aN_a\). For this, it is enough to verify \((F_aN, F_aP) = (F_aN, F_aN_a)\).

\[
(F_aN, F_aP) = (PF_a, PN) \quad \text{triangle } NF_aP \text{ isosceles}
\]

\[
= (PF_a, PF_e) + (PF_e, PN) \quad N = \text{center of circle } PF_eF_a
\]

\[
= \frac{1}{2}(NF_a, NF_e) + (PF_e, PN) \quad N = \text{center of circle } PF_eF_a
\]

\[
= \frac{1}{2}(NI_a, NI) + (PF_e, PN) \quad \text{Proposition 6}
\]

\[
= (OI, OI_a) + (NF_e, N_iF_e) \quad \text{triangle } NPF_e \text{ isosceles}
\]

\[
= (OI, OI_a) + (NO, N_iO) \quad O, N, F_e, N_i \text{ concyclic}
\]

\[
= (OI, ON_a) + (ON, OI) \quad O, N, F_e, N_i \text{ concyclic}
\]

\[
= (ON, ON_a) \quad O, N, F_e, N_i \text{ concyclic}
\]

The same reasoning shows that \(P\) also lies on the lines \(F_bN_b\) and \(F_cN_c\). □
We summarize the main results in this note in Figure 5 below, and conclude with an identification of the triangle centers \( N_i \) and \( P \). According to the Encyclopaedia of Triangle Centers \([3]\), the intouch triangle has orthocenter \( X(65) \). Its nine-point center \( N_i \), being the midpoint of \( IX(65) \), is \( X(942) \). This point lies on the line through \( X(11) = F_e \) and \( X(113) \), which is on the nine-point circle. Therefore \( P \) is \( X(113) \), which is also the midpoint of \( HX(110) \).

\[
\begin{align*}
A & \quad B \\
& \quad C \\
I_a & \quad I_b \\
N_i & \quad P
\end{align*}
\]

Figure 5.

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Generalized Archimedean Arbelos Twins

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Abstract. We generalize the well known Archimedean arbelos twins by extending the notion of arbelos, and we construct an infinite number of Archimedean circles.

1. Archimedean arbelos

On a segment $AB$ we take an arbitrary point $P$ and with diameters $AP$, $PB$, $AB$ we construct the semicircles $O_1(R_1)$, $O_2(R_2)$, $O(R)$, where $R = R_1 + R_2$. If we cut from the large semicircle the small ones then the resulting figure is called from antiquity (Archimedes) arbelos (the shoemaker’s knife). The perpendicular at $P$ to $AB$ meets the large semicircle at $Q$ and divides the arbelos in two mixtilinear triangles with equal incircles.

Theorem 1 (Archimedean arbelos twins). The two circles $K_1(r_1)$ and $K_2(r_2)$ that are inside the arbelos and are tangent to the arbelos and the line $PQ$ have equal radii $r_1 = r_2 = \frac{R_1 R_2}{R_1 + R_2}$ (see Figure 1). Equivalently,

$$\frac{1}{r_1} = \frac{1}{R_1} + \frac{1}{R_2}.$$ 

A circle in the arbelos with radius $r_1$ is called an Archimedean circle ([6, p.61]). We can find infinitely many such twin incircles where the above case is a limit special case. We generalize the notion of the arbelos as a triangle whose sides are in general circular arcs. We will investigate some special cases of these arbeloi.

Figure 1.
2. Generalized arbelos

2.1. Soddy type arbelos with 3 vertices tangency points of the arcs. Let $U_X, U_Y, U_Z$ (counter clockwise direction) be the vertices of the arbelos whose sides are arcs of circles tangent to each other at the vertices, with centers $X, Y, Z$ and radii $r_X, r_Y, r_Z$. If the rotation from $U_Y$ to $U_Z$ on the arc with radius $r_X$ is clockwise relative to $U_X$, then $r_X$ is positive, otherwise it is negative. Similarly we characterize the radii $r_Y, r_Z$ relative to the movement on the appropriate arcs from $U_Z$ to $U_X$, and from $U_X$ to $U_Y$. Hence, we can have 8 different cases of arbeloi with the same vertices and the same radii in absolute value. If we change the sign of the radii of the arbelos, then we get the complementary arbelos. Denote by $\Delta$ the area of the triangle $XYZ$ and $r$ the radius of a circle tangent to the arcs-sides of the arbelos.

**Theorem 2.** The radius $r$ of the circle that is tangent to the sides of the above arbelos is given by

$$\frac{1}{r} = \frac{1}{r_X} + \frac{1}{r_Y} + \frac{1}{r_Z} + \frac{2\Delta}{r_X r_Y r_Z},$$

or

$$\frac{1}{r} = \frac{1}{r_X} + \frac{1}{r_Y} + \frac{1}{r_Z} - \frac{2\Delta}{r_X r_Y r_Z}.$$  

One of these corresponds to the complementary arbelos.

**Proof:** If $K(r)$ is the circle tangent to the sides of the arbelos, then in Figure 2a, the radius $r_X$ is negative, the sides of triangle $XYZ$ are $a = |r_Y + r_Z|$, $b = |r_Z + r_X|$, $c = |r_X + r_Y|$, and the tripolar coordinates of $K$ relative to $XYZ$ and the area $\Delta$ of triangle $XYZ$ are

$$KX = \lambda = |r_X + r|, \quad KY = \mu = |r_Y + r|, \quad KZ = \nu = |r_Z + r|,$$

$$\Delta = \sqrt{r_X r_Y r_Z (r_X + r_Y + r_Z)}.$$
If we substitute these values into the equality that the tripolar coordinates satisfy ([3]):

\[(\mu^2 + \nu^2 - a^2)^2 \lambda^2 + (\nu^2 + \lambda^2 - b^2)^2 \mu^2 + (\lambda^2 + \mu^2 - c^2)^2 \nu^2 - (\mu^2 + \nu^2 - a^2)(\nu^2 + \lambda^2 - b^2)(\lambda^2 + \mu^2 - c^2) = \lambda^2 \mu^2 \nu^2,\]

we get

\[\left(\frac{1}{r} - \frac{1}{r_X} - \frac{1}{r_Y} - \frac{1}{r_Z}\right)^2 = \frac{4\Delta^2}{r_X^2 r_Y^2 r_Z^2}.\]

Therefore,

\[\frac{1}{r} = \frac{1}{r_X} + \frac{1}{r_Y} + \frac{1}{r_Z} \pm \frac{2\Delta}{r_X r_Y r_Z}.\]

□

Remarks. (1) The Archimedean arbelos is of Soddy type with collinear vertices where \(r_X = R_1, r_Y = R_2, r_Z = -R = -(R_1 + R_2),\) \(\Delta = 0.\) In this case, there is a double solution. Hence, the inradius of the Archimedean arbelos is given by

\[\frac{1}{r} = \frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_1 + R_2}.\]

(2) If the radii \(r_X, r_Y, r_Z\) are positive, then the radii \(r\) refer to the inner and outer Soddy circles ([2]).

In the sequel, we shall adopt the following notation: For a line \(\ell, \Pi_\ell\) denotes the orthogonal projection map onto \(\ell.\)

**Lemma 3.** Let \(K(r)\) be a circle tangent externally or internally to the circles \(O_1(R_1)\) and \(O_2(R_2),\) where \(R_1, R_2\) may assume any real values. For a point \(F\) on the radical axis of the circles,

\[r = \frac{O_1 \overrightarrow{O_2} \cdot \Pi_{O_1 O_2} (\overrightarrow{FK})}{R_1 - R_2}.\]

\[1\]

Figure 3.
2.2. From this the result follows.

□

The radical center of \( K \) power \( U \)

Let \( r \) be the incircle of this arbelos and \( K \) on a line \( \mathcal{L} \) and \( K \) be the incircle of the arbelos.

\[
2r(R_1 - R_2) = 2O_1 \overrightarrow{O_2} \cdot (\overrightarrow{FM} + \overrightarrow{MK}) = 2O_1 \overrightarrow{O_2} \cdot \Pi_{O_1O_2}(\overrightarrow{FK}),
\]

and the result follows.

\[\square\]

Remark. For the Archimedean twins (1) gives

\[
r_1 = \frac{(2R_1 - r_1)R_2}{R_1 + R_2 + R_1} \implies \frac{1}{r_1} = \frac{1}{R_1} + \frac{1}{R_2}.
\]

Theorem 4. Let \( U_XU_YU_Z \) be an arbelos with collinear centers \( X, Y, Z \) on a line \( \mathcal{L} \), and \( K \) be the incircle of the arbelos.

\[
r = \left| \frac{\Pi_{\mathcal{L}}(U_YU_Z)}{\overrightarrow{XY} - \overrightarrow{XZ}} \right|.
\]

Proof. Since \( U_Z, U_Y \) are points on the radical axes of the circle pairs \( X(r_X), Y(r_Y), \) and \( X(r_X), Z(r_Z) \) respectively, by Lemma 3 we have

\[
r = \frac{\overrightarrow{XZ} \cdot \Pi_{\mathcal{L}}(U_ZK)}{r_X - r_Y} = \frac{\overrightarrow{XZ} \cdot \Pi_{\mathcal{L}}(U_YK)}{r_X - r_Z}.
\]

Since \( X, Y, Z \) are all on the line \( \mathcal{L} \), \( \Pi_{\mathcal{L}}(U_ZK) = \frac{r_X - r_Y}{XY} \overrightarrow{XZ} \) and \( \Pi_{\mathcal{L}}(U_YK) = \frac{r_X - r_Z}{XZ} \overrightarrow{XZ} \), where \( \overrightarrow{XZ} \) is a unit vector on \( \mathcal{L} \). Hence,

\[
\Pi_{\mathcal{L}}(U_YU_Z) = \Pi_{\mathcal{L}}(U_YK) \times \Pi_{\mathcal{L}}(U_ZK)
\]

\[
= r \left( \frac{r_X - r_Z}{XZ} - \frac{r_X - r_Y}{XY} \right) \overrightarrow{XZ} = \overrightarrow{XZ}
\]

From this the result follows.

\[\square\]

2.2. Arbelos of type A. On a line \( \mathcal{L} \) we take the consecutive points \( U'_Z, U'_Y, U_Z, U'_X \), and construct on the same side of the line the semicircles \( (U'_ZU_Z) \), \( (U'_YU'_Y) \), and \( (U'_ZU'_Z) \). Let \( U_X \) be the intersection of the first two semicircles (see Figure 4). The arbelos \( U_XU_YU_Z \) is of type A. It has arc \( U_YU_Z \) positive, and \( U_ZU_X, U_XU_Y \) both negative. The diameter \( U_YU_Z \) is the base of the arbelos.

2.2.1. The incircle. Let \( K(r) \) be the incircle of this arbelos and \( A, B, C \) the points of tangency. If \( S \) is the external center of similitude of the semicircles \( (U'_ZU_Z) \) and \( (U'_YU'_Y) \), then the line \( BC \) passes through \( S \), and the inversion with pole \( S \) and power \( d^2 = SU_Y \cdot SU_Z = SU_X \cdot SB \cdot SC \) swaps the semicircles \( (U'_ZU_Z) \) and \( (U'_YU'_Y) \), and leaves the circle \( K(r) \) and the semicircle \( (U'_YU'_Y) \) invariant. Hence, \( SA^2 = SB \cdot SC \), and the \( SA \) is tangent at \( A \) to both \( K(r) \) and \( (U'_YU'_Y) \). If the line \( SA \) meets the perpendiculars to \( \mathcal{L} \) at \( U_Y, U_Z \) at \( D \) and \( E \) respectively, then \( D \) is the radical center of \( K(r), (U_YU_Z), (U'_YU'_Y) \), and \( E \) is the radical center of \( K(r), (U_YU_Z), (U'_ZU'_Z) \). Hence, \( DC = DA \) and \( EB = EA \).
Construction of the incircle. We construct the external center of similitude $S$ of the semicircles $(U'ZU)$ and $(UYU')$, and take a point $A$ on the semicircle $(UYU)$ such that $SA = SU_X$. Let the line $SA$ meet the perpendiculars to $L$ through $UY$, $UZ$ at $D$, $E$ respectively. We take on $(U'ZU)$ the point $B$ such that $EB = EA$, and on $(UYU')$ the point $C$ such that $DC = DA$. The circumcircle of $ABC$ is the incircle of the arbelos.

The radius of the incircle. If we take $U'ZU = 2y$, $UYU = 2R$, $UZU' = 2z$, then $r_X = R$, $r_Y = -R - y$, $r_Z = -R - z$, $XY = -y$, and $XZ = z$. From Theorem 4, we have

$$r = \frac{2R}{\frac{2R+y}{-y} - \frac{2R+z}{z}} = \frac{Ryz}{R(y + z) + yz}.$$  

or

$$\frac{1}{r} = \frac{1}{R} + \frac{1}{y} + \frac{1}{z}.$$  \hspace{1cm} (2)

2.3. Arbelos of type B. On a line $L$, we take the consecutive points $U'Y$, $U'Z$, $UY$, $UZ$, and construct on the same side of the line the semicircles $(U'ZU)$, $(UYU')$, and $(UYU)$. Let $U_X$ be the intersection of the first two semicircles (see Figure 5). The arbelos $UXUYUZ$ is of type B. It has arc $UYU$ positive and the arcs $UZUX$, $UXUY$ of different signs. The base of the arbelos is $UYUZ$.

Construction of the incircle. The construction is the same as in type A, but now $S$ is the internal point of similitude of the semicircles $(U'ZU)$ and $(UYU')$.

The radius of the incircle. We have the same formula as (2), but now since $U'_Y$ is not on the right hand side of $UZ$, the distance $z$ must be negative, and so we have

$$\frac{1}{r} = \frac{1}{R} + \frac{1}{y} - \frac{1}{z}.$$  \hspace{1cm} (3)
Remark. For the incircle $K(r)$ of the Archimedean arbelos, since $U_X = A, U_Y = P, U_Z = B$ with base $2R_2$, $y = R_1, z = -R_1 - R_2$, (3) gives
\[
\frac{1}{r} = \frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_1 + R_2}.
\]

3. Generalized Archimedean arbelos twins

We shall construct in the Archimedean arbelos a generalized pair of inscribed equal circles.

**Theorem 5.** In the Archimedean arbelos (Figure 6) where $AP = 2R_1$, $PB = 2R_2$, $AB = 2R_1 + 2R_2$, we extend $AB$ (to left and right) with equal segments $AA' = 2x = BB'$. The semicircle $(A'P)$ divides the arbelos in two arbeloi with incircles $K_1(r_1), K_3(r_3)$, and the semicircle $(PB')$ divides the arbelos in two arbeloi with incircles $K_2(r_2), K_4(r_4)$. We have a couple of twin circles: $r_1 = r_2$ and $r_3 = r_4$, with
\[
\frac{1}{r_1} = \frac{1}{x} + \frac{1}{R_1} + \frac{1}{R_2} \quad \text{and} \quad \frac{1}{r_3} = \frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_1 + R_2 + x}.
\]

**Proof.** The circle $K_1(r_1)$ is the incircle of an arbelos of type A with base $AP$. Hence, $\frac{1}{r_1} = \frac{1}{R_1} + \frac{1}{x} + \frac{1}{R_2}$. 
The circle $K_2(r_2)$ is the incircle of an arbelos of type A with base $PB$. Hence, 
$$\frac{1}{r_2} = \frac{1}{R_2} + \frac{1}{x} + \frac{1}{R_1}.$$ 

The circle $K_3(r_3)$ is the incircle of an arbelos of type B with base $PB$. Hence, 
$$\frac{1}{r_3} = \frac{1}{R_2} + \frac{1}{x} - \frac{1}{R_1 + R_2 + x}.$$ 

The circle $K_4(r_4)$ is the incircle of an arbelos of type B with base $AP$. Hence, 
$$\frac{1}{r_4} = \frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_1 + R_2 + x}.$$ 

Therefore, $r_1 = r_2$ and $r_3 = r_4$. □

Remark. If $x \to \infty$, then the semicircles $(A'P)$ and $(PB')$ tend to the perpendicular semiline $PQ$, and the four incircles tend to the Archimedean twin circles.

4. Bisectors of the Archimedean Arbelos

We have seen that in the Archimedean arbelos, the semiline $PQ$ divides the arbelos in two arbeloi with equal incircles. This semiline as a degenerate semicircle is like a bisector from $P$ of the arbelos that produces the twins (Archimedean circles) with radius $r_P$ such that 
$$\frac{1}{r_P} = \frac{1}{R_1} + \frac{1}{R_2}.$$ 

We find the other two bisectors of the arbelos from the vertices $A$ and $B$ (Figure 7).

Let the semicircle $(AO_2)$ bisector of the arbelos meet the semicircle $(PB)$ at $Q_A$, and $A_1(r_1), A_2(r_2)$ be the incircles of the arbeloi $Q_AAP, Q_AAB$.

The arbelos $Q_AAP$ with base $AP$ is of type B, so we have $\frac{1}{r_1} = \frac{1}{R_1} + \frac{1}{x} - \frac{1}{R_1 + R_2}$. In order to have $r_1 = r_2 = r_A$, we set $2x = R_2$. Hence the point $O_2$ is the midpoint of $PB$, and the bisector semicircle $(AO_2)$ meets $PQ$ at the point $Q'$ such that $PQ' = \sqrt{AP \cdot PO_2} = \sqrt{2R_1R_2}$, and 
$$\frac{1}{r_A} = \frac{1}{R_1} + \frac{2}{R_2} - \frac{1}{R_1 + R_2}.$$ 

Similarly, if $O_1$ is the midpoint of $AP$, then the semicircle $(O_1B)$ is the bisector of the arbelos from $B$ that passes also from $Q'$, and 
$$\frac{1}{r_B} = \frac{2}{R_1} + \frac{1}{R_2} - \frac{1}{R_1 + R_2}.$$ 

Hence, the three bisectors of the Archimedean Arbelos are concurrent at $Q'$. 
5. Infinitely many Archimedean circles

There are many exciting constructions of Archimedean circles or families of these; see [4, 5]. Here we construct a more natural family of Archimedean circles that contains the original Archimedean twins. In the Archimedean arbelos with diameters $AP$, $PB$, $AB$, and semicircles $O_1(R_1)$, $O_2(R_2)$, $O(R)$, we take at the left of $A$ the point $X$, and at right of $B$ the point $Y$ such that $AX = BY = 2x$. We rotate clockwise $P$ around $A$ by an angle $\pi/2$ at $A_1$, and counterclockwise $P$ around $B$ by an angle $\pi/2$ at $B_1$. The line $XA_1$ meets the line $PQ$ at $D_1$. The line $YB_1$ meets the line $PQ$ at $C_1$. We take at the left of $A$ the point $C$ such that $CA = PC_1 = 2d_1$, and at right of $B$ the point $D$ such that $BD = PD_1 = 2d_2$. The semicircles $(CP)$, $(AY)$ meet at $M$, and the semicircles $(PD)$, $(XB)$ meet at $N$. We show that the incircles $K_1(r_1)$, $K_2(r_2)$ of the arbeloi $MAP$, $NPB$ are both Archimedean circles.

**Proof.** Since in triangle $XPD_1$, $PA_1$ is a bisector and $AA_1$ is parallel to $PD_1$ (Figure 8), we know that $\frac{1}{PD_1} + \frac{1}{XP} = \frac{1}{AA_1}$. Hence,

$$\frac{1}{d_2} + \frac{1}{x + R_1} = \frac{1}{R_1}. \quad (4)$$

Similarly, from triangle $PYC_1$, we have

$$\frac{1}{d_1} + \frac{1}{x + R_2} = \frac{1}{R_2}. \quad (5)$$

![Figure 8.](image-url)
The arbelos $MAP$ is of type A; so we have \( \frac{1}{r_1} = \frac{1}{R_1} + \frac{1}{d_1} + \frac{1}{x+R_2} = \frac{1}{R_1} + \frac{1}{R_2} \) from (5).

The arbelos $NPB$ is also of type A; so we have \( \frac{1}{r_2} = \frac{1}{R_2} + \frac{1}{d_2} + \frac{1}{x+R_1} = \frac{1}{R_2} + \frac{1}{R_1} \) from (4).

Hence, \( r_1 = r_2 \) is the radius of the Archimedean circle. \( \square \)

For \( x = 0 \) the circles \( K_1(r_1) \), \( K_2(r_2) \) coincide with the original Archimedean twin circles since the semicircles \( (AY) \), \( (XB) \) coincide with the semicircle \( (AB) \), and the semicircles \( (CP) \), \( (PD) \) coincide with the line \( PQ \). If \( M_1, M_2 \) are the midpoints of \( CP, AY \), then applying Stewart’s theorem to triangle \( MM_1M_2 \), we have

\[
O_1M^2 \cdot M_1M_2 = M_1M_2^2 \cdot O_1M_2 + M_2M^2 \cdot O_1M_1 - O_1M_1 \cdot O_1M_2 \cdot M_1M_2,
\]

or

\[
(d_1 + R_2 + x)O_1M^2 = (d_1 + R_1)^2(R_2 + x) + (R_1 + R_2 + x)^2d_1 - d_1(R_2 + x)(d_1 + R_2 + x).
\]

Substituting \( d_1 = \frac{R_2(x + R_2)}{x} \), we get

\[
O_1M^2 = R_1^2 + 4R_1R_2 = O_1P^2 + PQ^2 = O_1Q^2.
\]

Hence, the locus of \( M \) is the circular arc \( O_1(Q) \) from the point \( Q \) to \( Q_A \) on the perpendicular \( Q_AA \) to \( AB \). Similarly, we can prove that the locus of \( N \) is the circular arc \( O_2(Q) \) from the point \( Q \) to \( Q_B \) on the perpendicular \( Q_BB \) to \( AB \).

6. A special generalization of arbelos with arcs of angle \( 2\phi \)

If we substitute the semicircles in Archimedean arbelos with arcs of angle \( 2\phi \), i.e., \( AP = 2R_1 \sin \phi \), \( PB = 2R_2 \sin \phi \), \( AB = 2(R_1 + R_2) \sin \phi \) (Figure 9) [1], the points \( A, O_1, O \) are collinear; so are the points \( O, O_2, B \). The tangent to the arc \( (AP) \) at \( P \) meets the arc \( (AB) \) at \( Q_A \), and the tangent to the arc \( (PB) \) at \( P \) meets the arc \( (AB) \) at \( Q_B \). Let \( K_1(r_1) \), \( K_2(r_2) \) be the incircles of the arbeloi \( Q_BAP \) and \( Q_APB \). If \( 2\phi = \pi \), then we have the classical arbelos, and \( Q_A, Q_B \) coincide with the point \( Q \). We prove that \( r_1 = r_2 \).

**Proof:** The point \( A \) is on the radical axis of \( O_1(R_1), O_2(R_1 + R_2) \). From (1), we have

\[
r_1 = \frac{O_1O \cdot \Pi_{O_1O}(AK_1)}{r_{O_1} - r_O} = \frac{R_2 \cdot \Pi_{O_1O}(AK_1)}{R_1 + R_1 + R_2}.
\]

Also, \( r_1 = \Pi_{O_1O}(K_1P) \). Hence,

\[
2R_1 \sin^2 \phi = \Pi_{O_1O}(AK_1) + \Pi_{O_1O}(K_1P) = r_1 \cdot \frac{2R_1 + R_2}{R_2} + r_1 = r_1 \cdot \frac{2(R_1 + R_2)}{R_2},
\]

or \( r_1 = \frac{R_1R_2 \sin^2 \phi}{R_1 + R_2} \). Similarly,

\[
r_2 = \frac{O_2O \cdot \Pi_{O_2O}(BK_2)}{r_O - r_{O_2}} = \frac{R_1 \cdot \Pi_{O_2O}(BK_2)}{-R_1 - R_2 - R_2},
\]

or \( r_2 = \frac{R_1R_2 \sin^2 \phi}{R_1 + R_2} \).
and \( r_2 = \Pi_{OO_2}(PK_2) \). Hence,

\[
2R_2 \sin^2 \phi = \Pi_{OO_2}(PK_2) + \Pi_{OO_2}(K_2B) = r_2 + r_2 \cdot \frac{R_1 + 2R_2}{R_1} = r_2 \cdot \frac{2(R_1 + R_2)}{R_1},
\]

or

\[
r_2 = \frac{R_1 R_2 \sin^2 \phi}{R_1 + R_2}.
\]

If \( 2\phi = \pi \), then we have the radius of the Archimedean circle.

Construction of the twins. The perpendicular from \( P \) to \( AB \) meets \( O_1O_2 \) at the point \( C \), and the parallel from \( C \) to \( PO_1 \) meets \( PO_2 \) at \( D \). Since \( PC \) is a bisector in triangle \( PO_1O_2 \), we have \( \frac{1}{CD} = \frac{1}{R_1} + \frac{1}{R_2} \). Hence we need the construction of \( r_1 = CD \cdot \sin^2 \phi \). The perpendicular from \( D \) to \( AB \) meets \( AB \) at \( E \) and the perpendicular from \( E \) to \( PO_1 \) meets this line at the point \( F_1 \). The symmetric of \( F_1 \) in \( PC \) is the point \( F_2 \) on the line \( PO_2 \). The perpendicular at \( F_2 \) to \( PF_2 \) meets the circle \( O_1(F_1) \) at the point \( K_1 \) and the line \( EF_1 \) meets the circle \( O_2(F_2) \) at the point \( K_2 \). These points are the centers of the twin incircles and the construction of these circles is obvious.

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