Forum Geometricorum
Volume 15 (2015) 1–4.

FORUM GEOM
ISSN 1534-1178

Generalized Tucker Circles

Nikolaos Dergiades

Abstract. It is known that if we cut the sides of the angles of a triangle, with six consecutive alternating antipallel and parallel segments to the sides of the triangle then we get a closed hexagram that is inscribed in a circle, the Tucker circle. Since the above hexagram has sides parallel to the sides of the pedal triangles of O and H that are isogonal conjugate points, we generalize the Tucker circles by considering two isogonal conjugate points on the McCay cubic.

Given a reference triangle ABC, a Tucker hexagon $A_bA_cB_cB_aC_aC_b$ has vertices, two on each sideline (see Figur 1), such that B_cC_b, C_aA_c, A_bB_a are parallel to the sidelines BC, CA, AB respectively, whereas B_aC_a, C_bA_b, A_cB_c are antiparallel to these sidelines. It is well known that these six vertices all lie on a circle whose center is a point on the Brocard axis.

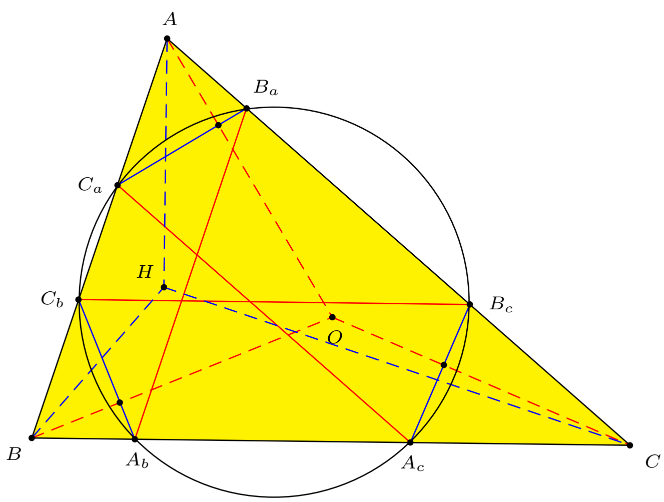

Figure 1.

The segments B_aC_a, C_bA_b, A_cB_c are parallel to the sides of the orthocenter H of triangle ABC, and the segments B_cC_b, C_aA_c, A_bB_a are parallel to the sides of the pedal triangle of the circumcenter O. Since O, H are isogonal conjugates, the segments B_aC_a, C_bA_b, A_cB_c are perpendicular to OA, OB, OC respectively, and the segments B_cC_b, C_aA_c, A_bB_a are perpendicular to HA, HB, HC respectively. From this aspect we shall generalize the Tucker hexagons and circles by requiring

Publication Date: January 12, 2015. Communicating Editor: Paul Yiu.

the sides B_aC_a, C_bA_b, A_cB_c of a hexagon $A_bA_cB_cB_aC_aC_b$ to be perpendicular to AP, BP, CP, and the sides B_cC_b, C_aA_c, A_bB_a perpendicular to AQ, BQ, CQ for a pair of isogonal conjugates P and Q. In Theorem 2 below we shall establish the necessary and sufficient that the line containing P and Q must pass through the circumcenter O. In other words, P and Q are points on the McCay cubic which has barycentric equation

$$\sum_{\text{cyclic}} a^2 S_A \mathbb{X}(c^2\mathbb{Y}^2 - b^2\mathbb{Z}^2) = 0. \tag{1}$$

We shall make use of the notion of directed angle of two lines. For two lines L_1, L_2, denote by $(L_1,\ L_2)$ the directed angle from L_1 to L_2. The basic properties of directed angles can be found in Johnson [2, §§16–19]. Here is a characterization of the points on the McCay cubic in terms of directed angles.

Lemma 1 ([1])**.** *The point P lies on the McCay cubic if and only if*

$$(BC,\ AP) + (CA, BP) + (AB,\ CP) = \frac{\pi}{2} \pmod{\pi}.$$

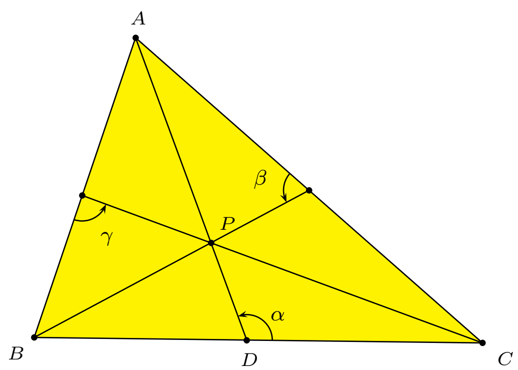

Figure 2.

Proof. Let $\alpha = (BC,\ AP)$, $\beta = (CA,\ BP)$, and $\gamma = (AB,\ CP)$ be the directed angles, and $x = \cot\alpha$, $y = \cot\beta$, $z = \cot\gamma$. It is known that

$$\alpha + \beta + \gamma = \frac{\pi}{2} \pmod{\pi} \quad \text{if and only if} \quad x + y + z = xyz. \tag{2}$$

If D is the trace of AP on BC (see Figure 2), then the law of sines in triangle ADC gives

$$\frac{\sin(\alpha + C)}{\sin\alpha} = \frac{DC}{AC} \implies \cot\alpha + \cot C = \frac{DC}{2R\sin B\sin C}. \tag{3}$$

Similarly, from triangle ABD we have

$$\frac{\sin(\pi - \alpha + B)}{\sin(\pi - \alpha)} = \frac{BD}{AB} \implies -\cot\alpha + \cot B = \frac{BD}{2R\sin B\sin C}. \tag{4}$$

From (3) and (4) we get

$$\frac{x+\frac{S_C}{S}}{-x+\frac{S_B}{S}}=\frac{DC}{BD}=\frac{v}{w} \implies x=\frac{S_Bv-S_Cw}{(v+w)S}.$$

Similarly we have $y=\frac{S_Cw-S_Au}{(w+u)S}$ and $z=\frac{S_Au-S_Bv}{(u+v)S}$. By substitution into (2), we get

$$\sum_{\text{cyclic}} a^2S_Au(c^2v^2-b^2w^2)=0.$$

Comparison with (1) shows that P is a point on the McCay cubic. □

Theorem 2. *Let P be a point on the plane of triangle ABC with cevians not perpendicular to the sides of ABC at their vertices, and Q be its isogonal conjugate. Beginning with an arbitrary point B_a on CA, construct points C_a on AB, A_c on BC, B_c on CA, C_b on AB, A_b on BC such that $B_aC_a \perp AP$, $C_aA_c \perp BQ$, $A_cB_c \perp CP$, $B_cC_b \perp AQ$, $C_bA_b \perp BP$. The following statements are equivalent.*

(a) *The perpendicular from A_b to CQ passes through the initial point B_a.*
(b) *The six points B_a, C_a, A_c, B_c, C_b, A_b are concyclic.*
(c) *The points P and Q lie on the McCay cubic.*

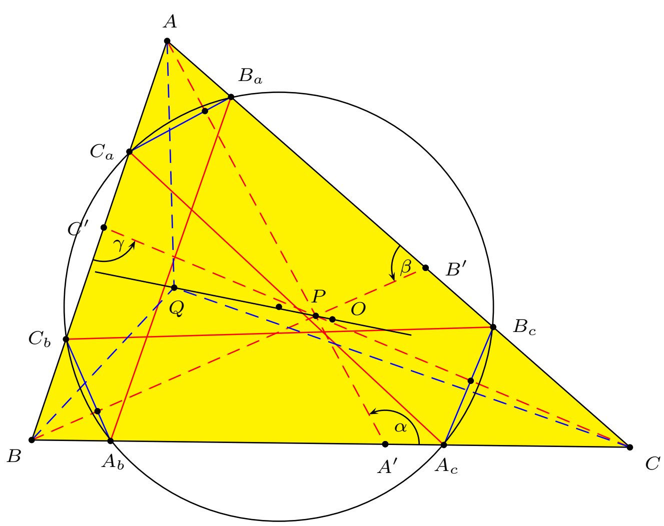

Figure 3.

Proof. Let $A'B'C'$ be the cevian triangle of P and $\alpha=(BC,\ AP)$, $\beta=(CA,\ BP)$, $\gamma=(AB,\ CP)$ the directed angles (see Figure 3).

(a) ⇒ (b) If the perpendicular from A_b to CQ passes through the initial point B_a, then since the segments B_aC_a, B_cC_b are perpendicular to the isogonal lines PA, QA relative to AB, AC, they are antiparallels relative to AB, AC and the points B_c, B_a, C_a, C_b are on a circle $\mathscr{C}_a$. Similarly the points C_a, C_b, A_b, A_c are on a circle $\mathscr{C}_b$, and the points A_b, A_c, B_c, B_a are on a circle $\mathscr{C}_c$. The three circles coincide if any two of them do. Now, if they are distinct, then pairwise, they have a sideline of triangle ABC for radical axes, and the three radical axes are nonconcurrent, an impossibility. Therefore, the six points are concyclic.

(b) $\Rightarrow$ (a) If the circumcircle of triangle $B_aC_aA_c$ passes through B_c, then obviously it also passes through C_b and A_b. The lines A_cB_c, A_bB_a are antiparallels relative to CA, CB, and the lines CP, CQ are isogonal relative to CA, CB. Since $A_cB_c \perp PC$, we conclude that $A_bB_a \perp CQ$.

(b) $\Leftrightarrow$ (c) It is easy to see the equivalence of the following statements.
(b) The six points B_a, C_a, A_c, B_c, C_b, A_b are concyclic.
(b1) B_a, C_a, A_c, B_c are concyclic.
(c1) $(B_cB_a,\ B_aC_a) = (B_cA_c,\ A_cC_a)$.
(c2) $(BC, A_cB_c) + (B_cB_a,\ B_aC_a) + (A_cC_a, BC) = 0$.

Now, referring to Figure 3, we have

$$\begin{aligned}
(BC,\ A_cB_c) &= (BC,\ AB) + (AB,\ CP) + (CP,\ A_cB_c) = B + \gamma + \frac{\pi}{2},\\
(B_cB_a,\ B_aC_a) &= (CA,\ BC) + (BC,\ AP) + (AP,\ B_aC_a) = C + \alpha + \frac{\pi}{2},\\
(A_cC_a,\ BC) &= (A_cC_a,\ BQ) + (BQ,\ BC)\\
&= \frac{\pi}{2} + (AB,\ BP)\\
&= \frac{\pi}{2} + (AB,\ AC) + (AC,\ BP) = \frac{\pi}{2} + A + \beta.
\end{aligned}$$

From this,

$$\begin{aligned}
0 &= (BC,\ A_cB_c) + (B_cB_a,\ B_aC_a) + (A_cC_a,\ BC)\\
&= (A + B + C) + (\alpha + \beta + \gamma) + \frac{3\pi}{2}\\
&= \alpha + \beta + \gamma - \frac{\pi}{2} \pmod{\pi}.
\end{aligned}$$

It follows that (c1) and (c2) are equivalent to $\alpha+\beta+\gamma = \frac{\pi}{2} \pmod{\pi}$. By Lemma 1, this is equivalent to (c). This completes the proof of the theorem. □

For $P = O$, the above circles are the known Tucker circles. For $P = I$, the incenter, these circles are all concentric at I. For a fixed $P \neq I$ on the McCay cubic, the centers of the circles lie on a line through the Lemoine point. We leave it to the reader to find clever barycentric coordinates for the initial point B_a to obtain elegant cyclic formulas for the barycentric coordinates of the other vertices of the hexagon, and to expose interesting properties of these generalized Tucker circles.

References

[1] B. Gibert, *Cubics in the Triangle Plane*, McCay cubic, Prorerty 4;
`http://bernard.gibert.pagesperso-orange.fr/Exemples/k003.html`

[2] R. A. Johnson, *Advanced Euclidean Geometry*, Dover reprint, 2007.

Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
E-mail address: `ndergiades@yahoo.gr`