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Heronian Triangles of Class K: Congruent Incircles Cevian Perspective

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Abstract. We relate the properties of a cevian that divides a reference triangle into two sub-triangles with congruent incircles to the system of inner and outer Soddy circles of the same reference triangle. We show that if constraints are placed on the reference triangle then relationships exist between the Soddy circles, the incircle of the reference triangle and the congruent incircles of the sub-triangles. In particular, we show that a class of Heronian triangles exists with inradius equal to integer multiples of their inner and outer Soddy circle radii.

1. Congruent incircles cevian

It has been shown by Yiu [4, pp.127–132] that if a triangle ABC (with side-lengths a, b, c) is divided by a cevian through A into two subtriangles with congruent incircles of radius ρ, then the length of the congruent incircles cevian AD is $\sqrt{s(s-a)}$, and

$$\rho = \frac{r}{1+\sqrt{t_b t_c}} = \frac{r}{a}(s - \sqrt{s(s-a)}), \tag{1}$$

where s is the semiperimeter and r the inradius of triangle ABC, and $t_a = \tan\frac{A}{2} = \frac{r}{s-a}$, $t_b = \tan\frac{B}{2} = \frac{r}{s-b}$, $t_c = \tan\frac{C}{2} = \frac{r}{s-c}$ are the tangents of the half angles of the triangle (see Figure 1). These numbers satisfy the basic relation

$$t_a t_b + t_b t_c + t_c t_a = 1. \tag{2}$$

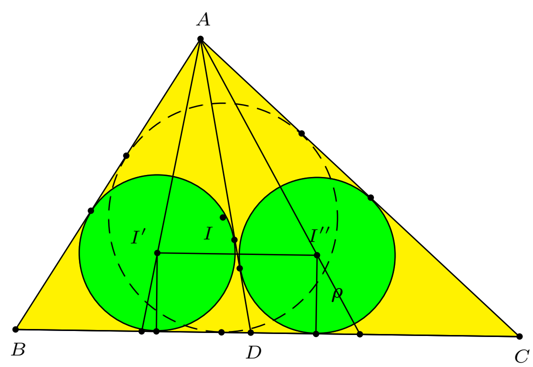

Figure 1.

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Proposition 1. *If θ denotes angle ADB for the congruent incircle cevian AD, then*

$$\cos\theta = \frac{t_b - t_c}{t_b + t_c} = \frac{b-c}{a}, \tag{3}$$

$$\sin\theta = \frac{2\sqrt{t_b t_c}}{t_b + t_c} = \frac{2\sqrt{(s-b)(s-c)}}{a}. \tag{4}$$

Proof. This follows from the formula $\tan\frac{\theta}{2} = \sqrt{\frac{t_c}{t_b}}$ in [4, p.131], and the identities $\cos\theta = \frac{1-t^2}{1+t^2}$ and $\sin\theta = \frac{2t}{1+t^2}$ where $t = \tan\frac{\theta}{2}$. □

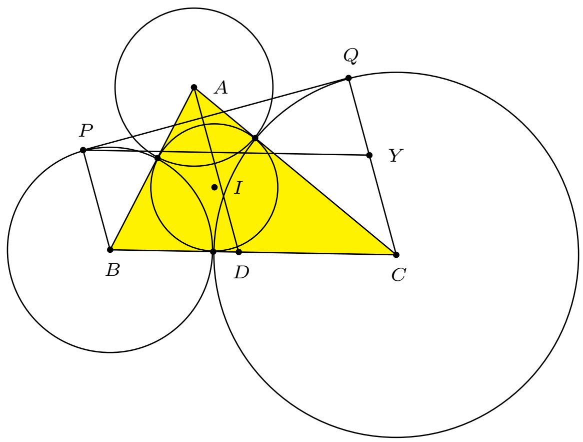

Figure 2.

Now consider the triad of mutually tangent circles with centers at the vertices A, B, C. These have radii $s-a$, $s-b$, $s-c$ respectively. Without loss of generality we may assume $b \geq c$. If the external common tangent of the B- and C- circles on the same side of A touches these circles at P and Q respectively, then $\cos PYQ = \frac{(s-c)-(s-b)}{(s-c)+(s-b)} = \frac{b-c}{a}$ (see Figure 2). It follows from (3) that PQ is perpendicular to the congruent incircle cevian AD. This leads to a simple ruler and compass construction of the congruent incircles cevian.

Theorem 2. *The congruent incircle cevian AD is the perpendicular through A to external common tangent of the B- and C- circles* (of the triad of mutually tangent circles with centers at the vertices) *on the same side of BC as vertex A.*

2. Radii of Soddy circles

The standard configuration for the Soddy circles of a reference triangle is shown in Figure 3. It has been shown by Dergiades [3] that the radii of $S(r_{\text{i}})$ and $S'(r_{\text{o}})$ are given by the formulas:

$$r_{\text{i}} = \frac{\Delta}{4R + r + 2s} \quad \text{and} \quad r_{\text{o}} = \frac{\Delta}{4R + r - 2s}. \tag{5}$$

where Δ is the area of the reference triangle, R its circumradius and r its inradius.

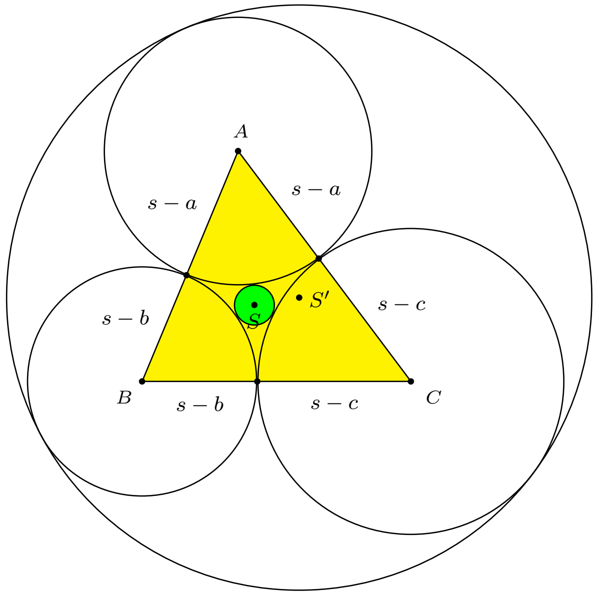

Figure 3.

Here are two well-known identities associated with the radii of the Soddy circles:

$$\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}+\frac{2}{r}=\frac{1}{r_{\rm i}}, \tag{6}$$

$$\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{2}{r}=\frac{1}{r_{\rm o}}. \tag{7}$$

If we write $K := t_a + t_b + t_c$, these can be put in the form

$$\frac{r}{r_{\rm i}} = K+2, \qquad \frac{r}{r_{\rm o}} = K-2.$$

From these,

$$\frac{r_{\rm o}}{r_{\rm i}} = \frac{K+2}{K-2}. \tag{8}$$

3. Soddyian triangles

The case $K = 2$ has been considered by Jackson [2]. In this case, the outer Soddy circle has degenerated into a straight line, and the triangle is called *Soddyian*. It has the property that if the sides are $a \geq b \geq c$, then

$$\frac{1}{\sqrt{s-a}} = \frac{1}{\sqrt{s-b}} + \frac{1}{\sqrt{s-c}}.$$

By multiply through by $\sqrt{r}$ and converting to tangent half angles we get:

$$t_a = 1 + \sqrt{t_b t_c}.$$

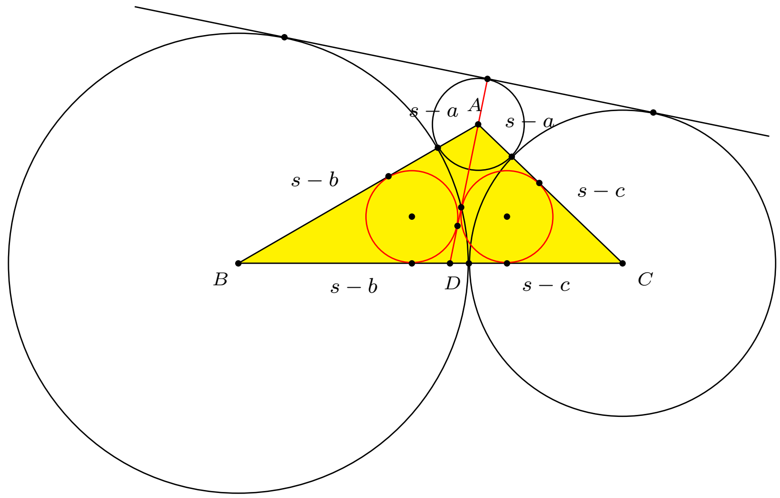

Figure 4.

Comparing this with the radius of the congruent incircles in (1), we obtain the following theorem.

Theorem 3. *In the triad of mutually tangent circles with centers at the vertices of a Soddyian triangle, the smallest circle is congruent to the incircles of the sub-triangles divided by the congruent incircle cevian through its center* (see Figure 4).

We prove another interesting property of the congruent incircles cevian triangle of a Soddyian triangle.

Theorem 4. *In a Soddyian triangle ABC with $a \geq b \geq c$, the congruent incircle cevian AD is parallel to the Soddy line* (joining the incenter to the Gergonne point); see Figure 5.

Proof. Set up a rectangular coordinate system with B as the origin, and positive x-axis along the line BC, so the the coordinates of the vertices and the incenter are

$$A = (c\cos B,\ c\sin B), \quad B = (0,\ 0), \quad C = (a,\ 0), \quad I = (s-b,\ r) = \left(\frac{r}{t_b},\ r\right).$$

The Gergonne point has homogeneous barycentric coordinates

$$G_{\rm e} = \left(\frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c}\right) = (t_a : t_b : t_c).$$

Since $t_a + t_b + t_c = 2$, this has Cartesian coordinates

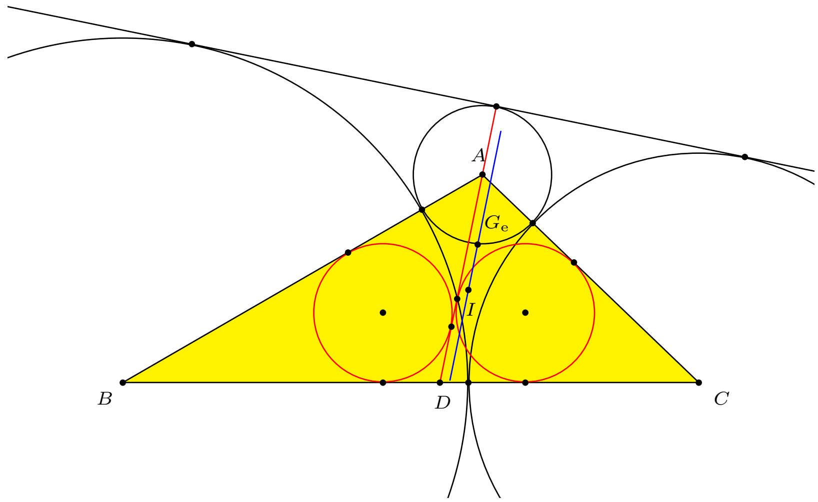

Figure 5.

$$G_e = \frac{1}{2}(t_a \cdot A + t_b \cdot B + t_c \cdot C) = \left(\frac{t_a c \cos B + t_c a}{2}, \frac{t_a c \sin B}{2}\right)$$

$$= \left(\frac{t_a\left(\frac{r}{t_a}+\frac{r}{t_b}\right)\cdot\frac{1-t_b^2}{1+t_b^2} + t_c\left(\frac{r}{t_b}+\frac{r}{t_c}\right)}{2}, \frac{t_a\left(\frac{r}{t_a}+\frac{r}{t_b}\right)\cdot\frac{2t_b}{1+t_b^2}}{2}\right)$$

$$= \left(r \cdot \frac{(t_a+t_b)(1-t_b^2)+(t_b+t_c)(1+t_b^2)}{2t_b(1+t_b^2)}, r \cdot \frac{t_a+t_b}{1+t_b^2}\right)$$

$$= \left(r\left(\frac{1-t_b^2}{2t_b(t_b+t_c)} + \frac{t_b+t_c}{2t_b}\right), \frac{r}{t_b+t_c}\right).$$

Let ψ be the angle between the Soddy line and the base line BC.

$$\tan\psi = -\frac{\frac{r}{t_b+t_c} - r}{r\left(\frac{1-t_b^2}{2t_b(t_b+t_c)} + \frac{t_b+t_c}{2t_b}\right) - \frac{r}{t_b}}$$

$$= \frac{2t_b(t_b+t_c-1)}{(1-t_b^2)+(t_b+t_c)(t_b+t_c-2)}$$

$$= \frac{2t_b(t_b+t_c-1)}{(1-t_b^2)-t_a(t_b+t_c)}$$

$$= \frac{2t_b(t_b+t_c-1)}{(1-t_b^2)-(1-t_bt_c)}$$

$$= \frac{2(1-t_a)}{t_c-t_b} = \frac{2(t_a-1)}{t_b-t_c}.$$

However, from Proposition 1, we have

$$\tan\theta = \frac{2\sqrt{t_bt_c}}{t_b - t_c} = \frac{2(t_a - 1)}{t_b - t_c}.$$

This shows that the Soddy line is parallel to the congruent incircles cevian. □

4. Heron triangles from Soddy circles

Soddyian triangles with integer sides are always Heronian [2, §4].

We shall say that a triangle has class K if the sum of the tangents of its half angles is equal to K. Thus, Soddyian triangles have class 2. Heronian triangles of class 2 are constructed in [2]. Let K be a positive integer. An integer triangle of class K is Heronian if and only if the tangents of its half angles are rational. Let θ be the angle ADB for the congruent incircle cevian AD. We have $t_b - t_c = (t_b + t_c)\cos\theta$. Together with $t_a + t_b + t_c = K$ and the basic relation (2), we have

$$\begin{aligned} t_a &= \frac{K(1+\cos^2\theta) + 2\varepsilon\sqrt{K^2 - 3 - \cos^2\theta}}{3 + \cos^2\theta}, \\ t_b &= \frac{(1+\cos\theta)(K - \varepsilon\sqrt{K^2 - 3 - \cos^2\theta})}{3 + \cos^2\theta}, \\ t_c &= \frac{(1-\cos\theta)(K - \varepsilon\sqrt{K^2 - 3 - \cos^2\theta})}{3 + \cos^2\theta} \end{aligned} \tag{9}$$

for $\varepsilon = \pm 1$. Clearly, t_a, t_b, t_c are rational if and only if $K^2 - 3 - \cos^2\theta = v^2$ for a rational number v, i.e., $K^2 - 3$ is a sum of two squares of rational numbers. Equivalently, $K^2 - 3$ is a sum of squares of two integers.

Lemma 5. *An integer is a sum of two squares of rational numbers if and only if it is a sum of squares of two integers.*

Proof. We need only prove the necessity part, for square-free integers. Let $n = u^2 + v^2$ for two rational numbers. Writing $u = \frac{h}{q}$ and $v = \frac{k}{q}$ for integers h, k, q, we have $nq^2 = h^2 + k^2$ for integers h, k, q. Here, h and k must be relatively prime, since any common divisor must be prime to q, and so its square must divide n, contrary to the assumption that n is square-free. Let p be a prime divisor of n. Modulo p, $h^2 + k^2 \equiv 0$. Since at least one of h and k, say, k, is nonzero modulo p, we have -1 is a quadratic residue modulo p, and $p \equiv 1 \pmod 4$. Thus, p is a sum of two squares of integers. This being true for every prime divisor of n, the number n is itself a sum of two squares of integers. □

Theorem 6. *Let $K > 1$ be a positive integer. Heronian triangles of class K exists if and only if $K^2 - 3$ is a sum of two squares of integers.*

The necessity part follows from Lemma 5 above. We shall construct Heronian triangles of class 4 in the next section. The construction clearly applies to class K with $K^2 - 3$ equal to a sum of two squares of integers.

5. Heronian triangles of class 4

The ratio of the radii of the Soddy circles of a triangle is given by (8). For integer values of $K := t_a + t_b + t_c$, this ratio is an integer only when $K = 3, 4, 6$, and is equal to 5, 3, 2 respectively. By Theorem 6 above, there is no Heronian triangle of class $K = 3, 6$.

We construct Heronian triangles with $K = 4$. Without loss of generality, assume $a \geq b \geq c$. The parameters t_a, t_b, t_c are given in (9) with $K = 4$. Here, $K^2 - 3 = 13$, and we require $\cos\theta$ and $v := \sqrt{13 - \cos^2\theta}$ to be rational numbers. Since $13 = 3^2 + 2^2$, we rewrite $v^2 = 13 - \cos^2\theta$ as

$$(3 - \cos\theta)(3 + \cos\theta) = (v - 2)(v + 2).$$

Since all factors involved are rational, we assume $3 - \cos\theta = w(v + 2)$ for a rational number w. It follows that $w(3 + \cos\theta) = v - 2$. Solving these for $\cos\theta$ and v, we have

$$\cos\theta = \frac{3 - 4w - 3w^2}{1 + w^2}, \qquad v = \frac{2 + 6w - 2w^2}{1 + w^2}. \tag{10}$$

Note that $t_b = t_c$ if and only if $\cos\theta = 0$. In this case, w cannot be rational. We shall assume $b > c$, so that θ is an acute angle, and $0 < \cos\theta < 1$. For this, $\frac{\sqrt{3}-1}{2} < w < \frac{\sqrt{13}-2}{3}$. Substitution of $\cos\theta$ and $v = \sqrt{13 - \cos^2\theta}$ given in (10) into (9) (with $K = 4$), we obtain, for $\varepsilon = 1$,

$$t_a = \frac{3w^2 + 12w + 11}{w^2 + 3w + 3}, \qquad t_b = \frac{-w^2 - 2w + 2}{w^2 + 3w + 3}, \qquad t_c = \frac{2w^2 + 2w - 1}{w^2 + 3w + 3}, \tag{11}$$

and, for $\varepsilon = -1$,

$$t_a = \frac{11w^2 - 12w + 3}{3w^2 - 3w + 1}, \qquad t_b = \frac{-w^2 - 2w + 2}{3w^2 - 3w + 1}, \qquad t_c = \frac{2w^2 + 2w - 1}{3w^2 - 3w + 1}. \tag{12}$$

In the latter case, t_a cannot be greater than both t_b and t_c for $w \in \left(\frac{\sqrt{3}-1}{2}, \frac{\sqrt{13}-2}{3}\right)$. Therefore, Heronian triangles of class 4 are given by (11). Writing $w = \frac{m}{n}$ for relatively prime integers m and n, and using $s - a : s - b : s - c = \frac{1}{t_a} : \frac{1}{t_b} : \frac{1}{t_c}$, we may take

$$\begin{aligned}
s - a &= (2m^2 + 2mn - n^2)(-m^2 - 2mn + 2n^2),\\
s - b &= (2m^2 + 2mn - n^2)(3m^2 + 12mn + 11n^2),\\
s - c &= (-m^2 - 2mn + 2n^2)(3m^2 + 12mn + 11n^2).
\end{aligned}$$

This gives

$$\begin{aligned}
a &= (m^2 + n^2)(3m^2 + 12mn + 11n^2),\\
b &= (-m^2 - 2mn + 2n^2)(5m^2 + 14mn + 10n^2),\\
c &= (2m^2 + 2mn - n^2)(2m^2 + 10mn + 13n^2).
\end{aligned}$$

For integers $m, n \leq 15$ giving w in the range, we obtain *primitive* Heronian triangles of class 4 by dividing a, b, c by their greatest common denominator, as presented in the table below. An example is shown in Figure 6.

m	n	a	b	c	s	Δ	r	R
1	2	355	219	148	361	8094	$\frac{426}{19}$	$\frac{13505}{38}$
2	5	11803	10660	1299	11881	3460314	$\frac{31746}{109}$	$\frac{2574185}{218}$
3	7	47444	38963	9515	47961	92616414	$\frac{422906}{219}$	$\frac{20795465}{438}$
3	8	74387	72491	2180	74529	39502554	$\frac{144698}{273}$	$\frac{40620485}{546}$
4	9	132987	103156	33235	134689	856373214	$\frac{2333442}{367}$	$\frac{97695005}{734}$
5	11	301636	225235	84747	305809	4767644154	$\frac{8621418}{553}$	$\frac{333914045}{1106}$
5	12	2379	2035	388	2401	197274	$\frac{4026}{49}$	$\frac{233285}{98}$
5	13	526516	498675	31867	528529	3971806014	$\frac{5463282}{727}$	$\frac{765749525}{1454}$
6	13	595115	432452	179891	603729	19430005434	$\frac{25006442}{777}$	$\frac{925691645}{1554}$
7	15	1063668	757315	338059	1079521	63941458494	$\frac{61541346}{1039}$	$\frac{2212473965}{2078}$
8	15	1186923	620500	608179	1207801	94234794654	$\frac{85745946}{1099}$	$\frac{2611873625}{2198}$

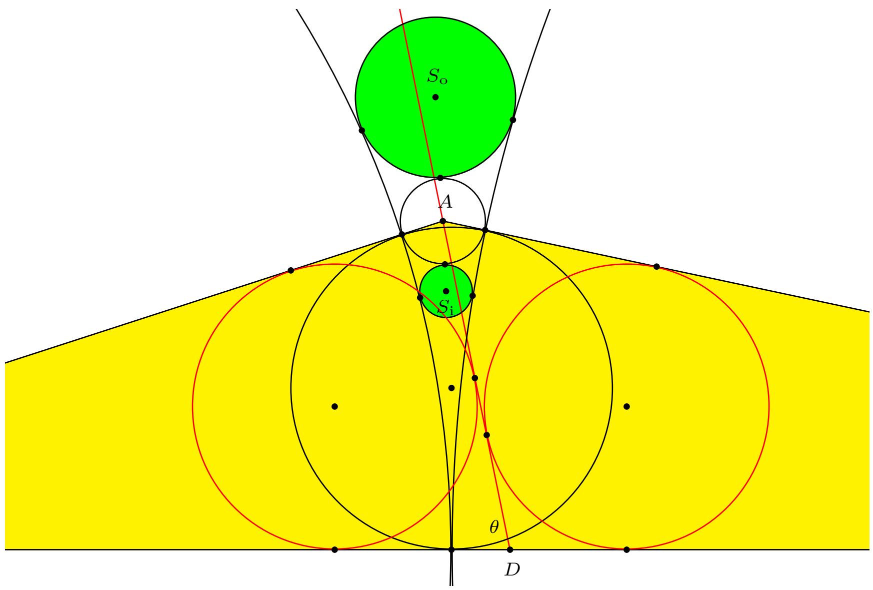

Figure 6.

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