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Volumes of Solids Swept Tangentially Around General Surfaces

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Abstract. In Part I ([2]) the authors introduced solid tangent sweeps and solid tangent clusters produced by sweeping a planar region S tangentially around cylinders. This paper extends [2] by sweeping S not only along cylinders but also around more general surfaces, cones for example. Interesting families of tangentially swept solids of equal height and equal volume are constructed by varying the cylinder or the planar shape S. For most families in this paper the solid tangent cluster is a classical solid whose volume is equal to that of each member of the family. We treat many examples including familiar quadric solids such as ellipsoids, paraboloids, and hyperboloids, as well as examples obtained by puncturing one type of quadric solid by another, all of whose volumes are obtained with the extended method of sweeping tangents. Surprising properties of their centroids are also derived.

1. Tangential sweeping around a general cylinder

Figure 1 recalls the concepts of solid tangent sweep and solid tangent cluster introduced in [2]. Start with a plane region S between two graphs in the same

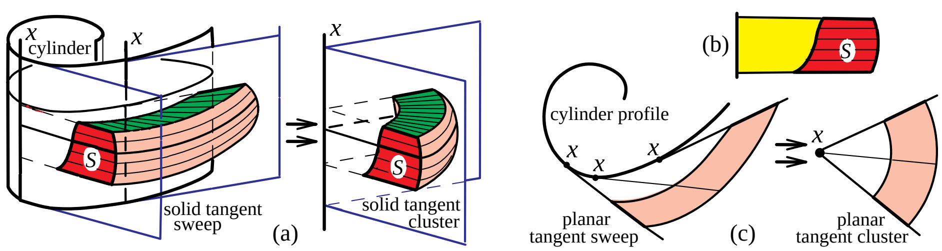

Figure 1. (a) Volume of a solid tangent sweep is equal to that of its solid tangent cluster. (b) Region S lies between two ordinate sets. (c) Top view of a horizontal cross section.

half-plane. To be specific, assume S consists of all points (x, y) satisfying the inequalities

$$f(x) \le y \le g(x), \quad a \le x \le b$$

where f and g are nonnegative functions related by the inequality $0 \le f(x) \le g(x)$ for all x in an interval $[a, b]$. In Figure 1a, the x axis is oriented vertically, and S is in the upper half-plane having the x axis as one edge. If we rotate S around the x axis we obtain a solid of revolution swept by region S, as indicated in the right portion of Figure 1a. More generally, place the x axis along the generator of a general cylinder (not necessarily circular or closed) and, keeping the upper

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half-plane tangent to the cylinder, move it along the cylinder. Then S generates a tangentially swept solid we call a *solid tangent sweep*. The corresponding *solid tangent cluster* is that obtained by rotating S around the x axis.

When the smaller function f defining S is identically zero, the swept solid is called a *bracelet*. Examples are shown in Figures 2 and 3. Clearly, by Figure 1b, any swept solid can be produced by removing one bracelet from another. In [2] we proved:

Theorem 1. *The volume of the solid tangent sweep does not depend on the profile of the cylinder, so it is equal to the volume of the solid tangent cluster, a portion of a solid of revolution.*

The proof used the fact that the shaded band and annulus in Figure 1c have equal areas, together with the slicing principle: *Two solids have equal volumes if their horizontal cross sections taken at any height have equal areas*.

Families of solid tangent sweeps with the same solid tangent cluster. For a given region S we can allow the cylinder to vary and thus obtain a *family* of solid tangent sweeps, all with the same solid tangent cluster. Thus, from Theorem 1 we obtain the following corollary:

Corollary 1. *Each member of the family has the same volume as their common solid tangent cluster.*

Moreover, from such a family one can obtain infinitely many new families with the same property by slicing the solids of the given family by two horizontal planes at given distance apart. Not only are the volumes of the slices equal because of the slicing principle, but we also have the following corollary:

Corollary 2. *For any such family of slices, the altitudes of the volume centroids above a fixed horizontal base plane are also equal.*

This property of centroids is another consequence of the slicing principle (see [3; p.150]). In Section 6 we use Corollary 2 to locate centroids of many solids.

2. CONIC SECTIONS SWEEPING AROUND CIRCULAR CYLINDERS

In Figure 2a, S is a semielliptical disk, and the swept solid is an ellipsoidal bracelet whose volume is that of its solid cluster, an ellipsoid of revolution. In

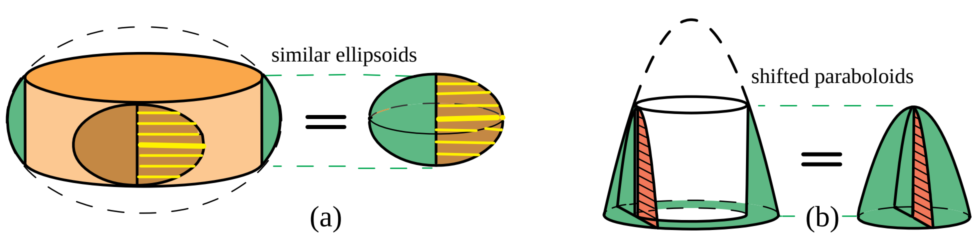

Figure 2. (a) All ellipsoidal bracelets have the same volume as the ellipsoid. (b) All paraboloidal bracelets have the same volume as the paraboloid of revolution.

Figure 2b, S is half a parabolic segment, and the solid sweep is a paraboloidal bracelet whose volume is that of its solid cluster, part of a paraboloid of revolution.

If Figure 3a, S is a double right triangle sweeping around a circular cylinder. The swept solid is a hyperboloidal bracelet of one sheet whose volume is that of its solid cluster, a portion of a solid cone. In Figure 3b, S is a portion of a hyperbolic segment sweeping around a circular cylinder. The solid sweep is a hyperboloidal bracelet of two sheets (only one of which is shown), whose volume is that of its solid cluster, a portion of a hyperboloid of revolution.

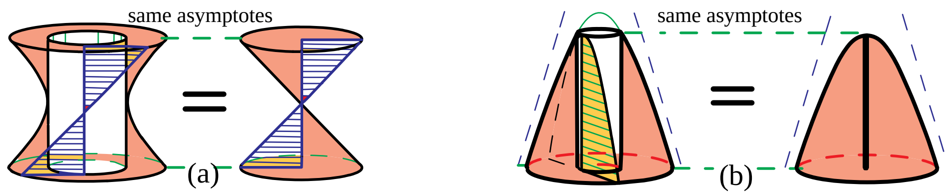

Figure 3. (a) Double triangle sweeps a hyperboloidal bracelet with the same volume as its solid cluster, a portion of a solid cone with the same volume. (b) Hyperbolic segment sweeps a hyperboloidal bracelet. The solid cluster is part of a hyperboloid of revolution of the same volume.

These results are summarized in Figure 4, where E and P denote the ellipsoid and paraboloid in Figure 2, H_1 is a hyperboloid of one sheet in Figure 3a (a degenerate case shown), and H_2 is a hyperboloid of two sheets in Figure 3b. Vertical sections of a circular cylinder, C, are also included, regarded as swept by a degenerate conic.

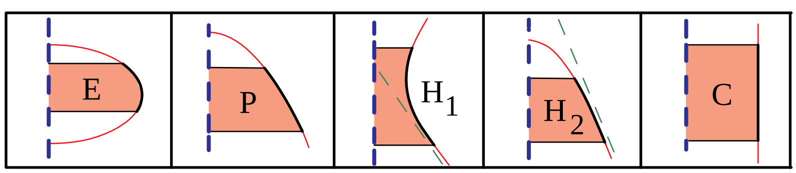

Figure 4. Special sweeping regions S bounded by conics.

Because the tangent sweeps in the foregoing examples are taken around a circular cylinder, the same solids can be obtained by using this cylinder to drill a hole through the axis of a solid bounded by a quadric surface. The volume of each drilled solid depends only on the height of the cylindrical hole and not on its radius. When the radius is zero, the drilled solid is the solid cluster, a quadric surface of revolution. A classical case is when the solid being drilled is a sphere, a result usually treated by integral calculus. All spherical bracelets of a given height have equal volume.

Solids sweeps and clusters whose outer lateral boundary is a quadric surface. When a conic is rotated around one of its axes of symmetry, the solid of revolution has a lateral surface that is a portion of a quadric surface. Rotation around a different axis will not produce a quadric surface. For example, rotating a circle around a line not through its center produces a torus, which is a *quartic* surface. When a conic is swept tangentially around a circular cylinder, with the symmetry axis of the conic lying on a generator of the cylinder, the solid tangent sweep and its solid cluster have outer lateral surfaces that are similar quadric surfaces.

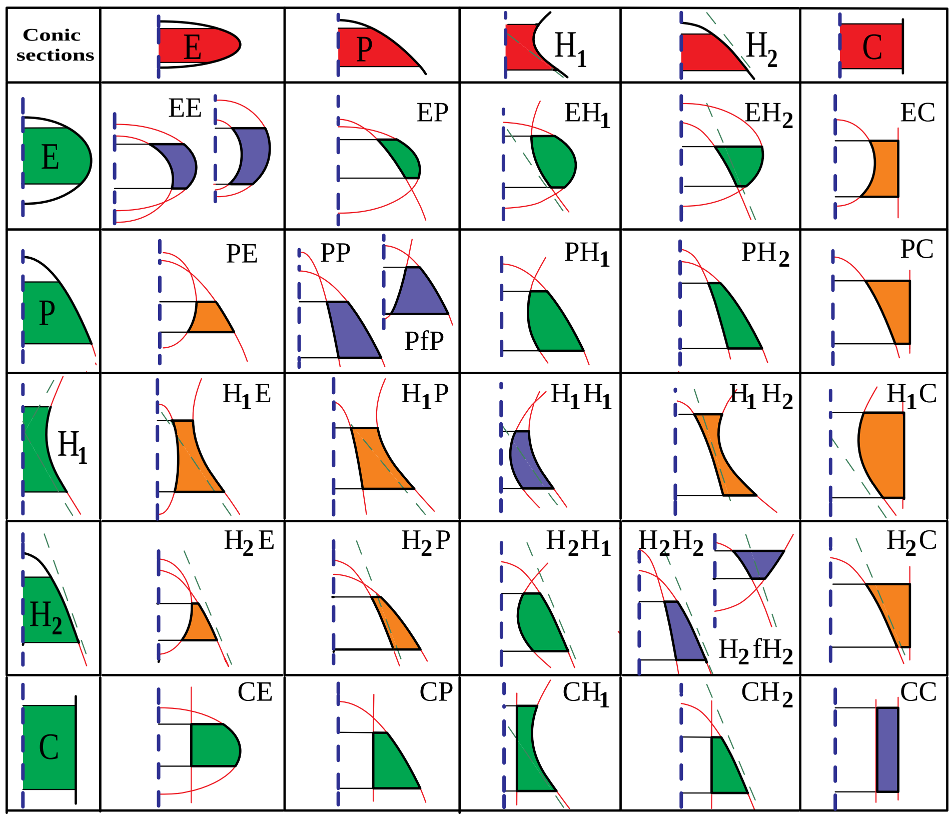

Figure 5. Table summarizing sweeping regions S bounded by two conics.

Solids swept by combinations of conics. Now we consider solids swept by regions S in Figure 1 where both functions f and g that define S have portions of conic sections as their graphs. The table in Figure 5 shows various possible combinations. The examples in Figure 4 are used as the top row and leftmost column of the table, with E meaning ellipse, P meaning parabola, H_1 a hyperbola whose rotation about its axis produces a hyperboloid of one sheet, H_2 a hyperbola whose rotation about its axis produces a hyperboloid of two sheets, and C meaning circular cylinder. Conics in the top row form the outer boundary of S and those in the left column form the inner boundary. The dashed vertical line in each entry of the table is a common axis of symmetry of the two conics.

The first diagonal entry in the table shows two possible cases when both boundaries are ellipses, one when the ellipses intersect, and another when they do not intersect. The second diagonal entry shows two possible cases when both boundaries are parabolas, one when they open in the same direction, the other when they open in opposite direction, with fP indicating 'flipped' parabola. Similarly, the next-to-last diagonal entry shows two possible cases of two hyperbolas of type H_2 opening in the same or opposite direction, with fH_2 indicating 'flipped' hyperbola.

When a region S from the table is rotated about the common fixed vertical axis of symmetry it generates a solid of revolution, a solid cluster, whose inner and outer surfaces are quadric surfaces. When the axis of symmetry is allowed to move tangentially around a *circular* cylinder, S generates a solid tangent sweep having the solid of revolution as its solid tangent cluster. Because the cylinder is circular, the inner and outer surfaces of each tangent sweep are quadric surfaces, similar to the corresponding surfaces of the cluster. The sweep and cluster have equal volumes, and cross sections produced by any horizontal plane have equal areas. *As the radius of the cylinder changes, a family of solid tangent sweeps is produced, each with the same volume as the solid tangent cluster.*

Dual solids. Figure 6a shows a family of spherical bracelets of a given height. They are of type CE in Figure 5, where ellipse E is a circle. Figure 6b shows a family of circular cylinders of given height from which inscribed spherical portions have been removed. They are of type EC in Figure 5, where again E is a circle. When swept solids of type CE and EC have the same height, we call them *dual* solids. The solid cluster in Figure 6a is a sphere, and its dual in Figure 6b is a cylinder with a spherical hole. Archimedes showed that the volume of a sphere is 2/3 that

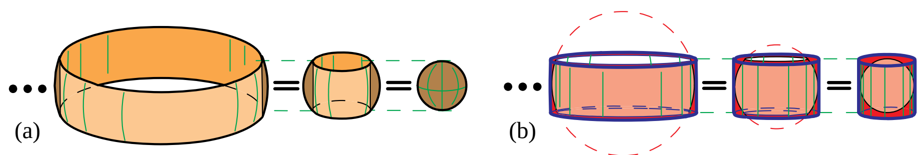

Figure 6. (a) Family of spherical bracelets of given height. (b) Family of solids dual to those in (a).

of its smallest circumscribing cylinder (a result inscribed on his tombstone), so the volume of the solid cluster in Figure 6b is exactly half that of the solid cluster in Figure 6a. The same ratio holds for any two dual members of these families. The term *dual* is used more generally to refer to two solids of the same height swept by regions S in Figure 5 that are symmetrically located with respect to the main

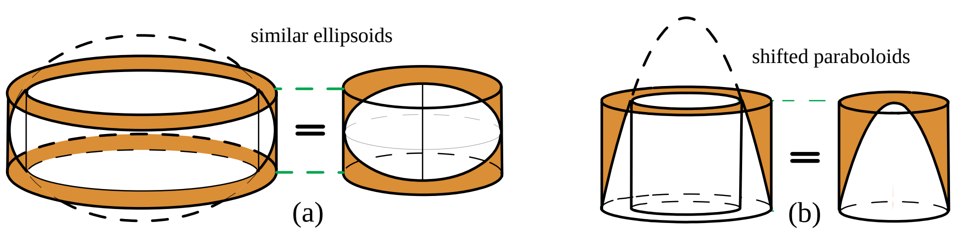

Figure 7. Family dual (a) to ellipsoidal bracelets and (b) to paraboloidal bracelets.

diagonal. In dual solids the types of outer and inner surfaces are interchanged. Figure 7a shows two members of a family of ellipsoidal bracelets dual to those in Figure 2a, and Figure 7b displays two members of a family dual to the paraboloidal bracelets in Figure 2b.

Figures 8a and 8b show two members of families of hyperboloidal bracelets dual to those in Figure 3a and 3b, respectively. In a given family of dual bracelets the volume of each punctured cylinder depends only on the height of the cylinder and not on its radius.

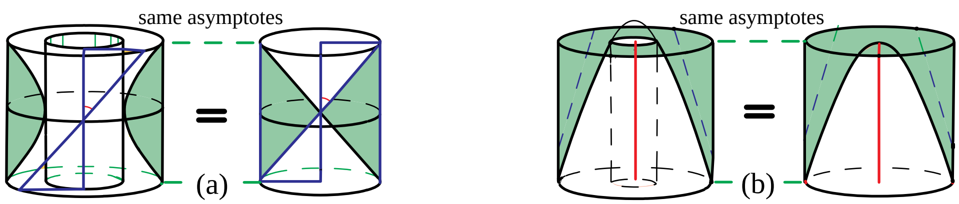

Figure 8. Family dual to hyperboloidal bracelets in Figure 3. (a) Bracelets of one sheet. (b) Bracelets of two sheets.

In all the foregoing examples, the volume of a swept solid plus that of its dual solid is equal to the volume of the circumscribing cylinder.

Solids swept by combinations of regions bounded by conics. Theorem 1 can be extended to include any solid swept by a suitable combination of regions S of the type shown in Figure 1. Figure 9 indicates several examples obtained by combining re-

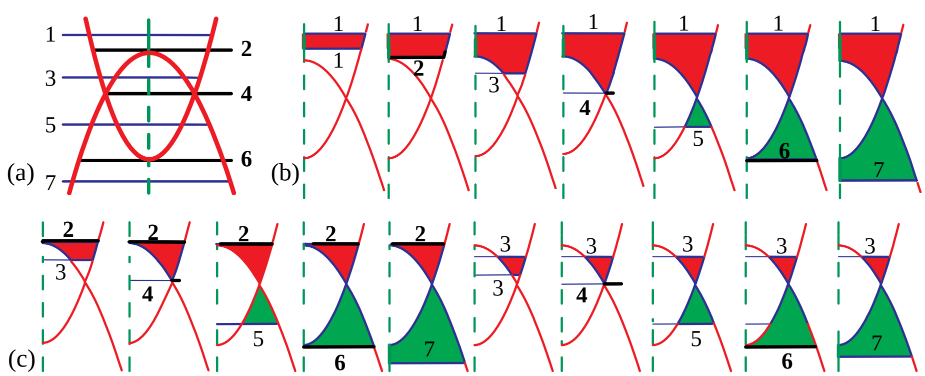

Figure 9. (a) Sweeping regions S bounded by two parabolas. (b) Strips with line 1 as upper boundary. (c) Strips with lines 2 and 3 as upper boundaries.

gions of type PfP in Figure 5, a parabola and an intersecting flipped parabola. There are seven numbered horizontal lines in Figure 9a. The even numbered lines, shown darker, are fixed. Line 2 passes through the vertex of one parabola, line 4 passes through the intersection points of the two parabolas, and line 6 passes through the vertex of the flipped parabola. They divide the plane into four horizontal strips, and the odd numbered lines lie somewhere inside these strips as indicated. As the odd numbered lines vary in position they generate different types of plane regions between the two parabolas that can be swept around the common axis of symmetry. Samples are shown in Figures 9b and 9c. Images symmetric with respect to line 4 are not shown.

3. TANGENTIAL SWEEPING BY VARIABLE PLANE REGIONS ALONG SPECIAL CYLINDERS

In Figure 1, solid sweeps and their clusters were generated by sweeping a fixed plane region S tangentially along a general cylinder. This section treats special cylinders and includes cases in which S is allowed to vary. Further examples of variable sweeping regions are given in Section 8.

Tractrix as profile of the cylinder. Figure 10 shows a *tractrix cylinder*, whose profile is a tractrix, with various regions swept tangentially along the same tractrix cylinder. In Figure 10a a rectangle of fixed size is swept tangentially along a tractrix cylinder; the corresponding solid tangent cluster is part of a circular cylinder.

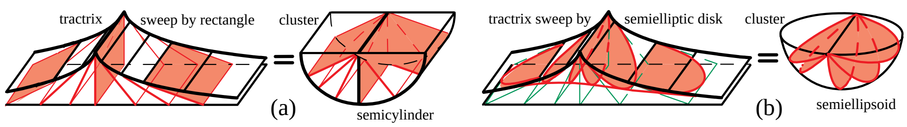

Figure 10. Tangentially swept solids generated by (a) a rectangle, and (b) a semielliptical disk moving tangentially along a tractrix cylinder.

In Figure 10b a semielliptical disk inscribed in the rectangle of Figure 10a is swept along the same tractrix cylinder; the corresponding solid tangent cluster is part of an ellipsoid of revolution. (The ellipsoid in Figure 10b is almost spherical.) Both solid tangent clusters are familiar solids whose volumes are well known or are easily calculated. The corresponding tangentially swept solids are not well known, and integral calculus does not easily yield their volumes. But Theorem 1 does the job with little effort! The volume of each solid sweep is simply equal to that of its solid tangent cluster, which is easily calculated.

Exponential as profile of the cylinder. Figure 11 shows two solids swept tangentially along an *exponential cylinder*, whose profile is an exponential curve. The solid in Figure 11a is swept by a rectangle whose base is of fixed length and whose altitude

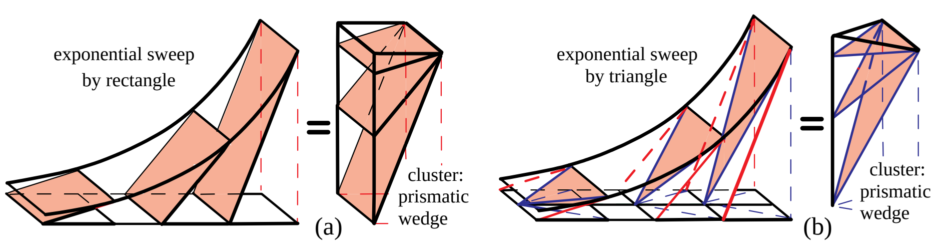

Figure 11. Solids swept tangentially along an exponential cylinder by (a) variable rectangle, and (b) variable isosceles triangle.

is the length of the tangent segment from the exponential curve to its asymptote. Because the subtangents of an exponential have constant length, the solid tangent cluster is a portion of half a rectangular prism. The solid in Figure 11b is swept by an isosceles triangle inscribed in the tangential rectangle of Figure 11a. Its solid tangent cluster is a portion of a triangular prism.

The solid in Figure 12a is swept by a semielliptical disk inscribed in the rectangle of Figure 11a. Its solid cluster is part of a cylindrical wedge with a semielliptical base. In Figure 12b the semielliptical disk of Figure 12a is flipped over. The corresponding solid cluster is the complementary part of the cylindrical wedge in Figure 12a.

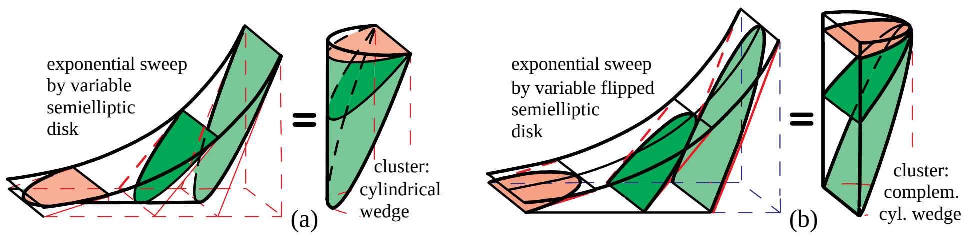

Figure 12. Solids swept tangentially along an exponential cylinder by (a) variable semielliptical disk, and (b) flipped variable semielliptical disk.

Cycloid as profile of the cylinder. In Figure 13a, a solid is swept by a variable rectangle moving tangentially along a *cycloidal cylinder*, whose profile is a cycloid. Figure 13b shows the solid swept by an isosceles triangle inscribed in the rectangle of Figure 13a.

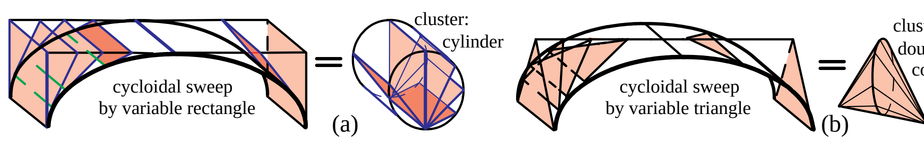

Figure 13. Solids swept tangentially along a cycloidal cylinder by (a) variable rectangle, and (b) variable isosceles triangle.

The solid in Figure 14a is swept by a variable elliptical disk inscribed in the rectangle of Figure 13a, and that in Figure 14b is swept by a semielliptical disk inscribed in the same rectangle.

The foregoing examples show that many infinite families of tangentially swept solids can be generated by plane regions moving along various cylinders. We have discussed a few special cases for which the volume of the solid tangent cluster is known or, as we shall see presently, can be easily determined without using integral calculus.

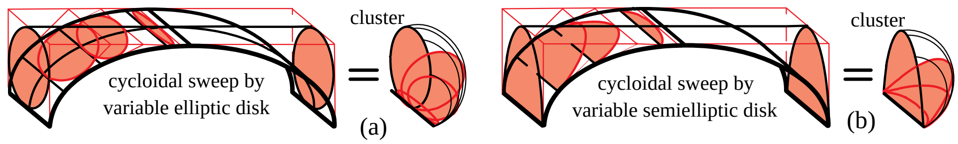

Figure 14. Solids swept tangentially along a cycloidal cylinder by a full elliptical disk in (a) and a semielliptical disk in (b).

Calculating the volumes of solid clusters. Each solid cluster is a portion of a solid of revolution. In the examples treated above we can calculate the volume of the solid cluster directly or by invoking a new comparison lemma that extends Pappus' rule on volumes of solids of revolution.

Take a plane region that may change its shape as it rotates about an axis. Let $A(\theta)$ denote the area of the region and let $c(\theta)$ denote the distance of its area centroid from the axis when the region has rotated through an angle θ from some initial position. By Pappus' rule, the volume ΔV of the solid of revolution generated by rotating through a small angle $\Delta\theta$ is given by

$$\Delta V = A(\theta)c(\theta)\Delta\theta.$$

This implies the following comparison lemma for volumes generated by two plane regions of areas $A_1(\theta)$ and $A_2(\theta)$ that change their shapes in a special way as they rotate about the same axis:

Comparison Lemma. *If at any stage of the rotation the ratio of their areas $A_1(\theta)/A_2(\theta)$ is a constant α, and the ratio of their centroidal distances $c_1(\theta)/c_2(\theta)$ is a constant γ, then the corresponding ratio of their volumes $V_1(\theta)/V_2(\theta)$, when swept through the same angle, is the constant $\alpha\gamma$, just as if the regions did not change their shapes. This ratio does not depend on the shape of the tangential cylinder.*

The comparison lemma allows us to calculate the volumes of the solid clusters treated in Figures 10 through 14.

In Figure 10b the solid cluster is a portion of an ellipsoid of revolution inscribed in the circular cylinder in Figure 10a, both rotated through the same angle. In this case we easily find that $\alpha = \pi/4$ and $\gamma = 8/(3\pi)$ giving $\alpha\gamma = 2/3$ for the ratio of their volumes, ellipsoid to cylinder. This is the famous 2/3 ratio for the volumes of a sphere and cylinder found on Archimedes' tombstone.

In Figure 11b the solid cluster is a portion of half a triangular prism inscribed in half the rectangular prism in Figure 11a. The two solid clusters are also solids of revolution for which the comparison lemma can be applied. In this case we find that $\alpha = 1/2$ and $\gamma = 2/3$, so the ratio of their volumes is $\alpha\gamma = 1/3$.

Similarly, we determine the volume of the solid cluster in Figure 12b by comparing it with that in Figure 11a. In this case we have $\alpha = \pi/4$ and $\gamma = 2(1 - \frac{4}{3\pi})$, giving $\alpha\gamma = \frac{\pi}{2} - \frac{2}{3}$. Figures 13 and 14 show four different solids swept along a cycloidal cylinder. The solid cluster in Figure 13a is a portion of a circular cylinder, so its volume is easily calculated. The other three solid clusters are not well known solids, but we can determine their volumes in terms of that of the cylindrical cluster in Figure 13a by applying the comparison lemma. Comparing the solid cluster in Figure 14a with that in Figure 13a we find $\alpha = \pi/4$ and $\gamma = 1$ giving us $\alpha\gamma = \pi/4$.

4. TANGENTIAL SWEEPING AROUND A CONE

Instead of generating solids tangentially swept around a cylinder, we replace the

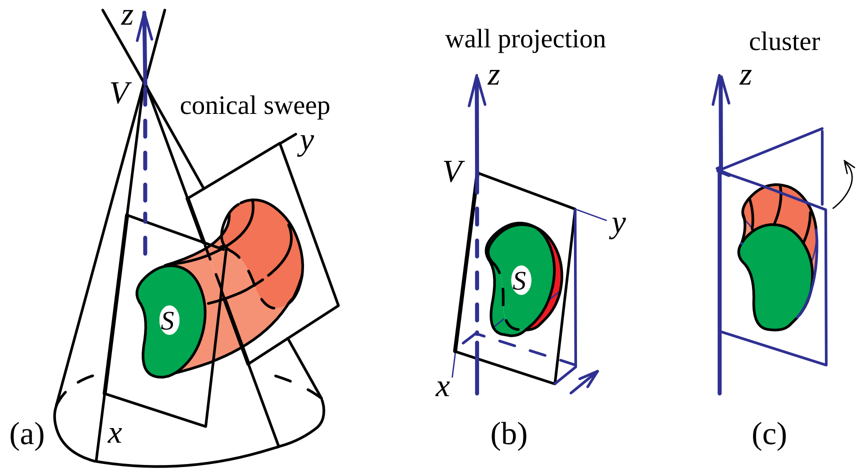

Figure 15. (a) Tangential sweeping by S around a cone. (b) Region S and its wall projection on the yz plane. (c) Rotation of wall projection in (b) around the z axis produces the cluster.

cylinder with a cone, as illustrated in Figure 15a. The cone can be quite general, not necessarily circular, but for the sake of simplicity we consider a right circular cone with vertex angle 2α, where $\alpha < \pi/2$.

Take a region S in the upper half of the xy plane tangent to the cone, with the x axis matching a generator of the cone. As this plane moves tangentially around the cone, region S sweeps out a toroid-like solid that we call a *conical sweep*. We are interested in determining the volume of the conical sweep.

Each cross section of the sweep cut by a plane perpendicular to the axis of the cone, which we call the z axis, is part of a planar ring whose area does not depend on the position of S, which can be near to or far away from the cone's vertex V. Consequently the volume of the conical sweep does not depend on the position of S. For convenience we take the origin of the xy plane to be the vertex V. Figure 15b shows a projection of S on the yz plane, called the *wall projection*, which makes an angle α with the x axis (half the vertex angle of the cone). The area of the wall projection of S is $\cos\alpha$ times the area of S. Figure 15c shows the solid of revolution obtained by rotating the wall projection around the z axis. We call this solid *the cluster of the conical sweep*. A plane perpendicular to the z axis cuts both the conical sweep and its cluster in regions of equal area so, by the slicing principle, their volumes are equal. This gives the following theorem, with the notation just introduced.

Theorem 3. (a) *The volume of a conical sweep of S does not depend on the distance of S from the vertex of the cone.*

(b) *The volume of a conical sweep is equal to the volume of its cluster.*

(c) *This common volume is equal to* cos α *times the volume of the solid of revolution obtained by rotating region S around a fixed axis.*

Ellipsoid of revolution. Our first application of Theorem 2 is to an ellipsoid of revolution shown in Figure 16b. When a semielliptical disk is swept tangentially

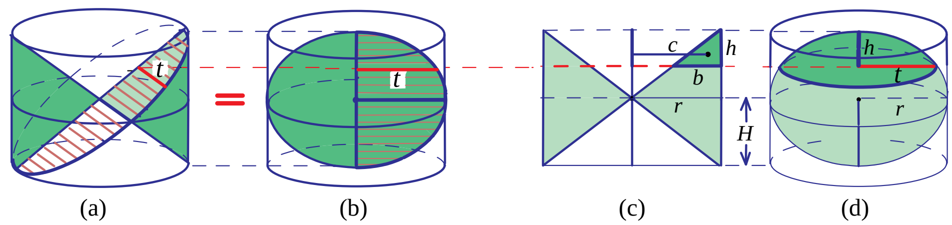

Figure 16. Finding the volume of an ellipsoid in (a) and (b), and of an ellipsoidal segment of altitude $h < H$ in (c) and (d).

around a circular cone as in Figure 16a it generates a punctured cylinder, a solid sweep lying between the cone and its circumscribing cylinder. The volume of this solid sweep is known to be 2/3 that of the cylinder. By Theorem 2 the volume of its cluster, the ellipsoid of revolution in Figure 16b, is also 2/3 that of its circumscribing cylinder. When the ellipsoid is a sphere this is Archimedes' tombstone result.

We shall determine, more generally, the volume $V(h)$ of the ellipsoidal segment of altitude h in Figure 16d in an alternative way by rotating the shaded triangle in Figure 16c about the vertical axis and applying Pappus's theorem. The shaded triangle of altitude h in Figure 16c sweeps out the upper portion of the punctured cylinder, which is the same as a portion of the solid sweep swept tangentially by the corresponding portion of the semielliptical disk in Figure 16a. The tangent cluster of this portion is the ellipsoidal segment of volume $V(h)$ in Figure 16d.

By Pappus, volume $V(h)$ is the product of the area of the triangle and the distance its centroid moves in one revolution. The area of the triangle is $bh/2$ and the area centroid of the triangle is at distance $c = r - b/3$ from the axis of rotation, hence

$$V(h) = 2\pi(r - \frac{b}{3})\frac{bh}{2} = \frac{\pi}{3}(3r - b)bh. \tag{1}$$

Archimedes [4; *On Conoids and Spheroids*, Proposition 27] showed that $V(h)$ bears a simple relation to the volume V_{cone} of the cone in Figure 17a with the same base and altitude (altitude h and base radius t, where t is the length of the chord in Figures 17a and 17b), namely

$$\frac{V(h)}{V_{\text{cone}}} = \frac{3H - h}{2H - h}, \tag{2}$$

where H is the length of the vertical semiaxis of the ellipsoid (half the height of the circumscribing cylinder) in Figure 16d, and $h < H$.

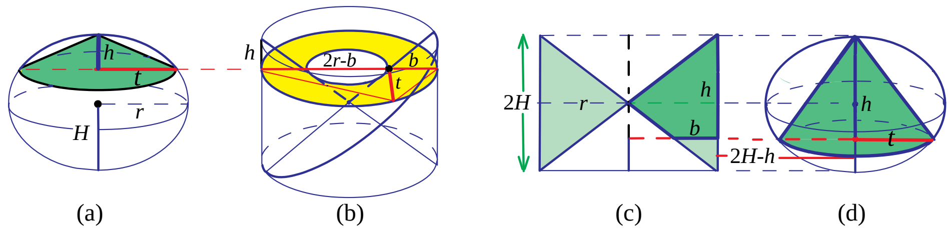

Figure 17. Proof of Archimedes's formula (2). In (d), $h \geq H$.

A simple proof of (2) can be given by observing that volume $V_{\text{cone}} = \pi t^2 h/3$. By similar triangles in Figure 17b, we find $b/t = t/(2r - b)$, so $t^2 = (2r - b)b$, hence

$$V_{\text{cone}} = \frac{\pi}{3} t^2 h = \frac{\pi}{3}(2r - b)bh. \tag{3}$$

Now divide (1) by (3) and use the similarity relation $r/b = H/h$ to obtain (2).

The same type of argument, using Figure 17d, proves (2) when $h \geq H$. (When h is replaced by $-h$, (2) becomes ratio (6) in [2] for a hyperboloidal segment.)

Paraboloid of revolution. Another result of Archimedes [4; *On Conoids and Spheroids*, Props. 21, 22], depicted in Figure 18b, states that *the volume of a paraboloidal solid of revolution is equal to half that of its circumscribing cylinder*. We shall deduce this by applying Theorem 2.

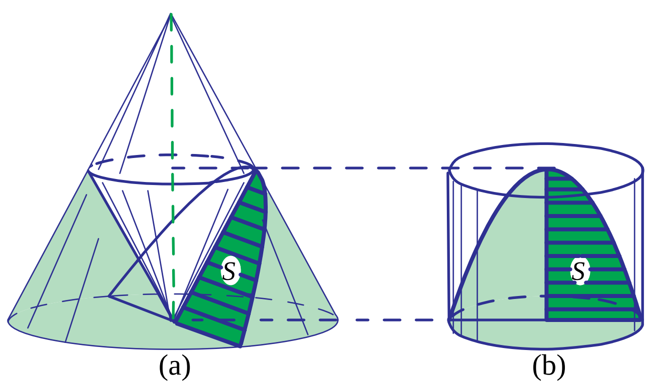

Figure 18. Tangential sweeping by half a parabolic sector around a cone produces a solid sweep in (a) whose cluster is a paraboloid of revolution in (b). The volume of the paraboloid is half that of its circumscribing cylinder.

Cut the large cone C in Figure 18a by a plane parallel to a generator through a point midway between the base and vertex of C. We take half the parabolic cross section as region S and and form a conical sweep by rotating S tangentially around the smaller cone c whose vertex is at the center of the base of the larger cone C. The corresponding conical cluster is the paraboloid of revolution in Figure 18b. By Theorem 2, its volume V is equal to that of the conical sweep in Figure 18a.

This solid sweep is inside the large cone C and outside the small cone c. Thus, $V = 6v(c)$, where $v(c)$ is the volume of the small cone c. But $v(c)$ is one-third the volume of the circumscribing cylinder through the base of c, so V is twice the volume of this cylinder which, in turn, is half that of the larger circumscribing cylinder in Figure 18b. This proves the result of Archimedes. The same result follows from (2) by keeping h fixed and allowing H to tend to ∞ so the ellipsoidal segment becomes paraboloidal.

General persoid of revolution. A torus is the surface of revolution generated by rotating a circle about an axis in its plane. The curve of intersection of a torus and a plane parallel to the axis of rotation is called a *curve of Perseus*, examples of which include the ovals of Cassini and leminscates of Booth and Bernoulli. Each such curve of Perseus has an axis of symmetry parallel to the axis of rotation. When the *persoidal region*, bounded by a curve of Perseus, is rotated about this axis of symmetry it generates a solid that we call a *persoid of revolution*.

In [2] we treated persoids of revolution obtained by rotating persoidal regions cut from a torus by planes parallel to the axis of the torus. Now we consider more general persoidal regions obtained by cutting planes that make an angle $\alpha < \pi/2$ with the toroidal axis. Examples are shown in Figures 19, 20, and 21.

In Figure 19a the axis of symmetry of the plane cross section S is designated as the x axis. In Figure 19b the x axis is oriented vertically and S is rotated about this fixed axis to generate a *general persoid of revolution*. By Theorem 2c, its

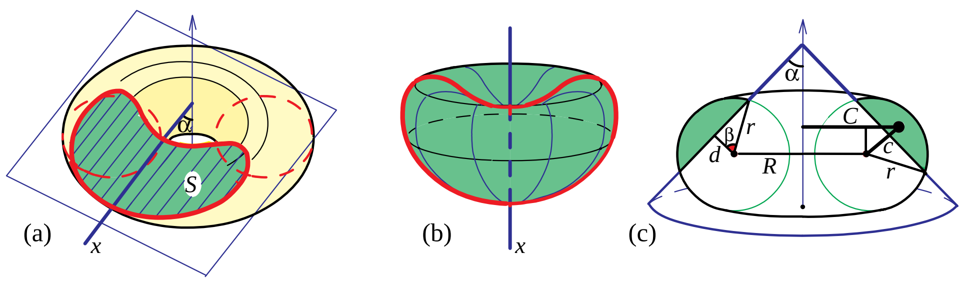

Figure 19. (a) Slanted toric section. (b) Its solid of revolution. (c) Diagram for calculating the volume by Pappus' theorem.

volume V is 1/cos α times the volume of the conical sweep obtained by tangential sweeping of S around a cone with vertex angle 2α. This cone is shown in Figure 19c together with a cross section of the torus through its axis. The tangential sweep is the portion of the solid torus outside the cone. This same solid is generated by rotating the circular segment in Figure 19c about the axis of the cone. By Pappus, the volume of this solid of revolution is $2\pi CA$, where A is the area of the circular segment, and C is the centroidal distance of the segment from the axis of rotation. Hence volume V of the solid of revolution in Figure 19b is given by

$$V = \frac{2\pi CA}{\cos\alpha}. \tag{4}$$

Now we show that

$$V = \frac{4}{3}\pi r^3 \sin^3 \beta + \frac{\pi R r^2 (2\beta - \sin 2\beta)}{\cos \alpha}, \tag{5}$$

where r is the radius of the circle that generates the torus as its center moves around a circle of radius R, α is half the vertex angle of the cone, and β is half the angle that subtends the circular segment of radius r. The first term in (5) is $\frac{4}{3}\pi(r \sin \beta)^3$, the volume of a spherical bracelet of altitude $r \sin \beta$.

Area A of the circular segment, expressed in terms of r and β, is

$$A = r^2(\beta - \sin \beta \cos \beta). \tag{6}$$

From Figure 19c we find $C = c \cos \alpha + R$, where c is the centroidal distance of the segment from the center of the circle of radius r. Hence $CA/\cos \alpha = cA + RA/\cos \alpha$. But $cA = \frac{4}{3}\pi(r \sin \beta)^3$ so (4) and (6) yield (5).

Figure 20a shows a vertical axial section of the torus and the end view of three parallel cutting planes that pass through the hole in the torus making an angle α with the vertical axis of the torus. They cut three curves of Perseus as indicated. We wish to find the volumes of each persoid of revolution about its own axis.

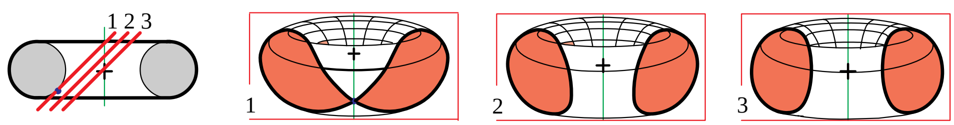

Figure 20. Slanted toric sections cut by parallel planes through the hole of the torus.

According to Theorem 2, this volume is equal to that of the conical sweep divided by $\cos \alpha$. In each of these example, the conical sweep is the entire solid torus, whose volume is $2\pi^2 r^2 R$, so the volume V of each persoid of revolution is

$$V = \frac{2\pi^2 r^2 R}{\cos \alpha}. \tag{7}$$

An interesting case occurs when the cutting plane is tangent internally to the inner part of the torus, as in Figure 21a. Here the curve of Perseus consists of two intersecting circles of radius R as seen from a direction perpendicular to the cutting plane. In Figure 21a, $r = R \cos \alpha$ so (7) gives $V = 2\pi^2 R^2 r$, which is the volume of a different torus generated by a circle of radius R rotated around a circle of radius r.

5. FAMILIES OF CONE-DRILLED SOLIDS OF EQUAL VOLUME

We turn next to examples of families of cone-drilled solids of equal volume obtained by sweeping simple shapes bounded by portions of conic sections (including degenerate conics) along a right circular cone. When the conic is attached to a generator of the cone along one of its axes of symmetry as in Figure 22, both the tangent sweep and the tangent cluster are solids bounded by quadric surfaces. Figure 22a shows a rectangular strip of given width attached tangentially to a right circular cone. Tangential sweeping produces a portion of a twisted cylinder outside

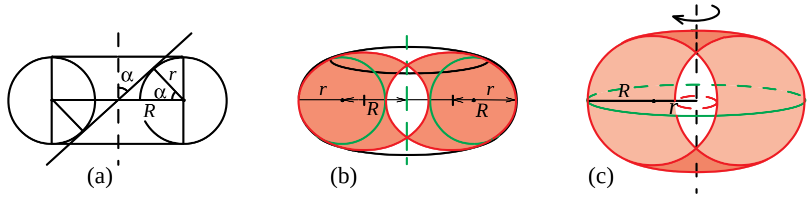

Figure 21. (a) Cross section showing cutting plane as a line doubly tangent internally to the inner part of the torus (side view). (b) Inclined view of the section in (a). (c) Normal view seen from a direction perpendicular to the cutting plane.

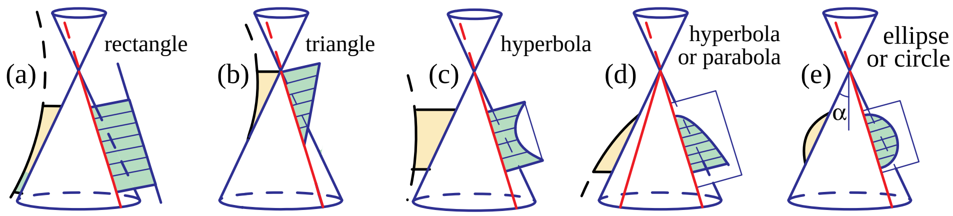

Figure 22. Regions bounded by conics sweeping tangentially around a cone.

the cone. A family (not shown) of equal height and equal volume is produced by shifting the rectangle up or down along the generator of the cone.

An interesting family is obtained by varying the vertex angle of the cone. By slicing all swept solids by parallel horizontal planes at distance H apart we get a family of slices of equal volume independent of the cone's vertex angle, as depicted in Figure 23. If the width of the strip is w, the volume of each slice in this family is equal to that of a circular cylinder of height H and radius w, or $\pi w^2 H$. In Figure

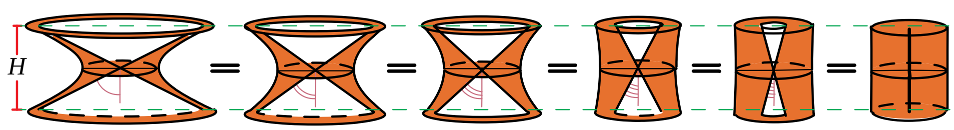

Figure 23. A family of cone-drilled hyperboloids of the same height and equal volume.

23 the swept solids are symmetric about the vertex of the central cone, but the same result holds if the tangential sweeping is done at any location relative to the vertex. The volume of each swept solid is equal to that of the circular cylinder.

One leg of a given right triangle can be attached tangent to a cone anywhere along a generator as in Figure 22b and rotated to sweep a portion of a twisted cylinder outside the cone. Varying the position of the right triangle produces a family of cone-drilled solids (not shown) having the same height and the same volume, that of the cone obtained by revolution of the vertical wall projection of the sweeping triangle.

We can attach a plane region bounded by a portion of a conic section as in Figures 22c, d and e, to produce more examples of interesting families of cone-drilled solids. A semielliptical disk will sweep a portion of an ellipsoid, or a paraboloid or hyperboloid of two sheets, depending on the proportions of the semiaxes of the ellipse.

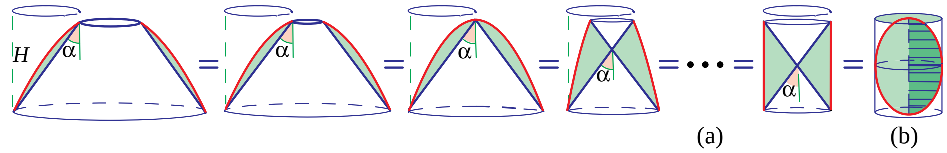

Figure 24. (a) Cone-drilled paraboloids of equal height and equal volume.
(b) Limiting case is a cylinder punctured by a cone.

If the ellipse is represented by a circle in its ceiling projection, then the solid is paraboloidal, drilled by a cone as in Figure 24. In this case all solids in this family have volumes equal to that of the ellipsoid obtained by revolution of the wall projection of the sweeping ellipse. If the lengths a and b of the semiaxes

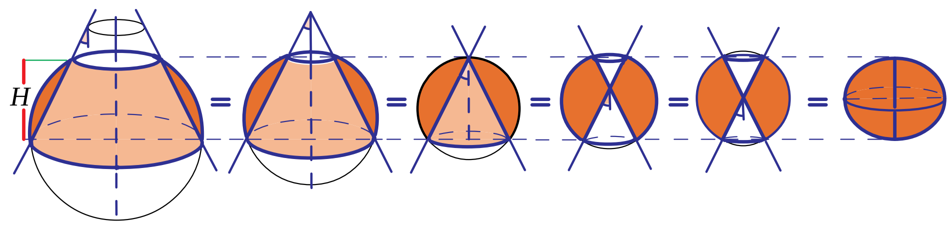

Figure 25. Cone-drilled spheres of equal height and equal volume.

of the ellipse satisfy $a/b < \sin\alpha$, where α is half the vertex angle of the cone, we obtain a family of cone-drilled similar ellipsoids of revolution. When $a = b$ the ellipse is a circle attached to a cone along its diameter as in Figure 22e and we obtain cone-drilled spheres of different sizes (Figure 25), all having the same height and the same volume. When a/b has a larger value we obtain a family of cone-drilled hyperboloids of two sheets, all having the same volume if their heights are equal.

The table in Figure 26 supplements that in Figure 5 by including a cone as a quadric surface. The first entry, labeled C, shows an axial vertical section of a cone punctured by a cylinder, and of a cylinder punctured by a cone. The second entry, labeled E, shows an axial vertical section of an ellipsoid punctured by a cone, and of a cone punctured by an ellipsoid. The remaining entries have analogous meanings, with P representing a paraboloid of revolution, H_1 a hyperboloid of one sheet, and H_2 one part of a hyperboloid of two sheets.

Combining families of drilled solids of equal height. Figure 27 shows a family of solids obtained by combining the families in Figure 24 and 25, where each member

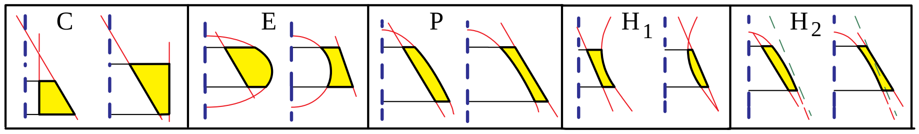

Figure 26. Axial vertical sections of quadric surfaces drilled by cones, and of cones punctured by quadric surfaces.

of the respective family is drilled by a congruent truncated cone of height H. The solids in this new family, shown with darker shading, also have the same volume, the difference of the volumes of those in Figure 24 and 25. That common volume, in turn, is that of a sphere.

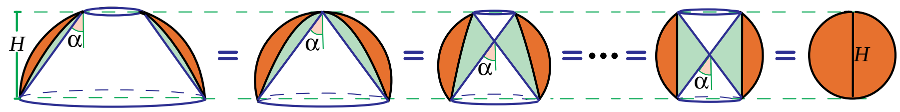

Figure 27. Paraboloidal-drilled spheres of height H and equal volume, that of a sphere of diameter H.

Similarly, Figure 28 shows a family of solids obtained by combining the families of the type in Figures 2 and 3 with the same height H. The limiting case in (c) is a portion of a sphere punctured by a cone, whose volume is that of the ellipsoid in (d).

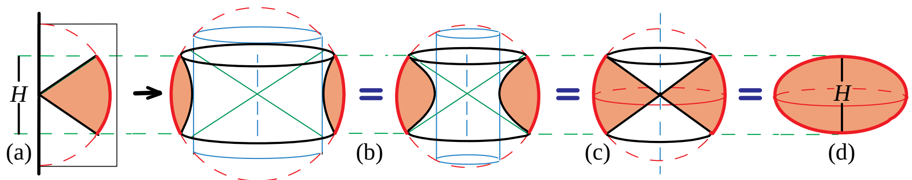

Figure 28. Spheres punctured by twisted cylinders of equal height produce solids of equal volume, that of an ellipsoid.

Special cases previously considered. Two special cases are treated by Polya [6; p. 202], where the volume of a conically and parabolically perforated sphere are obtained using integration. Polya's examples were extended by Alexanderson and Klosinski [1], who also used integration to calculate volumes of several solids obtained by rotating the region between two conic sections. They did not consider arbitrary horizontal slices as we did in Figure 5, but considered only special slices between common intersection points of the conics, so the entire boundary of each solid is made up of portions of quadric surfaces. Their examples are summarized in Figure 29, where now C represents a cone, a cylinder being a special case. Their list can be extended (without integration) as shown in Figure 30, where hyperboloids of two sheets H_2 are also considered. The notation H_2^2 indicates that both sheets are used. Similar examples were also treated in [5].

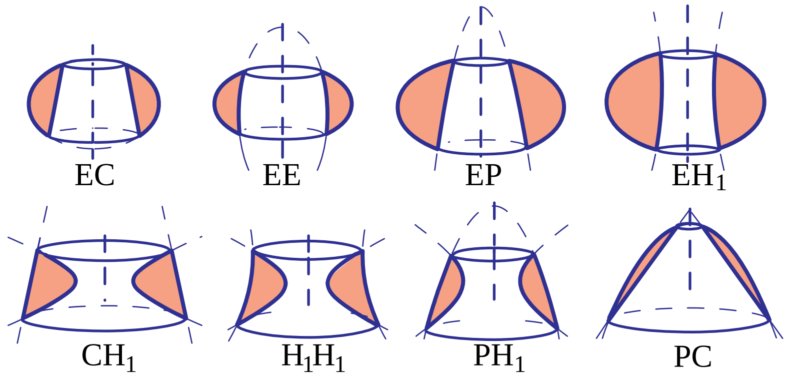

Figure 29. Solids generated by rotating regions between intersections of two conics.

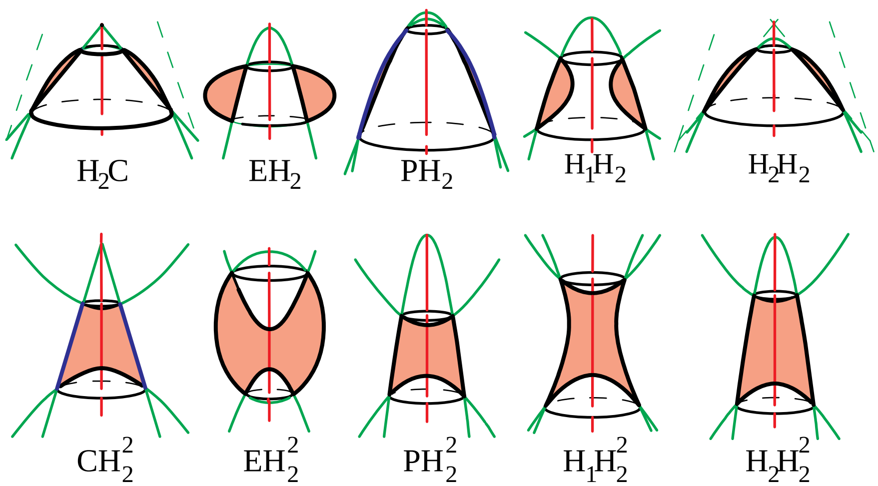

Figure 30. More solids like those in Figure 29 with hyperboloids of two sheets included.

In each entry of Figures 29 and 30 each conic can be scaled separately and shifted vertically so that the height of the hole in the punctured solid has a fixed value. Each entry yields a *family* of punctured solids having equal height and equal volume.

Alternative treatment. The equality of volumes for the families in Figures 27 or 28 can be obtained in an alternative manner, as illustrated in Figure 31 in a general setting. Take any plane region S between two graphs in the same half-plane as described in Section 2. Rotate this region tangentially along a cone (Figure 31a), or rotate its wall projection around a cylinder (Figure 31c). We generate a family of tangentially swept solids of equal volume by translating region S along the generator of the cone or by varying the radius of the cylinder. The common volume is that of the cluster in Figure 31b.

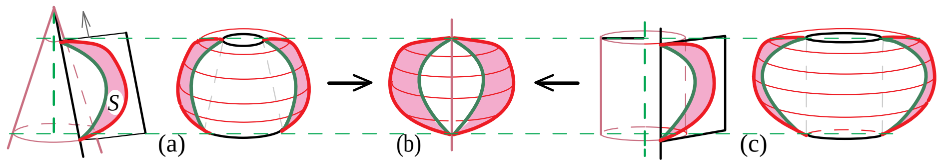

Figure 31. (a) Solid tangential sweep around a cone. (c) Solid tangential sweep of wall projection of (a) around a cylinder. (b) Solid tangent cluster of (a) and of (c).

Area balance of axial sections of swept solids. In [2] we showed that any vertical cross section of a general tangentially swept solid around a circular cylinder is in area balance with the corresponding vertical cross section of its solid cluster. This is a consequence of a balance-revolution principle introduced in [3; p. 410]. *The areas of two plane regions are in equilibrium with respect to a balancing axis if, and only if, the solids of revolution generated by rotating them about the balancing axis have equal volumes.* The same is true when the circular cylinder is replaced by a right circular cone. In fact, a stronger result holds. *Any vertical section of a tangentially swept solid around a circular cylinder or a right circular cone is in chord-by-chord balance with the corresponding vertical cross section of its solid cluster (with respect to the common axis of the cylinder or cone).*

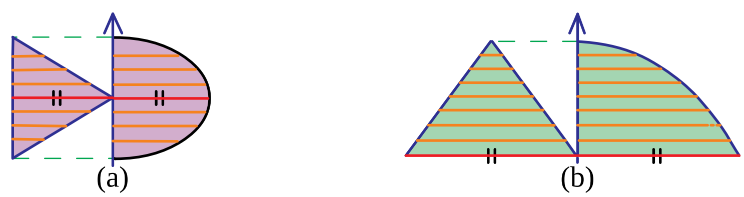

Figure 32. (a) Triangle and semielliptical disk in chord-by-chord balance. (b) Isosceles triangle and semiparabolic segment in chord-by-chord balance.

This follows from the fact that each horizontal cross section of the sweep is a circular ring whose area is equal to the corresponding circular cross section of the cluster, so by using Pappus we see that the horizontal chords are in balance. Examples of area balance of axial sections are exhibited by any two members of a family of solids of revolution generated by any entry in Figure 5. Figure 32a shows an example of area balance of axial sections of Figures 16a and 16b, a triangle and a semielliptical disk. Another example is shown in Figure 32b, area balance of axial sections of Figures 18a and 18b, an isosceles triangle and a semiparabolic segment. More examples appear in Figures 34 and 36.

6. CENTROIDS OF SOLID SWEEPS AND CLUSTERS

The property of volume centroids described in Corollary 2 of Section 1 also applies to solid sweeps obtained by sweeping around a cone instead of a cylinder. *The altitudes of the volume centroids of a solid sweep and its solid cluster are* *equal.* Now we use this property to locate the volume centroids of several solids of revolution. The first two are the ellipsoidal and paraboloidal segments of revolution in Figures 33a and b. Archimedes treated the centroid of a spherical segment (a special case of an ellipsoidal segment), and of a paraboloid of revolution. The next two, shown in Figure 34, were not treated by Archimedes. Figure 34a shows the upper half of a hyperboloid of one sheet, and Figure 34b shows the lower half of a hyperboloid of two sheets.

Centroids of ellipsoidal segment and paraboloidal segment. For the ellipsoidal segment of height h shown Figure 33a, z denotes the distance of its volume centroid from the top. We shall show that

$$z = h\frac{8H - 3h}{4(3H - h)}, \tag{8}$$

where H is the altitude of a full semiellipsoidal solid. For a spherical segment, Equation (8) is equivalent to Proposition 9 in [4; *Method*, p.35]. When $h = H$ the ellipsoidal segment is half an ellipsoid and (8) gives $z = \frac{5}{8}H$.

To prove (8), recall that in Figure 16 we observed that the ellipsoidal segment is the solid tangent cluster of a solid tangent sweep obtained by rotating a triangle tangentially around a circular cylinder. By Corollary 2, distance z is equal to that for the cylinder of altitude h and radius r punctured by an inverted truncated cone as in Figure 33a, whose volume we denote by V_{TC}, and whose centroidal distance from the top we denote by z_{TC}. Let $V(h)$ denote the volume of the ellipsoidal segment. The solid cylinder of radius r and altitude h has centroidal distance $h/2$ from the top, so by equating moments about the top we find

$$zV(h) + z_{\mathrm{TC}}V_{\mathrm{TC}} = \frac{h}{2}\pi r^2 h. \tag{9}$$

The term $z_{\mathrm{TC}}V_{\mathrm{TC}}$ is also the difference of moments of a large cone of altitude H and radius r, and a smaller cone of altitude $H - h$ and radius $r(H - h)/H$, which gives

$$z_{\mathrm{TC}}V_{\mathrm{TC}} = \frac{H^2}{4}\pi\frac{r^2}{3} - (\frac{H}{4} + \frac{3h}{4})\frac{\pi}{3}(H - h)(\frac{H - h}{H}r)^2. \tag{10}$$

From (1) we have $V(h) = \frac{\pi}{3}(3r - b)bh$, where $b = hr/H$. This becomes $V(h) = \frac{\pi}{3}(3H - h)(rh/H)^2$, which when used in (9) together with (10) leads, after much algebraic simplification, to (8).

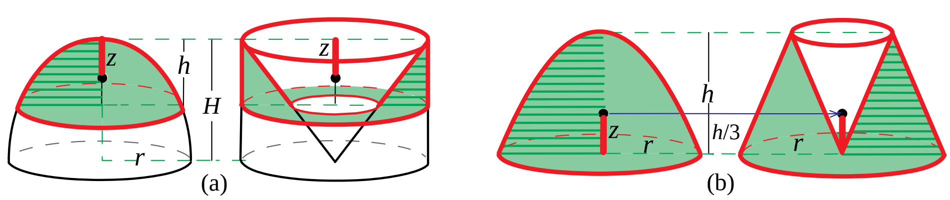

Figure 33. (a) Centroid of an ellipsoidal segment. (b) Centroid of a paraboloidal segment.

We treat next the paraboloidal segment of altitude h in Figure 33b. In this case the formula for centroidal distance z from its base is very simple:

$$z = \frac{h}{3}, \tag{11}$$

a result found by Archimedes in [4; *Method*, Proposition 5]. Figure 18 shows that the paraboloidal segment is the solid cluster of the tangential sweep, and we showed earlier that its volume V is six times the volume v of the small inverted cone of the same altitude in Figure 33b. To prove (11) we note that the solid tangent sweep in Figure 33b can be obtained by removing two smaller cones, each of volume v, from the large cone of altitude $2h$ in Figure 18a. Equating moments about the base of the configuration in Figure 18a we find

$$6zv + 2vh = 4vh,$$

which immediately gives (11).

Note that the centroidal distance $h/3$ of the paraboloidal segment is exactly the same as the planar centroidal distance of the isosceles triangle of base r and altitude h that sweeps out the punctured truncated cone in Figure 33b when rotated about the axis of the cone. As we will show later in this section, this is not a mere coincidence, but is a phenomenon shared by solids obtained by rotating planar regions with an axis of symmetry.

Centroids of hyperboloidal segments. First we treat a hyperboloidal segment of one sheet cut from the upper half of the unpunctured solid in Figure 3a. The segment has altitude h, lower circular base of radius r, and upper circular base of radius R, as shown in Figure 34a. We will show that its centroidal distance Z from the lower base is given by

$$Z = \frac{3}{4}h\frac{R^2 + r^2}{R^2 + 2r^2}. \tag{12}$$

We know that the punctured solid has the same volume and same centroidal distance from the base as its solid cluster, the cone of altitude h and radius t in Figure 34a.

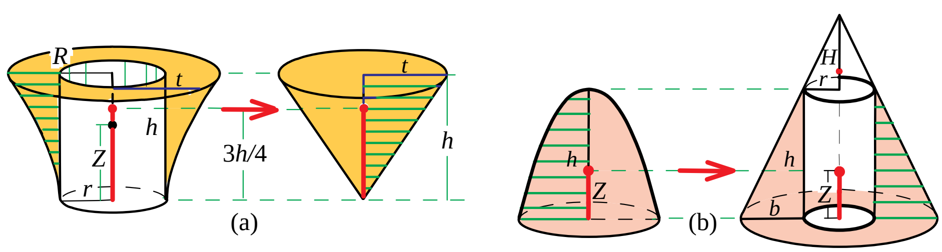

Figure 34. Centroid of hyperboloidal segments of (a) one sheet, and (b) one of two sheets.

Here $t^2 + r^2 = R^2$. Equating moments of the volume [cone] of the cone plus the volume [cyl] of the cylindrical hole with that of the volume [cone]+[cyl] of the unpunctured hyperboloidal segment, we find

$$\frac{3}{4}h[\text{cone}] + \frac{1}{2}h[\text{cyl}] = Z([\text{cone}] + [\text{cyl}]). \tag{13}$$

But [cone]= $\pi t^2 h/3$ and [cyl]= $\pi r^2 h$. Use these in (13) and solve for Z to obtain (12).

For the hyperboloidal segment of two sheets, one of which, of altitude h, is shown in Figure 34b, the centroidal distance Z from the base is given by

$$Z = \frac{h}{4}\frac{4H+h}{3H+h}, \tag{14}$$

where H is the altitude of the small cone in Figure 34b.

To prove (14), we use the fact that the punctured truncated cone in Figure 34b has the same volume $V(h)$ and same centroidal distance Z from the base as its solid cluster, the hyperboloidal segment. Volume $V(h)$ is equal to that of the solid swept by the triangle of base b and altitude h in Figure 34b. By Pappus, we have

$$V(h) = 2\pi(r + \frac{b}{3})\frac{bh}{2} = \pi r^2(3 + \frac{h}{H})\frac{h^2}{3H}, \tag{15}$$

where we have used the similarity relation $b/r = h/H$. Equating moments of the configuration in Figure 34b about the base, we have

$$\frac{1}{4}(H+h)[\text{Cone}] = \frac{1}{2}h[\text{cyl}] + (h + \frac{1}{4}H)[\text{cone}] + ZV(h), \tag{16}$$

where [Cone] denotes the volume of the large cone of radius $r + b$ and altitude $H + h$, [cyl] denotes the volume of the cylinder of radius r and altitude H, and [cone] denotes the volume of the small cone of radius r and altitude H. Now use the volume formulas

$$[\text{Cone}] = \frac{1}{3}\pi(r+b)^2(H+h) = \frac{1}{3}\pi r^2(1+\frac{h}{H})^2(H+h),\ \ [\text{cone}] = \frac{1}{3}\pi r^2 H,\ \ [\text{cyl}] = \pi r^2 h$$

together with (15) in (16), and solve for Z to get (14) after algebraic simplification.

Special centroidal altitude lemma. Figure 35a shows a right triangular region of altitude h rotated to generate a solid cone of the same altitude. The areal centroidal distance of the triangle above its base is $h/3$, but the volume centroidal distance of the cone is $h/4$ (Figure 35a). Earlier we observed that the volume centroidal distance $h/3$ of the paraboloidal segment in Figure 33b is equal to the areal centroidal distance of an isosceles triangle of altitude h that sweeps out the punctured truncated cone. This surprising result is explained by the following lemma on centroidal altitudes, illustrated in Figure 35b.

Centroidal altitude lemma. *The area centroid of any axially symmetric plane region has the same altitude above any fixed base as the volume centroid of the solid of revolution swept by the plane region around any axis in that plane disjoint from the region that is parallel to the axis of symmetry.*

The idea of the proof is very simple. Because each horizontal chord of the plane region has its centroid on the axis of symmetry, during the revolution it sweeps an area proportional to the chord length. Therefore, the volume centroid of the solid

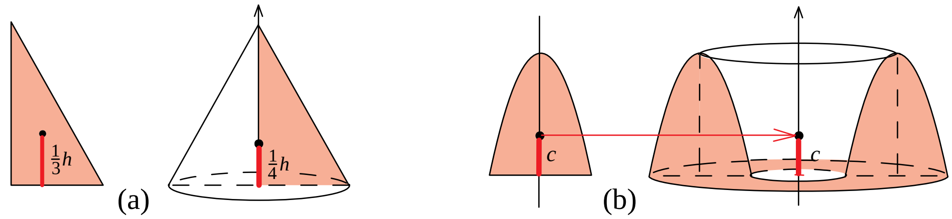

Figure 35. (a) Centroid of triangle and cone are different. (b) Centroidal altitude lemma.

having these areas as horizontal cross sections is at the same altitude as the areal centroid determined by the chords.

This idea can be converted into a rigorous proof by using integrals to represent the two centroids. If $l(h)$ denotes the length of the chord at altitude h, the altitudes of the area centroid and volume centroid are given, respectively, by

$$\text{area centroid} = \frac{\int l(h)hdh}{\int l(h)dh}, \quad \text{volume centroid} = \frac{\int A(h)hdh}{\int A(h)dh},$$

where $A(h)$ denotes the cross sectional area of the solid at altitude h. By Pappus, $A(h) = 2\pi Rl(h)$, where R is the distance between the two parallel axes. The constant factor $2\pi R$ cancels in the second ratio of integrals, and we see that the area centroid and volume centroid are at the same altitude.

Now we apply the lemma to the upper half of a torus (Figure 36b) generated by revolving a semicircular disk of radius r (Figure 36a) around any axis at distance $R \geq r$ from its center. According to the lemma, the volume centroid of the semitorus is at the same altitude as the area centroid of the semicircular disk, regardless of R. Figure 36c shows a vertical section of the semitorus by a plane internally tangent to the torus, the upper half of a persoidal region consisting of two congruent pieces with an axis of symmetry. When one piece is swept tangentially around a cylinder through the hole of the torus its tangent sweep is the torus itself. When rotated around its own axis of symmetry it produces the semipersoid of revolution (d). According to Corollary 2, the semitorus in Figure 36b and solid in (d) have the same centroidal altitude. Consequently, the volume centroid of the semipersoid of revolution in (d) has the same altitude as that of the semicircular disk in (a).

The same property holds for any horizontal slice of the configuration in Figure 36 because any horizontal slice of a semicircular disk has an axis of symmetry.

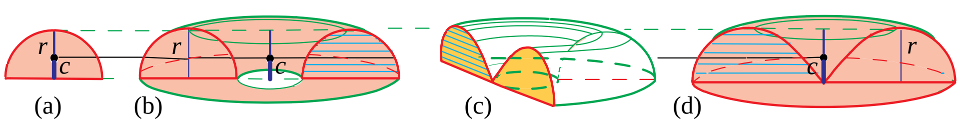

Figure 36. Semicircular disk (a), semitorus (b), and solid (d), obtained by revolution of the lemniscate in (c), all have their centroid at the same altitude.

More generally, the foregoing analysis applies to any persoidal region cut by a vertical plane passing through the hole of the torus. Moreover, it also applies when the circular disk in Figure 36a is replaced by any symmetric plane region generating a toroidal-like solid. For example, we can use an isosceles triangle as in Figure 37a and rotate it around any vertical axis disjoint from the triangle. The resulting toroidal-like solid will a punctured truncated cone, and the corresponding persoid-like solids will be bounded by two hyperboloids of revolution as shown in Figure 37b. Their centroids will be at the same altitude above the base, which in this case is one-third the altitude of the triangle.

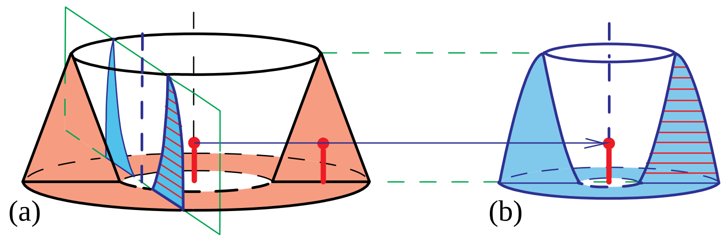

Figure 37. (a) Punctured truncated cone with a section bounded by two hyperbolas. (b) Hyperboloid of revolution punctured by another hyperboloid. The centroids of both solids have the same altitude as that of the triangle.

Finally, we note that the special centroidal altitude lemma is also valid when the symmetric plane region is swept tangentially along any cylinder whose generator is parallel to the symmetry axis of the region. In view of Corollary 2, the tangentially swept solid has its volume centroid at the same altitude as that of the plane region.

7. TANGENTIAL SWEEPING AROUND A GENERAL SURFACE

Earlier we generated solids by tangential sweeping along a cylinder or cone. Now we use a more general surface as depicted by the lightly shaded region in Figure 38, which we call a *tangency surface*. Take such a surface and slice it by a family of horizontal parallel planes, as indicated in Figure 38a. Take a typical curve of intersection as tangency curve, and construct a tangent sweep using vectors from the tangency curve to some free-end curve. The free-end curves lie on another surface, as illustrated in Figure 38a, which we call the *free-end surface*. For each horizontal tangent sweep, construct its planar tangent cluster by translating each tangent to a common point in that plane. The darker shaded regions in Figure 38 depict the tangent sweep and its cluster in the bottom plane. They have equal areas. As we move from the bottom horizontal plane to the top one in Figure 38a, the solid swept tangentially between the tangency surface and the free-end surface is called a *solid tangent sweep*. Now construct the *solid tangent cluster* as the union of the planar clusters with their common points lying on one vertical line P, as in Figure 38b.

Each horizontal plane intersects the solid tangent sweep and the solid tangent cluster along plane regions having equal area. By the slicing principle, we have:

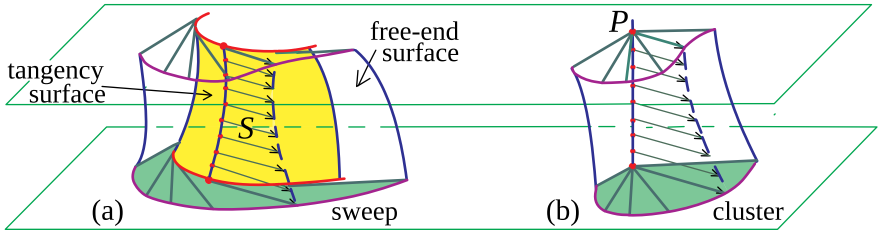

Figure 38. (a) Solid tangent sweep. (b) Solid tangent cluster.

Theorem 4. *The portion of a solid tangent sweep between any two horizontal planes has the same volume as its corresponding solid tangent cluster.*

Corollary 2 of Theorem 1 also is valid for solid sweeps and clusters in Theorem 3: *The altitudes of the volume centroids above a fixed horizontal base are equal.*

As in earlier examples, there is an alternative method for producing the solid tangent sweep. Choose an initial tangent vector in the bottom horizontal plane from the tangency surface to the free-end surface. As we move continuously from the bottom plane to the top, select those tangent vectors of the tangency curves parallel to the initial tangent vector. Their tangency points trace a curve which is a directrix for a cylindrical region containing all these parallel tangency vectors. This cylindrical region, which we call S, plays the same role as the plane region S used earlier for sweeping along a cylinder or cone. It can change its shape as it moves tangentially around the surface.

Tangency surfaces of revolution. In Figure 39a the tangency surface is a sphere of diameter H, and S is a rectangular strip of width w wrapped tangentially with one edge along a meridian joining the poles. As S rotates around the sphere, the opposite edge sweeps part of the surface of a larger sphere, so the solid tangent sweep is a solid spherical shell between two concentric spheres with the polar caps of the larger sphere removed, as depicted in Figure 39b. Figure 39c shows the

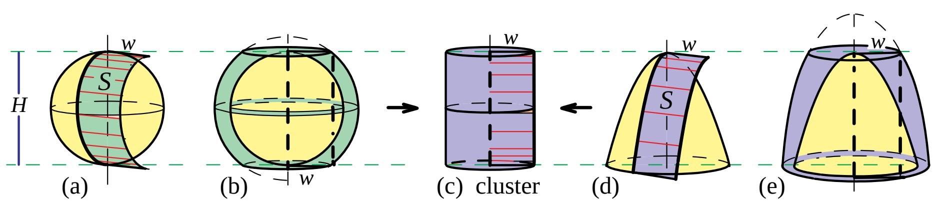

Figure 39. (a) Rectangle sweeping around a sphere. (b) Solid tangential sweep. (d) Rectangle sweeping around paraboloid. (e) Swept solid. (c) Common solid tangent cluster.

corresponding solid tangent cluster, a circular cylinder of radius w and altitude H. According to Theorem 3, the spherical shell and the cylinder have equal volumes. This was also noted in [3; Theorem 5.2]. When a rectangle of width w is wrapped around a paraboloid of altitude H as in Figure 39d, the corresponding solid tangent sweep is the solid paraboloidal shell in Figure 39e. The cylinder in Figure 39c is its solid tangent cluster. Surprise: The volume of the paraboloidal shell in (e) is equal to that of the spherical shell in (b)! Two more examples, with the paraboloid

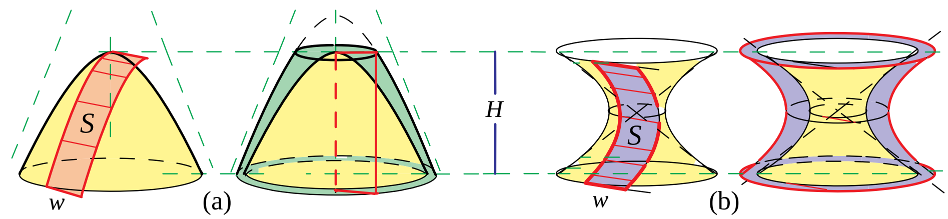

Figure 40. Rectangle sweeping around two types of hyperboloid in (a) and (c). Their solid tangent sweeps have the same volume as their common tangent cluster cylinder in (b).

replaced by two types of hyperboloid, are shown in Figure 40a and 40b. Another surprise: Each solid tangent sweep has volume equal to that of the cylinder in Figure 39c.

Figure 41a shows an example of Figure 39a in which the inner surface is produced by rotating a curve $y = y(x)$ around the x axis, and the outer surface is a coaxial cylinder of radius a. A horizontal tangent vector from the surface to the

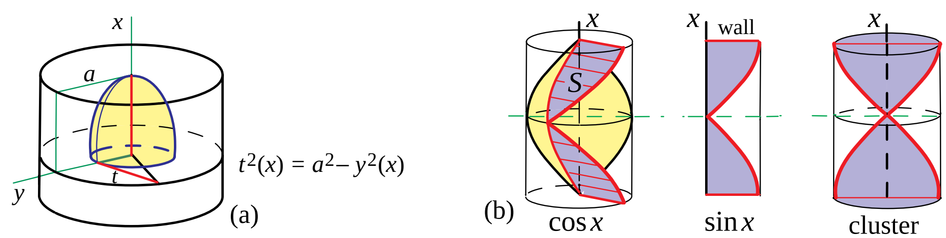

Figure 41. (a) Tangent of length $t(x)$ from inner surface of revolution to outer cylinder. (b) Tangential region S sweeps solid between rotated cosine curve and cylinder of radius 1. Its wall projection is the ordinate set of a sine curve; rotating it produces solid cluster.

cylinder has length $t = t(x)$ given by $t^2(x) = a^2 - y^2(x)$. The vectors of length t form a region S that generates a solid tangential sweep lying outside the surface and inside the cylinder, and the wall projection is a plane region formed by the ordinate set of $t(x)$. For example, if $a = 1$ and $y(x) = \cos x$ as shown in Figure 41b, then $t(x) = \sin x$, and the corresponding solid tangent cluster is a surface bounded by rotating a sine curve. Its volume is equal to that of the solid tangent sweep. Consequently, the solid between the cylinder and the rotated cosine has the same volume as the rotated cosine, each being half that of the circumscribing cylinder. The same is true, of course, for the solid between the cylinder and the rotated sine.

In Figure 42a, the inner surface is a paraboloid obtained by rotating the parabola $y^2(x) = x$ around the x axis, and the wall projection is bounded by a portion of the

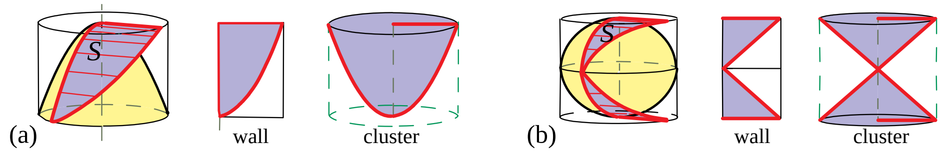

Figure 42. (a) Paraboloid inscribed in cylinder. (b) Ellipsoid inscribed in cylinder.

parabola $t^2(x) = a^2 - x$, which is a flipped version of the original parabola. (The flipped parabola also appears in another context in [3; p. 181].) The solid cluster, a flipped version of the original paraboloid, has the same volume of the solid tangent sweep. This example gives another proof of Archimedes' result that the volume of a paraboloid of revolution is half that of its circumscribing cylinder.

In Figure 42b an ellipsoid is inscribed in a cylinder, and the wall projection of the plane region that generates the solid tangent cluster is a right triangle, shown shaded. Rotating this triangle produces the tangent cluster, a right circular cone whose volume, one-third that of the cylinder, is also that of the solid tangent sweep. Finally, we remark that in the case of surfaces of revolution, the axial sections of the tangential sweep and cluster are in chord-by-chord balance and hence in area balance with respect to the axis of revolution.

8. CONCLUDING REMARKS

Start with any family of swept solids with circular horizontal cross sections. Figure 43a shows the cross section of a typical member of the family and of its

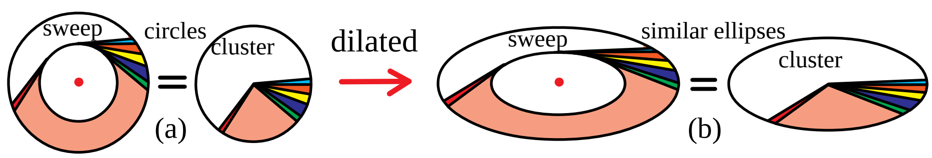

Figure 43. (a) Circular tangent sweep and cluster. (b) Horizontal dilation of (a).

tangent cluster, both cross sections having the same area. If all members of the family are dilated by a factor a in one horizontal direction, as indicated in Figure 44, all the cross sectional areas are multiplied by the factor a so all dilated cross sectional areas will be equal. Consequently, all the solids in the dilated family of equal height will have equal volumes and equal centroidal altitude above a fixed horizontal base. But now the cross sections are elliptic, as indicated in Figure 44b, and the dilated solids are elliptic quadrics punctured by elliptic quadrics.

We can also scale and dilate the elliptic cross sections so that they become parabolic, by moving one focus of the ellipse to infinity. This can be achieved on the cone that is cut by a plane to produce the ellipse. The plane that cuts an ellipse can be rotated so it becomes parallel to a generator of the cone to transform the ellipse to a parabola. Rotating the cutting plane further produces a hyperbola.

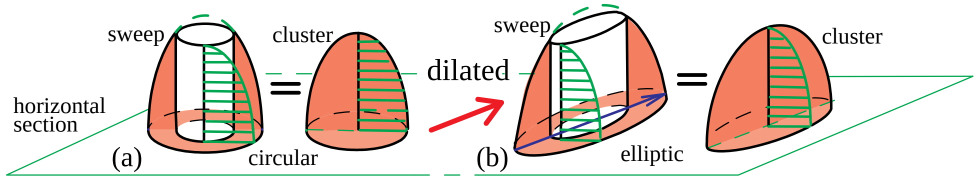

Figure 44. (a) Solid sweep and cluster. (b) Horizontal dilation of (a).

Consequently, proper scaling and dilation transforms punctured parabolic surfaces to punctured hyperbolic surfaces.

Thus we see that families of swept solids of equal height and equal volume can be extended to all types of quadric surfaces. These represent familiar examples of swept solids swept by *variable* plane regions S bounded by conic sections.

References

[1] G. L. Alexanderson and L. F. Klosinski, Some surprising volumes of revolution, *Two-Year College Mathematics Journal*, 6 (1975), 13–15.

[2] T. M. Apostol and M. A. Mnatsakanian, Volumes of solids swept tangentially around cylinders, *Forum Geom.*, 15 (2015) 13–44.

[3] T. M. Apostol and M. A. Mnatsakanian, *New Horizons in Geometry*, Dolciani Mathematical Expositions No. 47, Mathematical Association of America, 2012.

[4] T. L. Heath, *The Works of Archimedes*, Dover, New York, 1953.

[5] M. A. Mnatsakanian, On the area of a region on a developable surface, *Dokladi Armenian Acad. Sci.*, 73 (2) (1981) 97–101. (Russian);
`www.its.caltech.edu/~mamikon/Article.html`;
English translation: `http://mamikon.com/Article81.pdf`.

[6] G. Polya, *Mathematics and Plausible Reasoning, Vol. I, Induction and Analogy in Mathematics*, Princeton University Press, Princeton, 1954.

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