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Equilateral Triangles and Kiepert Perspectors in Complex Numbers

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Abstract. We construct two equilateral triangles associated with an arbitrary hexagon, and show that they are perspective.

1. Two equilateral triangles associated with a hexagon

Consider a hexagon $A_1A_2A_3A_4A_5A_6$ with equilateral triangles $B_jA_jA_{j+1}$ constructed on the six sides externally. Here we take the subscripts modulo 6. Let G_j be the centroid of triangle $B_jA_jA_{j+1}$. We first establish the following interesting result.

Theorem 1. *The midpoints of the segments G_1G_4, G_2G_5, G_3G_6 form an equilateral triangle.*

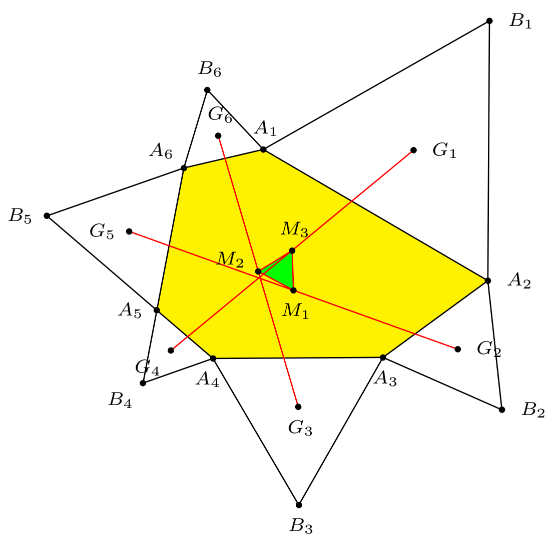

Figure 1.

We prove this theorem by using complex number coordinates of the points. Suppose the hexagon is in the complex plane. Each of the vertices A_j, $j = 1, 2, \ldots, 6$, has a complex affix α_j. We shall often simply identify a point with its complex

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affix. Throughout this note, ω denotes a complex cube root of unity. It satisfies $1+\omega+\omega^2=0$. The other complex cube root of unity is ω^2.

Lemma 2. (a) *A triangle with vertices z_1, z_2, z_3 is equilateral if and only if $z_1+\omega z_2+\omega^2 z_3=0$ for a complex cube root of unity ω.*

(b) *The center of an equilateral triangle with $\alpha_j\alpha_{j+1}$ as a side is γ_j, where*

$$(1-\omega)\gamma_j = -\omega\alpha_j+\alpha_{j+1}$$

for a complex cube root of unity ω.

Proof of Theorem 1. Let M_1, M_2, M_3 be the midpoints of G_2G_5, G_3G_6, G_1G_4 respectively. These have complex affixes $z_j=\frac{1}{2}(\gamma_{j+1}+\gamma_{j+4})$ for $j=1,2,3$. By Lemma 2(b),

$$\begin{aligned}
&2(1-\omega)(z_1+\omega^2 z_2+\omega z_3)\\
&=(1-\omega)((\gamma_2+\gamma_5)+\omega^2(\gamma_3+\gamma_6)+\omega(\gamma_4+\gamma_1))\\
&=(-\omega\alpha_2+\alpha_3)+(-\omega\alpha_5+\alpha_6)+\omega^2(-\omega\alpha_3+\alpha_4)\\
&\quad+\omega^2(-\omega\alpha_6+\alpha_1)+\omega(-\omega\alpha_4+\alpha_5)+\omega(-\omega\alpha_1+\alpha_2)\\
&=0.
\end{aligned}$$

Therefore, $z_1+\omega^2 z_2+\omega z_3=0$, and by Lemma 2(a), z_1, z_2, z_3 are the vertices of an equilateral triangle.

This completes the proof of Theorem 1.

By replacing ω by ω^2 in Lemma 2(b), we have an analogous result of Theorem 1 with the equilateral triangle constructed on the sides of the given hexagon internally. In other words, if for $j=1,2,\dots,6$, G'_j is the reflection of G_j in the side A_jA_{j+1}, then the midpoints M'_1 of $G'_2G'_5$, M'_2 of $G'_3G'_6$, and M'_3 of $G'_1G'_4$ also form an equilateral triangle (see Figure 2).

What is more interesting is that the two equilateral triangles $M_1M_2M_3$ and $M'_1M'_2M'_3$ are perspective. We shall prove this by explicitly computing the complex affix of the point of concurrency (Theorem 6 below).

Lemma 3. *The line joining α, β and the line joining γ, δ intersect at*

$$\theta=\frac{(\overline{\gamma}\delta-\overline{\delta}\gamma)(\alpha-\beta)-(\overline{\alpha}\beta-\overline{\beta}\alpha)(\gamma-\delta)}{(\overline{\gamma-\delta})(\alpha-\beta)-(\overline{\alpha-\beta})(\gamma-\delta)}.$$

Proof. Note that the denominator of θ is purely imaginary. Rewrite the numerator as

$$\begin{aligned}
&(\overline{\gamma}\delta-\overline{\delta}\gamma)(\alpha-\beta)+\overline{\beta}(\gamma-\delta)\alpha-\overline{\alpha}(\gamma-\delta)\beta\\
&=(\overline{\gamma}\delta-\overline{\delta}\gamma+\overline{\beta}(\gamma-\delta))\alpha-(\overline{\gamma}\delta-\overline{\delta}\gamma+\overline{\alpha}(\gamma-\delta))\beta\\
&=(\overline{\gamma}\delta-\overline{\delta}\gamma+\overline{\beta}(\gamma-\delta)-(\overline{\gamma-\delta})\beta)\alpha-(\overline{\gamma}\delta-\overline{\delta}\gamma+\overline{\alpha}(\gamma-\delta)-(\overline{\gamma-\delta})\alpha)\beta.
\end{aligned}$$

This is a linear combination of α and β with purely imaginary coefficients. It follows that θ is a real linear combination of α and β with coefficient sum equal to 1. It represents a point on the line joining α and β. Since θ is invariant under the

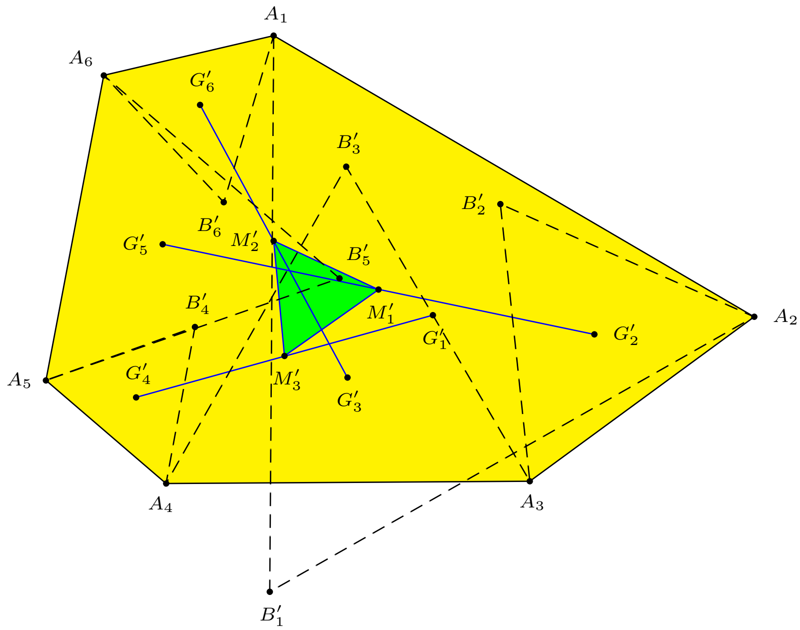

Figure 2.

permutation $(\alpha, \beta) \leftrightarrow (\gamma, \delta)$, it also represents a point on the line joining γ and δ. Therefore, it is the intersection of the two lines. □

We omit the proof of the next lemma.

Lemma 4. *Given two segments $\alpha\beta$ and $\alpha'\beta'$, let $\gamma(t)$ and $\gamma'(t)$ be the points dividing the segments $\alpha\beta$ and $\alpha'\beta'$ in the same ratio*

$$\alpha\gamma(t) : \gamma(t)\beta = \alpha'\gamma'(t) : \gamma'(t)\beta' = t : 1 - t,$$

the locus of the midpoint of $\gamma(t)\gamma'(t)$ is a straight line.

Consider the segments A_2A_3 and A_5A_6 with midpoints $\alpha = \frac{\alpha_2+\alpha_3}{2}$ and $\alpha' = \frac{\alpha_5+\alpha_6}{2}$. Let $\beta = \alpha + \frac{1}{2}(\alpha_2 - \alpha_3)i$ and $\beta' = \alpha' + \frac{1}{2}(\alpha_5 - \alpha_6)i$. These are vertices of isosceles right triangles constructed on the segments A_2A_3 and A_5A_6. Clearly, G_2 and G_5 divide the segment $\alpha\beta$ and $\alpha'\beta'$ in the same ratio; so do G'_2 and G'_5. An application of Lemma 4 identifies the line joining the midpoints of G_2G_5 and $G'_2G'_5$.

Corollary 5. *The line $M_1M'_1$ is the same as the line joining $\frac{\alpha_2+\alpha_5+\alpha_3+\alpha_6}{4}$ and $\frac{\alpha_2+\alpha_5+\alpha_3+\alpha_6}{4} + i \cdot \frac{\alpha_2+\alpha_5-\alpha_3-\alpha_6}{4}$.*

Theorem 6. *The lines $M_1M'_1$, $M_2M'_2$, and $M_3M'_3$ are concurrent at the point*

$$\frac{|\alpha_1+\alpha_4|^2(\alpha_2+\alpha_5-\alpha_3-\alpha_6)+|\alpha_2+\alpha_5|^2(\alpha_3+\alpha_6-\alpha_1-\alpha_4)+|(\alpha_3+\alpha_6|^2(\alpha_1+\alpha_4-\alpha_2-\alpha_5)}{2((\overline{\alpha_1+\alpha_4})(\alpha_2+\alpha_5-\alpha_3-\alpha_6)+(\overline{\alpha_2+\alpha_5})(\alpha_3+\alpha_6-\alpha_1-\alpha_4)+(\overline{\alpha_3+\alpha_6})(\alpha_1+\alpha_4-\alpha_2-\alpha_5))}.$$

Proof. Let $w_j = \frac{\alpha_j+\alpha_{j+3}}{2}$ for $j = 1, 2, 3$. By Corollary 5, $M_1M'_1$ is the line joining $\frac{w_2+w_3}{2}$ and $\frac{w_2+w_3}{2} + i \cdot \frac{w_2-w_3}{2}$. Similarly, $M_2M'_2$ is the line joining $\frac{w_3+w_1}{2}$

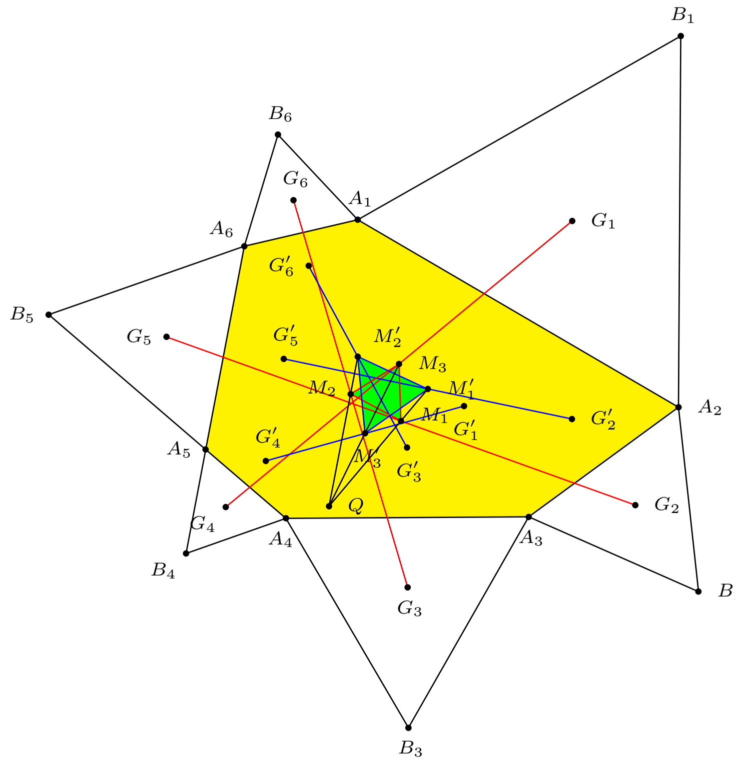

Figure 3.

and $\frac{w_3+w_1}{2}+i\cdot\frac{w_3-w_1}{2}$, and $M_3M'_3$ is the one joining $\frac{w_1+w_2}{2}$ and $\frac{w_1+w_2}{2}+i\cdot\frac{w_1-w_2}{2}$. By Lemma 3, the intersection of these last two lines is

$$Q=\frac{|w_1|^2(w_2+w_3)+|w_2|^2(w_3+w_1)+|w_3|^2(w_1+w_2)}{\overline{w_1}(w_2+w_3)+\overline{w_2}(w_3+w_1)+\overline{w_3}(w_1+w_2)}.$$

The cyclic symmetry of Q in w_1, w_2, w_3 shows that it lies also on the line $M_1M'_1$, and is therefore the point of concurrency of the three lines. Explicitly in terms of α_j for $j=1,2,\dots,6$, this is given in the statement of the theorem above. □

2. Kierpert perspectors

2.1. Theorem 1 is a generalization of Napoleon's theorem. If we put $A_1=A_4=A$, $A_2=A_5=B$, and $A_3=A_6=C$, then $B_1=B_4$, $G_1=G_4=M_1$. Similarly, $G_2=G_5=M_2$ and $G_3=G_6=M_3$. In this case, $M_1M_2M_3$ is the Napoleon triangle of triangle $A_1A_2A_3$. The vertices of the other Napoleon equilateral triangle $M'_1M'_2M'_3$ are the reflections of M_1, M_2, M_3 in BC, CA, AB respectively. The two equilateral triangles are perspective at the circumcenter O.

On the other hand, if we put $A_1=A_2=A$, $A_3=A_4=B$, and $A_5=A_6=C$, then $M_1M_2M_3$ and $M'_1M'_2M'_3$ are the inferior of the Napoleon triangles of ABC. They are perspective at the nine-point center.

2.2. Let ABC be a given triangle. Assume the circumcircle the unit circle in the complex plane, so that the vertices are unit complex numbers α, β, γ.

$$\alpha_1 = \alpha, \quad \alpha_2 = \frac{\alpha+\gamma}{2}, \quad \alpha_3 = \gamma, \quad \alpha_4 = \frac{\beta+\gamma}{2}, \quad \alpha_5 = \beta, \quad \alpha_6 = \frac{\beta+\alpha}{2}.$$

For $j = 1, 2 \dots, 6$, let G_j be the apex of an isosceles triangle with base A_jA_{j+1} and base angle θ. Thus,

$$G_j = \frac{\alpha_j + \alpha_{j+1}}{2} + \tan\theta \cdot \frac{\alpha_j - \alpha_{j+1}}{2} i.$$

In this case,

$$\begin{aligned} M_1 &= \frac{1}{2}(G_2 + G_5) \\ &= \frac{1}{2}\left(\frac{\alpha+3\gamma}{4} + \tan\theta \cdot \frac{\gamma-\alpha}{4} i + \frac{\alpha+3\beta}{4} + \tan\theta \cdot \frac{\alpha-\beta}{4} i\right) \\ &= \frac{1}{8}\left(2\alpha + 3\beta + 3\gamma - \tan\theta(\beta-\gamma)i\right) \\ &= \frac{1}{4}\alpha + \frac{3}{4}\left(\frac{\beta+\gamma}{2} - \frac{\tan\theta}{3} \cdot \frac{\beta-\gamma}{2} i\right) \end{aligned}$$

Note that $\frac{\beta+\gamma}{2} - \frac{\tan\theta}{3} \cdot \frac{\beta-\gamma}{2} i$ is the affix of the vertex of the isosceles triangle on BC with base angle $\arctan\left(\frac{1}{3}\tan\theta\right)$, on the same side as A. Similarly, M_2 and M_3 lie respectively on the lines joining B, C to the vertices of similar isosceles triangles on CA, and AB, constructed on the same sides of the vertices (see Figure 4).

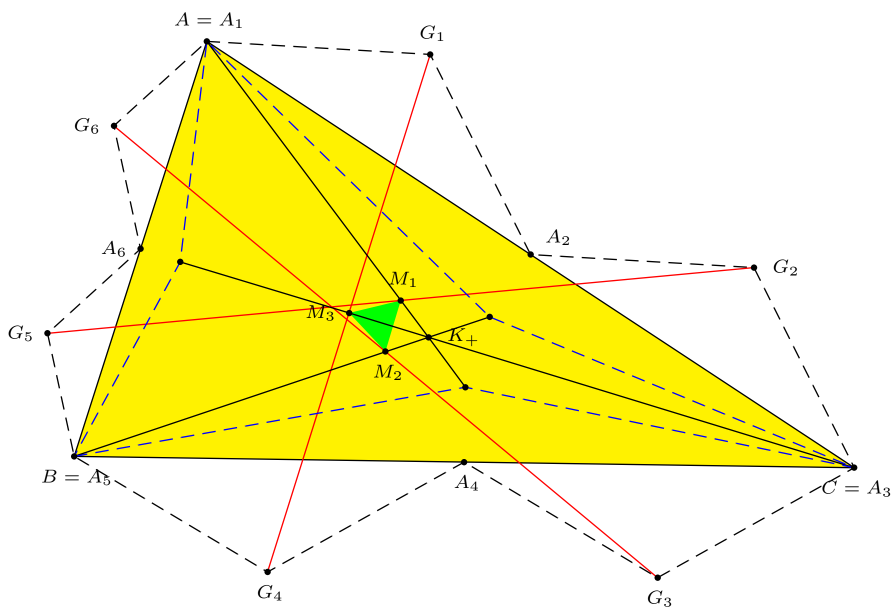

Figure 4.

Proposition 7. (a) *The triangle* $M_1M_2M_3$ *is perspective with* ABC *at the Kiepert perspector* $K\left(-\arctan\left(\frac{1}{3}\tan\theta\right)\right)$.

(b) *The triangle* $M'_1M'_2M'_3$ *is perspective with* ABC *at the Kiepert perspector* $K\left(\arctan\left(\frac{1}{3}\tan\theta\right)\right)$ (see Figure 5).

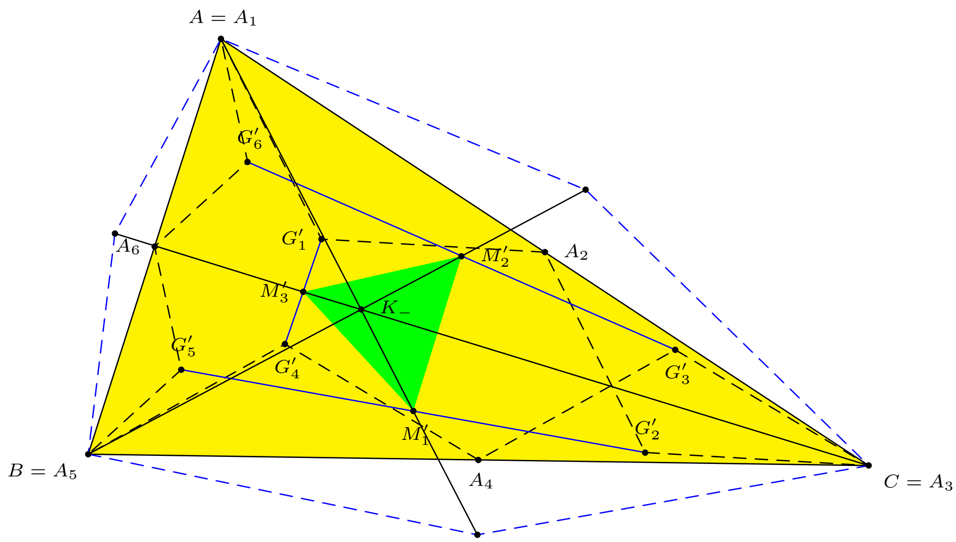

Figure 5.

Finally, we identify the perspector Q of the equilateral triangles $M_1M_2M_3$ and $M'_1M'_2M'_3$ (see Figure 6). The lines in question are

$M_1M'_1$	joining	$\frac{2\alpha+3\beta+3\gamma}{8}$	and	$\frac{2\alpha+3\beta+3\gamma}{8} - i\cdot\frac{\beta-\gamma}{8}$
$M_2M'_2$	joining	$\frac{3\alpha+2\beta+3\gamma}{8}$	and	$\frac{3\alpha+2\beta+3\gamma}{8} - i\cdot\frac{\gamma-\alpha}{8}$
$M_3M'_3$	joining	$\frac{3\alpha+3\beta+2\gamma}{8}$	and	$\frac{3\alpha+3\beta+2\gamma}{8} - i\cdot\frac{\alpha-\beta}{8}$

By Theorem 6, the perspector Q has complex affix $\frac{1}{4}(\alpha+\beta+\gamma)$. Since the orthocenter H of triangle ABC has complex affix $\alpha+\beta+\gamma$ (see, for example, [3, p.74]), Q is the point dividing OH in the ratio $OQ : OH = 1 : 4$. In terms of the nine-point center N and the centroid G, this satisfies $NG : GQ = 2 : 1$. Therefore, Q is the nine-point center of the inferior (medial) triangle. This is the triangle center $X(140)$ in [2] (see Figure 6).

2.3. Given triangle ABC, consider points X, X' on BC, Y, Y' on CA, and Z, Z' on AB such that

$$BX : XX' : X'C = CY : YY' : Y'A = AZ : ZZ' : Z'B = t : 1-2t : t$$

for some real number t. Construct similar isosceles triangles of base angles θ on the sides XX', $X'Y$, YY', $Y'Z$, ZZ', $Z'X$, all outside or inside the hexagon according as θ is positive or negative. Denote the new apices of the isosceles

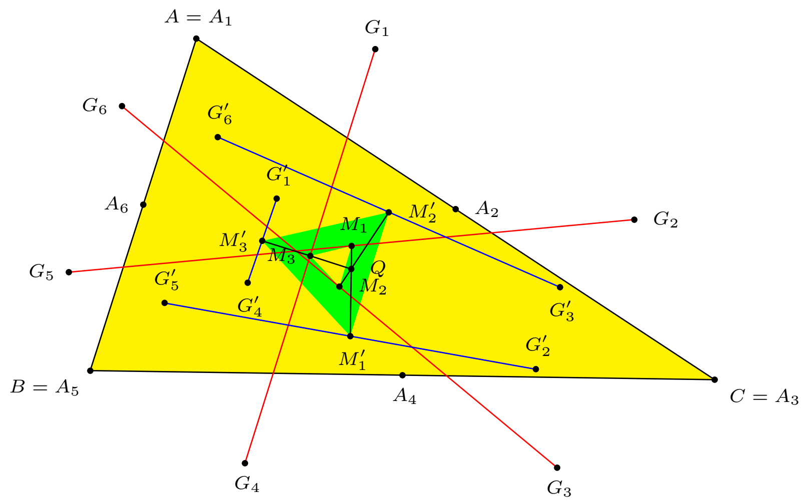

Figure 6.

triangles by A', C'', B', A'', C', B'' respectively. If the complex affixes of A, B, C are α, β, γ respectively, then

$$A' = \frac{\beta+\gamma}{2} + (1-2t)\tan\theta\cdot\frac{\beta-\gamma}{2}i,$$
$$A'' = (1-t)\alpha + t\cdot\frac{\beta+\gamma}{2} - t\tan\theta\cdot\frac{\beta-\gamma}{2}.$$

The midpoint of the segment $A'A''$ is

$$\begin{aligned} M_a = & \frac{1-t}{2}\alpha + \frac{1+t}{2}\cdot\frac{\beta+\gamma}{2} + \frac{1-3t}{2}\tan\theta\cdot\frac{\beta-\gamma}{2}i \\ = & \frac{1-t}{2}\alpha + \frac{1+t}{2}\left(\frac{\beta+\gamma}{2} + \frac{1-3t}{1+t}\tan\theta\cdot\frac{\beta-\gamma}{2}i\right) \end{aligned}$$

Note that $\frac{\beta+\gamma}{2} + \frac{1-3t}{1+t}\tan\theta\cdot\frac{\beta-\gamma}{2}i$ is the apex of the isosceles triangle on BC with base angle $\arctan\left(\frac{1-3t}{1+t}\tan\theta\right)$. Similar expressions hold for the coordinates of the midpoints M_b of $B'B''$ and M_c of $C'C''$. From these we conclude that the triangles $M_aM_bM_c$ and ABC are perspective at the Kiepert perspector $K\left(\arctan\left(\frac{1-3t}{1+t}\tan\theta\right)\right)$. (see Figure 7).

By reversing the sign of θ, we obtain $M'_aM'_bM'_c$ perspective with ABC at the Kiepert perspector $K\left(-\arctan\left(\frac{1-3t}{1+t}\tan\theta\right)\right)$. The line joining these two perspectors passes through the symmedian point of ABC.

These two triangles are equilateral if $\theta = \pm\frac{\pi}{6}$.

2.4. Given triangle ABC and an angle θ, consider the Kiepert triangle $A'B'C' := \mathcal{K}(\theta)$. On the sides of the hexagon $BA'CB'AC'$, construct, similar isosceles triangles of base angles ϕ. Let X_b be the apex of the triangle on CB' and X_c the one

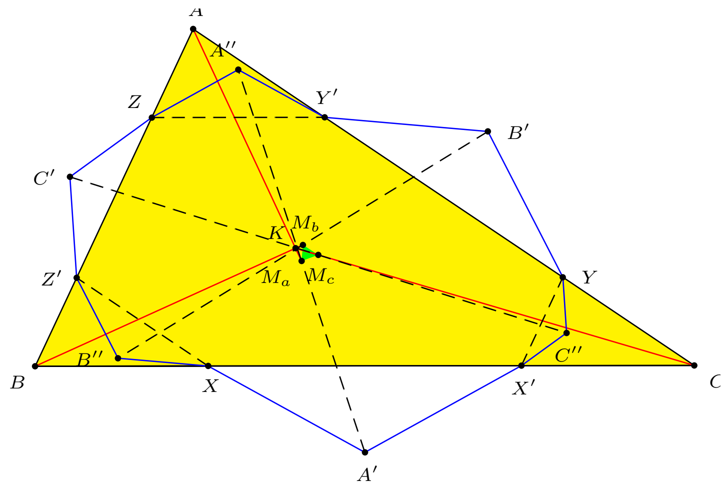

Figure 7

on $C'B$. The midpoint of X_bX_c has affix

$$\frac{\beta+\gamma}{2}+\frac{1}{8}(1-\tan\theta\tan\phi)(2\alpha-\beta-\gamma)-\frac{1}{8}(\tan\theta+\tan\phi)(\beta-\gamma)i$$
$$=\frac{1-\tan\theta\tan\phi}{4}\alpha+\frac{3+\tan\theta\tan\phi}{4}\left(\frac{\beta+\gamma}{2}-\frac{\tan\theta+\tan\phi}{3+\tan\theta\tan\phi}\cdot\frac{\beta-\gamma}{2}i\right)$$

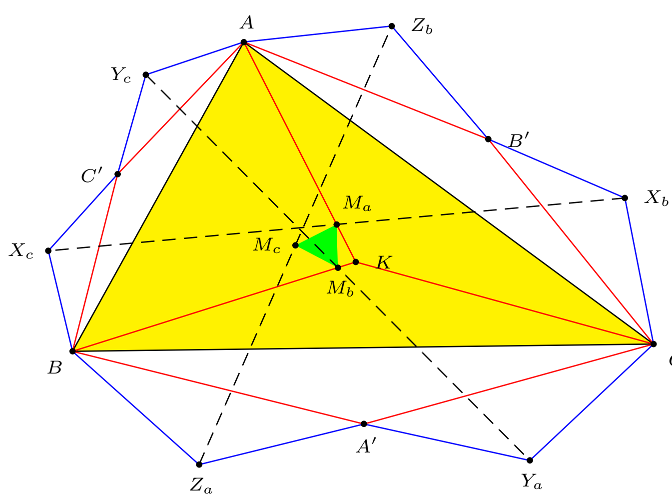

Figure 8

With similar expressions of the midpoints of the two other segments, we conclude that the midpoints of the three segments are perspective with ABC at the

Kiepert perspector

$$K\left(-\arctan\left(\frac{\tan\theta+\tan\phi}{3+\tan\theta\tan\phi}\right)\right).$$

3. Generalizations

Proposition 8 (Fritsch and Pickert [1])**.** *Given a quadrilateral $ABCD$, let A', B', C', D' be the centers of squares on the sides AB, BC, CD, DA, all constructed externally or internally of the quadrilateral. The midpoints of the diagonals of $ABCD$ and $A'B'C'D'$ form a square.*

Proposition 9 (van Aubel's theorem)**.** *Given an octagon $A_1A_2\cdots A_8$, let C_j, $j=1,2,\dots,8$* (indices taken modulo 8)*, be the centers of the squares on A_jA_{j+1}, all externally or internally of the octagon. The midpoints of C_1C_5, C_2C_6, C_3C_7, C_4C_8 form a quadrilateral with equal and perpendicular diagonals* (see Figure 9).

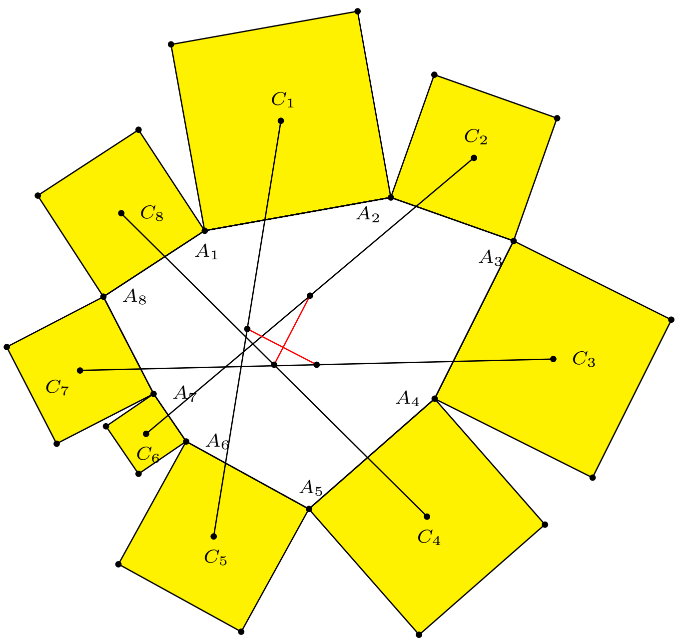

Figure 9

Proposition 10 (Thébault's theorem)**.** *Given an octagon $A_1A_2\cdots A_8$, let B_j be the midpoint of A_jA_{j+1} for indices $j=1,2,\dots,8$* (modulo 8)*. If C_j, $j=1,2,\dots,8$, are the centers of the squares on B_jB_{j+1}, all externally or internally of the octagon, then the midpoints of C_1C_5, C_2C_6, C_3C_7, C_4C_8 are the vertices of a square* (see Figure 10).

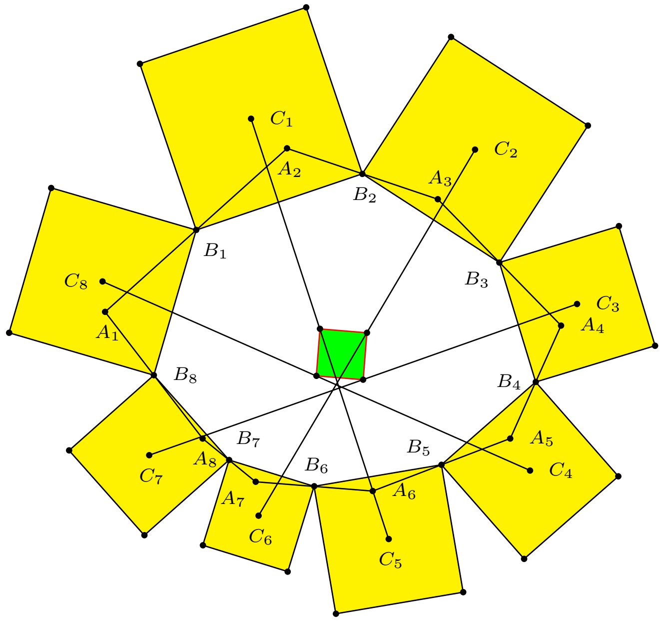

Figure 10

Proposition 8 is a special case of Proposition 10 with $A_1 = A_2$, $A_3 = A_4$, $A_5 = A_6$, $A_7 = A_8$.

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[1] R. Fritsch and G. Pickert, A quadrangle's centroid of vertices and van Aubel's square theorem, *Crux Math.*, 39 (2013) 362–367.

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