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Two Conjectures of Victor Thébault Linking Tetrahedra with Quadrics

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Abstract. We prove two of Thébault's conjectures. The first (1949) links four lines, that they are rulings of hyperbolic paraboloids or that they are coplanar, with orthocentric or isodynamic tetrahedra, respectively. The second (1953) links the radical center of four spheres with elements of tetrahedra.

1. Introduction

It is very well known that an Euclidean tetrahedron $T \equiv ABCD \subset \mathbb{A}^3$, where $\mathbb{A}^3$ is the Euclidean affine space, is called *orthocentric*, by definition, if the lines through the vertices which are orthogonal to the opposite faces are concurrent; and T is called *isodynamic*, by definition, if the segments that join the vertices with the incenter of the opposite faces are concurrent. It is also very well known that the radical center of four spheres is a point P such that the four powers of P with respect to the four spheres are are equal.

In [12] the famous French problemist Victor Thébault (1882-1960) conjectured the following: In a tetrahedron $T \equiv ABCD$, the planes tangent at A, B, C, D to the circumsphere of T cut the planes of the opposite faces in four lines. A necessary and sufficient condition for these four lines to be rulings of a hyperbolic paraboloid is that T be orthocentric, and a necessary and sufficient condition for these four lines to be coplanar is that T be isodynamic.

But the above conjecture, since 1949 has remained open.

Also, in [13] Victor Thébault conjectured the following: In a tetrahedron $ABCD$, let A', B', C', D' be the feet of the altitudes AA', BB', CC', DD'. The planes drawn through the midpoints of $B'C'$, $C'A'$, $A'B'$, $D'A'$, $D'B'$, $D'C'$ perpendicular to BC, CA, AB, DA, DB, DC respectively, are concurrent at a point P, which is the radical center of the spheres described with the vertices A, B, C, D as centers and with the altitudes AA', BB', CC', DD' as radii.

This conjecture, since 1953 has remained open.

In this paper we prove affirmatively these two results; we will call them theorems.

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Readers can see recent references of research papers about tetrahedra in [5], [6], [15], [16], [17], and about Thébault's problems in [2], [3], [4], [7], [8], [9], [10], [11] and [14].

2. Results

Theorem 1. *In a tetrahedron $T \equiv ABCD$, the planes tangent at A, B, C, D to the circumsphere of T cut the planes of the opposite faces in four lines. A necessary and sufficient condition for these four lines to be rulings of a hyperbolic paraboloid is that T be orthocentric, and a necessary and sufficient condition for these four lines to be coplanar is that T be isodynamic.*

Proof. To prove the result, we consider a Cartesian system of coordinates such that $A = (0, 0, 0)$, $B = (1, 0, 0)$, $C = (\alpha, \beta, 0)$, $D = (\gamma, \delta, \varepsilon)$ with $\alpha > 0$, $\beta > 0$ and $\varepsilon > 0$. Let π_A, π_B, π_C and π_D be the planes tangent at A, B, C, D to the circumsphere of T, respectively. Let σ_A, σ_B, σ_C, σ_D be the planes containing the faces BCD, ACD, ABD, ABC respectively. Let $r_A = \pi_A \cap \sigma_A$, $r_B = \pi_B \cap \sigma_B$, $r_C = \pi_C \cap \sigma_C$ and $r_D = \pi_D \cap \sigma_D$ be the four lines of the problem. We can calculate two points for every line r_A, r_B, r_C and r_D:

$$\begin{aligned}
r_A &\equiv \left(\tfrac{\alpha-\Phi}{1-\Phi}, \tfrac{\beta}{1-\Phi}, 0\right) \wedge \left(\tfrac{\Phi(\varepsilon+\gamma-1)+\Psi(1-\alpha)+\alpha(1-\varepsilon)-\gamma}{\varepsilon(\Phi-1)}, \tfrac{\beta(\Psi+(\varepsilon-1))+\delta(1-\Phi)}{\varepsilon(1-\Phi)}, 1\right), \\
r_B &\equiv \left(\tfrac{\alpha}{2\alpha-\Phi}, \tfrac{\beta}{2\alpha-\Phi}, 0\right) \wedge \left(\tfrac{\Phi\gamma-\alpha(\Psi+\varepsilon)}{\varepsilon(\Phi-2\alpha)}, \tfrac{\Phi\delta+\beta(2\gamma-\varepsilon-\Psi)-2\alpha\delta}{\varepsilon(\Phi-2\alpha)}, 1\right), \\
r_C &\equiv \left(\tfrac{\Phi}{2\alpha-1}, 0, 0\right) \wedge \left(\tfrac{\delta(2\beta-\Phi)-\Psi+\gamma}{\delta(1-2\alpha)}, 1, \tfrac{\varepsilon}{\delta}\right), \\
r_D &\equiv \left(\tfrac{\Psi}{2\gamma-1}, 0, 0\right) \wedge \left(\tfrac{\beta(2\delta-\Psi)-\Phi+\alpha}{\beta(1-2\gamma)}, 1, 0\right).
\end{aligned} \tag{1}$$

Here, we denote a line l through two points M and N as $l \equiv M \wedge N$, and $\Phi = \alpha^2 + \beta^2 = AC^2$, $\Psi = \gamma^2 + \delta^2 + \varepsilon^2 = AD^2$.

First, we note that this problem concerns the case of the Euclidean affine space but not the projective space. That is to say, the thesis of the problem is only true in the case that the four lines exist into the affine space and not into the plane of the infinite. For example: if $\alpha = \frac{1}{2}$, $\beta = 1$, $\gamma = \frac{1}{2}$, $\delta = \frac{1}{4}$ and $\varepsilon = \frac{1}{4}\sqrt{11}$, then T is isodynamic with $r_C \subseteq \pi_\infty$ (i.e. π_C is parallel to σ_C) and $r_D \subseteq \pi_\infty$, but $r_A \nsubseteq \pi_\infty$ and $r_B \nsubseteq \pi_\infty$. Therefore T is not equilateral, and we may assume that $\Psi \neq \Phi$ because T is not equilateral.

After a calculation, we find that the center of the circumsphere of T is $O = \frac{1}{2\beta}\left(\beta, \Phi - \alpha, \frac{\delta\alpha-\gamma\beta+\beta\Psi-\delta\Phi}{\varepsilon}\right)$. Because the four lines are affine, we have $\alpha \neq \frac{1}{2}$, $\gamma \neq \frac{1}{2}$, $\Phi \neq 1$ and $\Phi \neq 2\alpha$; see Equations (1). Also we may assume $\Psi \neq 1$ because if $\Psi = 1$, since $\Psi \neq \Phi$ we can choose another orientation of T with $AD = \Psi \neq 1$ and $AC = \Phi = 1$, and with Equations (1) the lines r_A and r_B are not affine. We note that T is isodynamic, by definition, if the segments that join the vertices with the incenter of the opposite faces are concurrent. It is very well known (see for example [1]) that T is isodynamic if and only if the three products of the pairs formed by the opposite edges are equal:

$$\begin{aligned}&AB \cdot CD = AC \cdot BD = AD \cdot BD\\ \Leftrightarrow &\sqrt{\Phi + \Psi - 2\alpha\gamma - 2\beta\delta} = \sqrt{\Phi}\sqrt{\Psi + 1 - 2\gamma} = \sqrt{\Psi}\sqrt{\Phi + 1 - 2\alpha};\end{aligned}$$

and this condition is true if and only if

$$\Psi = \Phi\frac{2\gamma - 1}{2\alpha - 1} = \frac{2\gamma\Phi - 2\alpha\gamma - 2\beta\delta}{\Phi - 1} = \frac{2\alpha\gamma + 2\beta\delta - \Phi}{2\alpha - \Phi}. \tag{2}$$

And we have the equivalencies

$$\begin{aligned}&\Phi\frac{2\gamma - 1}{2\alpha - 1} = \frac{2\gamma\Phi - 2\alpha\gamma - 2\beta\delta}{\Phi - 1} \Leftrightarrow \Phi\frac{2\gamma - 1}{2\alpha - 1} = \frac{2\alpha\gamma + 2\beta\delta - \Phi}{2\alpha - \Phi} \qquad (3)\\ &\Leftrightarrow \frac{2\gamma\Phi - 2\alpha\gamma - 2\beta\delta}{\Phi - 1} = \frac{2\alpha\gamma + 2\beta\delta - \Phi}{2\alpha - \Phi} \Leftrightarrow\\ &\Leftrightarrow 2\gamma\Phi\left(2\alpha - \Phi\right) + \Phi\left(\Phi - 1\right) + 2\left(1 - 2\alpha\right)\left(\alpha\gamma + \beta\delta\right) = 0.\end{aligned}$$

We note that T is orthocentric, by definition, if the lines through the vertices which are orthogonal to the opposite faces are concurrent. It is very well known (see for example [1]) that T is orthocentric if and only if the sum of the squares of the pairs formed by the opposite edges are equal:

$$\begin{aligned}&AB^2 + CD^2 = AC^2 + BD^2 = AD^2 + BC^2\\ \Leftrightarrow &1 - 2\alpha\gamma - 2\beta\delta + \Phi + \Psi = 1 - 2\gamma + \Phi + \Psi = 1 - 2\alpha + \Phi + \Psi,\end{aligned}$$

and this condition is true if and only if

$$\gamma = \alpha, \qquad \delta = \frac{\alpha - \alpha^2}{\beta}. \tag{4}$$

Now, if the four lines r_A, r_B, r_C, r_D are coplanar, then the three points $M_1 = \left(\frac{\alpha}{2\alpha - \Phi}, \frac{\beta}{2\alpha - \Phi}, 0\right)$, $M_2 = \left(\frac{\Phi}{2\alpha - 1}, 0, 0\right)$, $M_3 = \left(\frac{\Psi}{2\gamma - 1}, 0, 0\right)$ are collinear because the eight points in Equations (1) above are not in the plane $z = 0$; therefore $\frac{\Phi}{2\alpha - 1} = \frac{\Psi}{2\gamma - 1}$ because $\beta \neq 0$. Then if the four lines are coplanar we have $\Psi = \Phi\frac{2\gamma - 1}{2\alpha - 1}$. Also, the plane σ_1, which contains the four lines, also contains the points M_1, $M_2 = M_3$, and $M_4 = \left(\frac{\delta(2\beta - \Phi) - \Psi + \gamma}{\delta(1 - 2\alpha)}, 1, \frac{\varepsilon}{\delta}\right)$. Then, if we impose that the point $M_5 = \left(\frac{\Phi\gamma - \alpha(\Psi + \varepsilon)}{\varepsilon(\Phi - 2\alpha)}, \frac{\Phi\delta + \beta(2\gamma - \varepsilon - \Psi) - 2\alpha\delta}{\varepsilon(\Phi - 2\alpha)}, 1\right)$ lies in the plane σ_1 we find that $2\gamma\Phi\left(2\alpha - \Phi\right) + \Phi\left(\Phi - 1\right) + 2\left(1 - 2\alpha\right)\left(\alpha\gamma + \beta\delta\right) = 0$ because this is the condition that we find when we impose that $\det(\overrightarrow{M_2M_5}, \overrightarrow{M_2M_1}, \overrightarrow{M_2M_4}) = 0$ with $\Psi = \Phi\frac{2\gamma - 1}{2\alpha - 1}$. Therefore, using Equations (2) and (3), if the four lines are coplanar (in fact if r_B, r_C, r_D are coplanar) then T is isodynamic. Reciprocally, if T is isodynamic, the points $X = (x, y, z)$ in the plane σ_1 through the points M_1, $M_2 = M_3$ and M_4 verify $\sigma_1 \equiv \det(\overrightarrow{M_2X}, \overrightarrow{M_2M_1}, \overrightarrow{M_2M_4}) = 0$. An easy calculation, using Equations (2) and (3), proves that the eight points of Equations (1) are in σ_1. Therefore, the four lines are coplanar. Now, if we consider that T is orthocentric, then we calculate determinants and we find that r_A, r_B, r_C and r_D are parallel to the same plane. The plane by M_3 and r_C

is $\sigma_2 \equiv \det(\overrightarrow{M_3X}, \overrightarrow{M_3M_2}, \overrightarrow{M_3M_4}) = 0$, or equivalently using Equation (4), $\sigma_2 \equiv \varepsilon\beta y + \left(\alpha^2 - \alpha\right) z = 0$ and $\sigma_2 \cap r_A = P = \left(\frac{\Psi-\alpha}{\Psi-1}, \frac{\alpha(\alpha-1)}{\beta(\Psi-1)}, \frac{\varepsilon}{1-\Psi}\right)$. And with a calculation we have $P \in \sigma_3$, where σ_3 is the plane by M_3 and r_B because $\sigma_3 \equiv \det(\overrightarrow{M_3X}, \overrightarrow{M_3M_1}, \overrightarrow{M_3M_5}) = 0$, or using Equation (4),

$$\sigma_3 \equiv ax + by + cz + \Psi\beta^2\varepsilon = 0,$$

where

$$\begin{aligned}
a &= (1-2\alpha)\,\beta^2\varepsilon,\\
b &= ((2\alpha-1)\,\alpha + \Psi\Phi - 2\Psi\alpha)\,\varepsilon\beta,\\
c &= d + e,\\
d &= \Psi\left((\Phi-2\alpha)\,\alpha\,(\alpha-1) + \beta^2\,(\Psi-2\alpha)\right),\\
e &= \alpha\,(1-2\alpha)\left(\alpha - \left(\alpha^2+\beta^2\right)\right).
\end{aligned}$$

Therefore, the line l_1 through M_3 which cuts r_B and r_C is a line that also cuts r_A in P. As before, we can calculate the plane by $M_6 = \left(\frac{\beta(2\delta-\Psi)-\Phi+\alpha}{\beta(1-2\gamma)}, 1, 0\right)$ and r_C which is $\sigma_4 \equiv \det(\overrightarrow{M_6X}, \overrightarrow{M_6M_2}, \overrightarrow{M_6M_4}) = 0$, and using Equation (4), $\sigma_4 \cap r_A = Q = (Q_1, Q_2, Q_3)$ with

$$\begin{aligned}
Q_1 &= \tfrac{\Phi-\Phi\alpha+\Psi\beta+\alpha(2\alpha-\beta-2)}{(\Psi-1)\beta},\\
Q_2 &= \tfrac{\Phi\left(\alpha\left((1-\alpha)(\Phi+\beta+1)+2\alpha^2-2\right)-\beta^2\Psi+\beta^2\right)+\beta\alpha(\Psi(2\beta+\alpha-1)-2\beta)+2\alpha^2(1-\alpha)}{(\Phi-\Psi-\Psi\Phi+\Psi^2)\beta^2},\\
Q_3 &= \varepsilon\tfrac{2\alpha-\Phi-2\Phi\alpha+\Phi\beta-\Psi\beta+\Phi^2}{\beta(\Phi-\Psi-\Psi\Phi+\Psi^2)}.
\end{aligned} \tag{5}$$

Note that $\Psi \neq \Phi \Rightarrow \Phi - \Psi - \Psi\Phi + \Psi^2 \neq 0$. As before, using Equation (4), with a calculation we have $Q \in \sigma_5$, where σ_5 is the plane by M_6 and r_B with equation $\sigma_3 \equiv \det(\overrightarrow{M_3X}, \overrightarrow{M_3M_1}, \overrightarrow{M_3M_5}) = 0$. The line l_2 through M_6 which cuts r_B and r_C is a line that also cuts r_A in Q. Therefore, the four lines r_A, r_B, r_C, r_D cut the two lines l_1, l_2. These two lines l_1, l_2 are not parallel, for otherwise we find that $\Phi = 2\alpha$. Also, they do not intersect each other, for otherwise we find that $\Phi = 2\alpha$ or $\Phi = 1$. In fact, with a longer calculation we can prove that the four lines r_A, r_B, r_C, r_D cut the two lines l_1, l_2, without any condition; that is, without the condition of Equation (4). For example, $\sigma_2 \cap r_A = P = \left(\frac{\gamma-\Psi}{1-\Psi}, \frac{\delta}{1-\Psi}, \frac{\varepsilon}{1-\Psi}\right)$ and always the four lines r_A, r_B, r_C, r_D of Equations (1) cut the two lines l_1, l_2. Also, $r_D \cap r_C = \varnothing$ because $\Psi \neq \Phi$, and $r_D \cap r_B = \varnothing$ because $\Psi \neq 1$. Then the four lines are not in the same plane, for otherwise they should be parallel and they are in the sides of the tetrahedron T, which is impossible. In conclusion: the four lines r_A, r_B, r_C, r_D are parallel to the same plane, they are not in the same plane and they cut two lines l_1, l_2 which are not parallel and do not intersect each other. Therefore, they are four rulings of a hyperbolic paraboloid. Reciprocally, we consider that the four lines r_A, r_B, r_C, r_D are rulings of a hyperbolic paraboloid

H. First we make calculations and we have that

$$\begin{aligned}
r_A \cap r_B \neq \varnothing &\Rightarrow \Psi = \Phi\frac{2\gamma-1}{2\alpha-1},\\
r_A \cap r_C \neq \varnothing &\Rightarrow \Psi = \frac{2\alpha\gamma+2\delta\beta-\Phi}{2\alpha-\Phi},\\
r_A \cap r_D \neq \varnothing &\Rightarrow \Psi = \frac{2\gamma\Phi-2\alpha\gamma-2\delta\beta}{\Phi-1},\\
r_B \cap r_C \neq \varnothing &\Rightarrow \Psi = \frac{2\gamma\Phi-2\alpha\gamma-2\delta\beta}{\Phi-1},\\
r_B \cap r_D \neq \varnothing &\Rightarrow \Psi = \frac{2\alpha\gamma+2\delta\beta-\Phi}{2\alpha-\Phi},\\
r_C \cap r_D \neq \varnothing &\Rightarrow \Psi = \Phi\frac{2\gamma-1}{2\alpha-1}.
\end{aligned} \tag{6}$$

Note that $r_A \cap r_B \neq \varnothing$ and $r_B \cap r_C \neq \varnothing$ imply, using Equations (2) and (3), that T is isodynamic, and the four lines are in the same plane, in contradiction with that they are four rulings of H. If $r_B \cap r_C \neq \varnothing$, then $r_A \cap r_B = \varnothing$. The lines are rulings of H; if $r_B \cap r_C \neq \varnothing$, we must also have $r_A \cap r_C \neq \varnothing$. Therefore, using Equations (2) and (3), T is again isodynamic in contradiction. In conclusion $r_B \cap r_C = \varnothing$. Then $r_A \cap r_B = \varnothing$, because if not then $r_A \cap r_B \neq \varnothing$ and $r_A \cap r_C \neq \varnothing$, and, using Equations (2) and (3), T is again isodynamic in contradiction. Also $r_B \cap r_D = \varnothing$, because if not then $r_B \cap r_D \neq \varnothing$ and $r_B \cap r_C \neq \varnothing$, and, using Equations (2) and (3), T is again isodynamic in contradiction. Therefore r_A, r_B, r_C, r_D are four rulings of H parallel to the same plane. Then we impose that the director vectors of r_A, r_B, r_C, r_D are linearly dependent; we calculate determinants and we find that

$$\begin{aligned}
0 = {} & \Psi\left(\Phi\gamma+\alpha\left(2\alpha\gamma+2\beta\delta-3\gamma+1-\Phi\right)-\delta\beta\right)\\
& +\Phi\left(\beta\delta+\gamma\left(-2\alpha\gamma-2\beta\delta+3\alpha-1\right)\right)+2\alpha\left(\gamma^2-\alpha\gamma-\delta\beta\right)+2\beta\gamma\delta.
\end{aligned} \tag{7}$$

But since $\Psi = \gamma^2 + \delta^2 + \varepsilon^2$ for infinitely many $\varepsilon \in \mathbb{R}$,

$$0 = \Phi\gamma+\alpha\left(2\alpha\gamma+2\beta\delta-3\gamma+1-\Phi\right)-\delta\beta. \tag{8}$$

This implies that

$$\begin{aligned}
\gamma &= \frac{\alpha^3+\alpha\beta^2-2\delta\beta\alpha+\delta\beta-\alpha}{-3\alpha+3\alpha^2+\beta^2}, \qquad (9)\\
\delta &= -\frac{-\alpha^3-\alpha\beta^2+3\alpha^2\gamma+\gamma\beta^2-3\alpha\gamma+\alpha}{\beta\left(2\alpha-1\right)},\\
0 &= \Phi\left(\beta\delta+\gamma\left(-2\alpha\gamma-2\beta\delta+3\alpha-1\right)\right)+2\alpha\left(\gamma^2-\alpha\gamma-\delta\beta\right)+2\beta\gamma\delta.
\end{aligned}$$

The last equation of (9) implies that

$$\delta = -\gamma\frac{-3\alpha^3-3\alpha\beta^2+3\alpha^2-2\alpha\gamma+2\alpha^3\gamma+2\alpha\gamma\beta^2+\beta^2}{\beta\left(2\alpha^2\gamma+2\gamma\beta^2-\beta^2-2\gamma+2\alpha-\alpha^2\right)} \tag{10}$$

With (10), the second equation of (9), and the condition $\gamma \neq \frac{1}{2}$, we have $\gamma = \alpha$. Finally with this result and the first equation of (9), we obtain $\delta = \frac{\alpha-\alpha^2}{\beta}$. Then T is orthocentric. □

Theorem 2. *In a tetrahedron $ABCD$, let A', B', C', D' be the feet of the altitudes AA', BB', CC', DD'.* [1] *The planes drawn through the midpoints of $B'C'$, $C'A'$, $A'B'$, $D'A'$, $D'B'$, $D'C'$ perpendicular to BC, CA, AB, DA, DB, DC respectively, are concurrent at a point P, which is the radical center of the spheres described with the vertices A, B, C, D as centers and with the altitudes AA', BB', CC', DD' as radii.*

Proof. To prove the result, we consider a Cartesian system of coordinates such that $A = (0,0,0)$, $B = (1,0,0)$, $C = (\alpha,\beta,0)$, $D = (\gamma,\delta,\varepsilon)$ with $\alpha > 0$, $\beta > 0$ and $\varepsilon > 0$. In this system, the feet of the altitudes are

$$\begin{aligned}
A' &= \frac{\beta\varepsilon}{M_a}\left(\beta\varepsilon, (1-\alpha)\,\varepsilon, \delta\,(\alpha-1) - \beta\,(\gamma-1)\right),\\
B' &= \frac{\beta\varepsilon}{M_b+\beta^2\varepsilon^2}\left(\frac{M_b}{\beta\varepsilon}, \alpha\varepsilon, \beta\gamma-\alpha\delta\right),\\
C' &= \frac{\beta\delta}{\delta^2+\varepsilon^2}\left(\frac{\alpha}{\beta\delta}\left(\delta^2+\varepsilon^2\right), \delta, \varepsilon\right),\\
D' &= (\gamma,\delta,0)\,,
\end{aligned}$$

where

$$\begin{aligned}
M_a &= \left(\alpha^2-2\alpha+1\right)\left(\delta^2+\varepsilon^2\right) + 2\beta\delta\,(\alpha-1) + \beta^2\left(1+\varepsilon^2\right)\\
&\quad + \beta\gamma\,(\beta\gamma-2\alpha\delta+2\delta-2\beta)\,,\\
M_b &= \alpha^2\left(\delta^2+\varepsilon^2\right) + \beta\gamma\,(\beta\gamma-2\alpha\delta)\,.
\end{aligned}$$

Also we calculate the planes drawn through the midpoints of $B'C'$, $C'A'$, $A'B'$, $D'A'$, $D'B'$, $D'C'$ perpendicular to BC, CA, AB, DA, DB, DC respectively. They are

$$\begin{aligned}
\pi_{BC} &\equiv 2\,(1-\alpha)\,x - 2\beta y + \alpha^2 + \frac{\beta^2\delta^2}{\delta^2+\varepsilon^2} - \frac{M_b}{M_b+\beta^2\varepsilon^2} = 0,\\
\pi_{CA} &\equiv 2\alpha x + 2\beta y - \frac{\beta^2\delta^2}{\delta^2+\varepsilon^2} - \frac{\alpha^2 M_a+\beta^2\varepsilon^2}{M_a} = 0,\\
\pi_{AB} &\equiv 2x - \frac{\beta^2\varepsilon^2}{M_a} - \frac{M_b}{M_b+\beta^2\varepsilon^2} = 0,\\
\pi_{DA} &\equiv 2\gamma x + 2\delta y + 2\varepsilon z - \delta^2 - \gamma^2 - \frac{\beta^2\varepsilon^2}{M_a} = 0,\\
\pi_{DB} &\equiv 2\,(1-\gamma)\,x - 2\delta y - 2\varepsilon z + \delta^2 + \gamma^2 - \frac{M_b}{M_b+\beta^2\varepsilon^2} = 0,\\
\pi_{DC} &\equiv 2\,(\alpha-\gamma)\,x + 2\,(\beta-\delta)\,y - 2\varepsilon z - \alpha^2 + \gamma^2 + \delta^2 - \frac{\delta^2\beta^2}{\delta^2+\varepsilon^2} = 0.
\end{aligned}$$

[1] "Altitude AA'" means that AA' is the straight line segment which joins the vertex A with the point A' on the opposite side plane BCD such that the segment AA' is orthogonal to plane BCD; and this point A' is called "foot of the altitude".

Another calculation shows that all these planes are concurrent at the point

$$P = (P_1, P_2, P_3) = \left(\phi, -\frac{\xi + \alpha\phi}{\beta}, \frac{\beta\varphi + \delta\xi + (\alpha\delta - \beta\gamma)\,\phi}{\beta\varepsilon}\right),$$

with

$$2\xi = -\alpha^2 - \frac{\beta^2\delta^2}{\delta^2 + \varepsilon^2} + \frac{\beta^2\varepsilon^2}{M_a},$$
$$2\phi = \frac{M_b}{M_b + \beta^2\varepsilon^2} + \frac{\beta^2\varepsilon^2}{M_a},$$
$$2\varphi = \gamma^2 + \delta^2 + \frac{\beta^2\varepsilon^2}{M_a}.$$

Finally, we calculate the power of P with respect to the spheres described with the vertices A, B, C, D as centers and with the altitudes AA', BB', CC', DD' as radii respectively. These are

$$P_a = P_1^2 + P_2^2 + P_3^2 - \frac{\beta^2\varepsilon^2}{M_a},$$
$$P_b = (P_1 - 1)^2 + P_2^2 + P_3^2 - \frac{\beta^2\varepsilon^2}{M_b + \beta^2\varepsilon^2},$$
$$P_c = (P_1 - \alpha)^2 + (P_2 - \beta)^2 + P_3^2 - \frac{\beta^2\varepsilon^2}{\delta^2 + \varepsilon^2},$$
$$P_d = (P_1 - \gamma)^2 + (P_2 - \delta)^2 + (P_3 - \varepsilon)^2 - \varepsilon^2.$$

It is easy to check that $P_a - P_b = P_a - P_c = P_a - P_d = 0$. Therefore, P is the radical center of the four spheres. □

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