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Lemniscates and a Locus Related to a Pair of Median and Symmedian

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Abstract. We show that the lemniscate of Bernoulli arises from a locus problem related to the orthogonality of a pair of median and symmedian of a triangle with a given side, and study a generalization of the locus problem.

1. A locus problem on the orthogonality of a pair of median and symmedian

Given a segment BC, consider the locus of A such that the median AM and the symmedian AL of triangle ABC are orthogonal to each other.

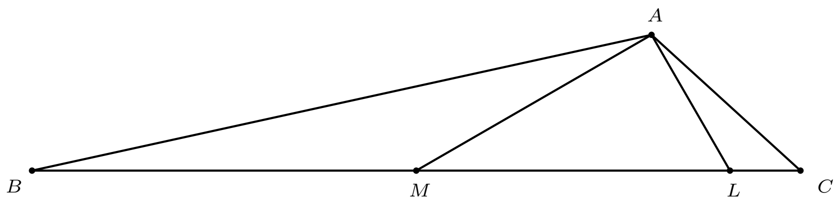

Figure 1.

In a Cartesian coordinate system, let the origin be the midpoint M of BC, and $B = (-a, 0)$ and $C = (a, 0)$ (see Figure 1). If $A = (u, v)$, the perpendicular to AM at A is the line $u(x-u) + v(y-v) = 0$. It intersects the x-axis at $L = \left(\frac{u^2+v^2}{u},\ 0\right)$. Now, AL is a symmedian if and only if $\frac{BL}{LC} = \frac{AB^2}{AC^2}$.

$$\frac{au + u^2 + v^2}{au - (u^2 + v^2)} = \frac{u^2 + v^2 + 2au + a^2}{u^2 + v^2 - 2au + a^2}.$$

Simplifying, we have

$$(u^2 + v^2)^2 = a^2(u^2 - v^2).$$

In polar coordinates, this is the curve $r^2 = a^2 \cos 2\theta$, the lemniscate with endpoints B and C (see Figure 2).

2. A generalization

Suppose instead of orthogonality, we require to A-median and A-symmedian to make a given angle $\alpha \neq 0$. If the *directed angle* $(AM,\ AL) = \alpha$, the slope of the line AL is

$$\tan\left(\alpha + \arctan\frac{v}{u}\right) = \frac{u \sin\alpha + v \cos\alpha}{u \cos\alpha - v \sin\alpha},$$

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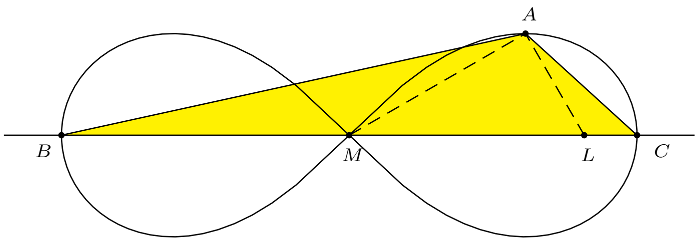

Figure 2.

and L is the point $\left(\frac{\sin\alpha(u^2+v^2)}{u\sin\alpha+v\cos\alpha},\ 0\right)$. The condition $\frac{BL}{LC}=\frac{AB^2}{AC^2}$ reduces in this case to

$$\sin\alpha(u^2+v^2)^2 = a^2(\sin\alpha(u^2-v^2)+\cos\alpha\cdot 2uv).$$

In polar coordinates, $(u,v)=(r\cos\theta,\ r\sin\theta)$, this becomes

$$\mathscr{L}(\alpha): \qquad r^2=\frac{a^2}{\sin\alpha}\cdot\sin(2\theta+\alpha)=\frac{a^2}{\sin\alpha}\cdot\cos(2\theta-\frac{\pi}{2}+\alpha).$$

In particular, $\mathscr{L}\left(\frac{\pi}{2}\right)$ is the lemniscate $r^2 = a^2\cos 2\theta$. $\mathscr{L}(\alpha)$ is the image of $\mathscr{L}\left(\frac{\pi}{2}\right)$ under a rotation by $\frac{\pi}{4}-\frac{\alpha}{2}$ about the center M, followed by a magnification of factor $\frac{1}{\sqrt{\sin\alpha}}$.

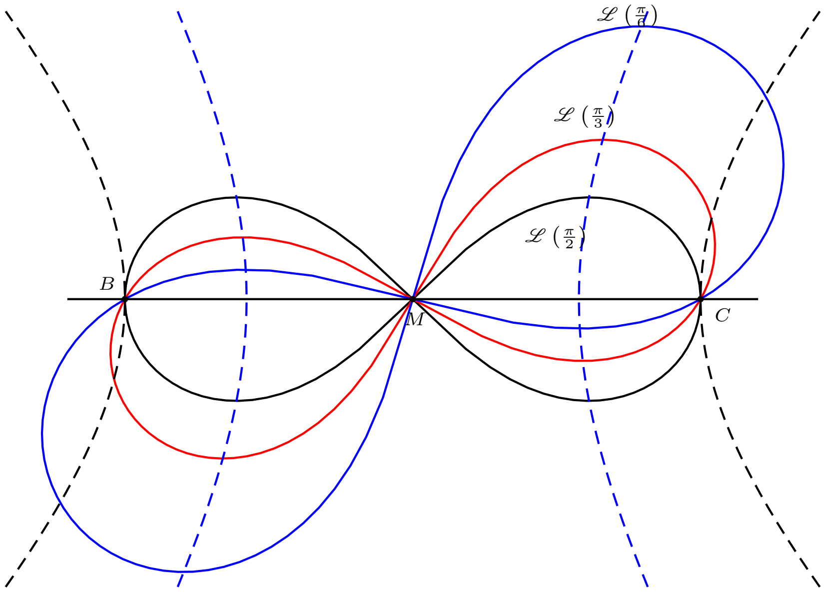

Figure 3.

3. On the family of lemniscates $\mathscr{L}(\alpha)$

For varying α, the extreme points of the lemniscate $\mathscr{L}(\alpha)$ are the points with polar coordinates $\left(\frac{a}{\sqrt{\sin\alpha}}, \frac{\pi}{4} - \frac{\alpha}{2}\right)$. This lies on the polar curve $r = \frac{a}{\sqrt{\cos 2\theta}}$ or $r^2 \cos 2\theta = a^2$. This is the rectangular hyperbola $x^2 - y^2 = a^2$, precisely the inverse of the lemniscate with respect to the circle with radius a and centered at the origin ([1, pp. 111–117], [2, pp. 143–147]; see Figure 3).

For each α, the "highest" point of the lemniscate $\mathscr{L}(\alpha)$ gives the largest triangle ABC with orthogonal A-median and A-symmedian. For points (x, y) on $\mathscr{L}(\alpha)$, $y_{\max}$ occurs at $\theta = \frac{\pi-\alpha}{3}$. Writing α in terms of θ, we have $\alpha = \pi - -3\theta$ and $2\theta + \alpha = \pi - \theta$. Therefore, this highest point lies on the polar curve

$$r^2 = \frac{a^2 \sin(\pi - \theta)}{\sin(\pi - 3\theta)} = \frac{a^2 \sin\theta}{\sin 3\theta}.$$

Further simplifying,

$$r^2 = \frac{a^2}{3 - 4\sin^2\theta} \implies 3r^2 - 4y^2 = a^2 \implies 3x^2 - y^2 = a^2.$$

Therefore, the locus of A for which triangle ABC is the *largest* among those with A-median and A-symmedian making a fixed angle is the hyperbola $3x^2 - y^2 = a^2$ (see Figure 3).

In particular, the largest triangle with orthogonal A-median and A-symmedian is constructible with ruler and compass. For $\alpha = \frac{\pi}{2}$, this is the point with polar coordinates $\left(\frac{a}{\sqrt{2}}, \frac{\pi}{6}\right)$. In Figure 4, O is the center of the square $MCDE$, A_0ME is an equilateral triangle. Construct the circular arc with center M and radius MO to intersect MA_0 at A, the highest point of the lemniscate $\mathscr{L}\left(\frac{\pi}{2}\right)$. In this case, $\cos BAC = -\frac{1}{\sqrt{3}}$.

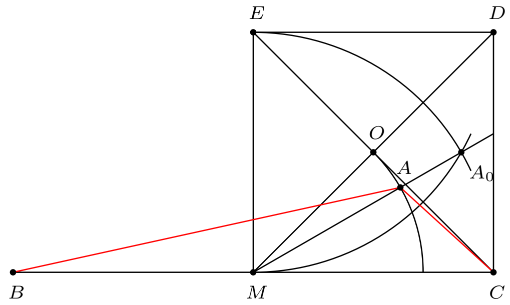

Figure 4.

References

[1] E. H. Lockwood, *A Book of Curves*, Cambridge University Press, 1967.

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