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Points on a Line that Maximize and Minimize the Ratio of the Distances to Two Given Points

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Abstract. Given a line ℓ and points B and C, we construct the two points on ℓ that maximize and minimize the ratio $\frac{XB}{XC}$ for X on ℓ.

In this note we solve a problem that generalizes the main result of [1]. Given triangle ABC with ℓ the line bisecting angle A, in [1] we asked to find the two points X_1 and X_2 that maximize and minimize $\frac{BX}{CX}$. It was proved that these two points are the incenter and excenter corresponding to angle A. It is worthwhile to notice that it is not difficult to prove a similar result where ℓ is the external bisector of A. In this case the two extremal points are the excenters which correspond to angles B and C. In this note we consider a more general problem where ℓ is an arbitrary line which does not contain the points B and C and find the two points X_1 and X_2 on ℓ which give the minimum and maximum of $\frac{BX}{CX}$.

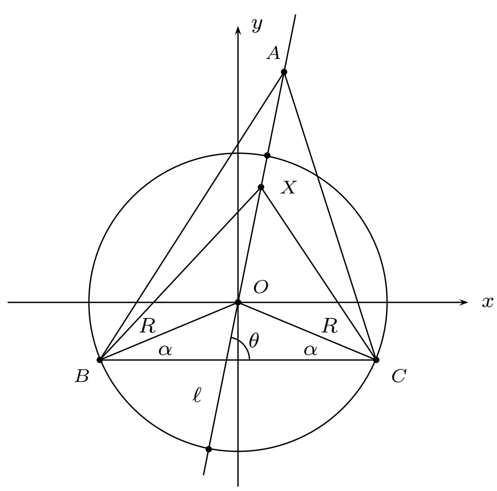

Figure 1

If ℓ is not perpendicular to BC, it intersects the perpendicular bisector of BC at a point O. Let R be the radius of the circle with center O and containing B and C. We make use of a Cartesian coordinate system with origin at O, and x-axis parallel to BC (see Figure 1). Thus, $B = R(-\cos\alpha, -\sin\alpha)$ and $C = R(\cos\alpha, -\sin\alpha)$, where $\alpha = \angle OBC = \angle OCB$.

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If the line ℓ makes an angle θ with the positive x-axis, then every point X on the line ℓ has coordinates $t(\cos\theta, \sin\theta)$ for some t. The ratio $\frac{BX}{CX}$ is a function of t. It is more convenient to consider

$$F(t) = \frac{BX^2}{CX^2} = \frac{t^2 + 2Rt\cos(\theta - \alpha) + R^2}{t^2 - 2Rt\cos(\theta + \alpha) + R^2}.$$

Differentiating with respect to t, we have

$$F'(t) = \frac{4\cos\theta\cos\alpha(R^2 - t^2)}{(t^2 - 2Rt\cos(\theta - \alpha) + R^2)^2}.$$

It is clear that $F'(t) = 0$ for $t = \pm R$. Therefore, F has two critical points which are on the circle, center O and containing B and C. In fact, $F(R) = \frac{1+\cos(\theta-\alpha)}{1-\cos(\theta+\alpha)}$ is maximum and $F(-R) = \frac{1-\cos(\theta-\alpha)}{1+\cos(\theta+\alpha)}$ is minimum.

Therefore, the points maximizing and minimizing the ratio $\frac{BX}{CX}$ are the intersections of ℓ and the circle through B and C, with center on ℓ.

If ℓ is the perpendicular through A to BC, then $\frac{BX}{CX}$ is a maximum (or minimum) at the intersection of ℓ and BC. It approaches 1 as X moves on ℓ away from BC.

Reference

[1] A. Bialostocki and D. Bialostocki, The incenter and an excenter as solutions to an extremal problem, *Forum Geom.*, 11 (2011) 9–12.

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