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Reciprocal Jacobi Triangles and the McCay Cubic

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Abstract. Given a triangle and a set of three angles, the celebrated geometrical theorem of Jacobi produces a new triangle in perspective with the first. If this second triangle is related to the first by another set of three angles then these two triangles are said to be reciprocal Jacobi triangles. It is shown that the locus of the perspector is then the McCay cubic.

1. Jacobi Triangles.

With ABC being any triangle, construct the points P, Q, R so that

$$\angle RAB = \angle QAC = \alpha,\ \angle PBC = \angle RBA = \beta \text{ and } \angle QCA = \angle PCB = \gamma.$$

These points form a *Jacobi triangle* for ABC and Jacobi's theorem states that the lines $AP,\ BQ$ and CR are concurrent (at the point K), see Figure 1. To quote [5], this result 'was seemingly discovered by Carl Friedrich Andreas Jacobi (not to be confused with Carl Gustav Jacob Jacobi), and published in 1825 in Latin'.

Many proofs of this result are available, e.g. [4] and [1, pp. 55–56], but one is given here because some results from it will be needed later.

1.1. *A Proof of Jacobi's Theorem.* With reference to Figure 1, let the lines AP and BC meet at P'. The sine rule applied to triangles BPP' and CPP' gives

$$\frac{BP'}{P'C} = \frac{\sin\gamma}{\sin\beta}\frac{\sin\angle BPP'}{\sin\angle CPP'} \tag{1}$$

and applied to triangles ABP and ACP,

$$\frac{\sin\angle BPA}{\sin\angle CPA} = \frac{c}{b}\frac{\sin(B+\beta)}{\sin(C+\gamma)} = \frac{\sin\angle BPP'}{\sin\angle CPP'}. \tag{2}$$

Ceva's theorem now implies that AP, BQ, CR are concurrent at K, say.

Furthermore, (1) and (2) give

$$\frac{\cot C + \cot\gamma}{\cot B + \cot\beta} = \frac{BP'}{P'C} = \frac{\Delta BAP'}{\Delta CAP'} = \frac{\Delta BKP'}{\Delta CKP'} = \frac{\Delta ABK}{\Delta ACK},$$

and so the relative areal coordinates (x, y, z) of K may be chosen to be

$$(x, y, z) = \left(\frac{1}{\cot A + \cot\alpha},\ \frac{1}{\cot B + \cot\beta},\ \frac{1}{\cot C + \cot\gamma}\right).$$

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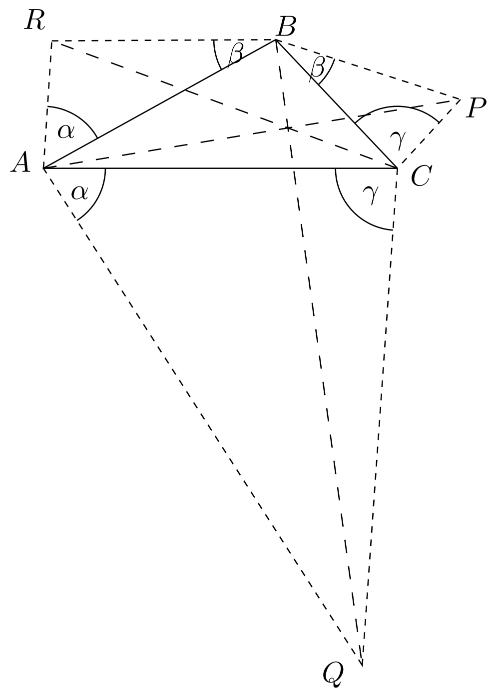

Figure 1. The points P, Q, R are constructed on a base triangle ABC with pairs of angles equal as shown. Jacobi's theorem states that AP, BQ, CR are concurrent and K will be used for the common point.

It can be seen that if $\alpha = \beta = \gamma$ then K lies on the rectangular hyperbola

$$yz(\cot B - \cot C) + zx(\cot C - \cot A) + xy(\cot A - \cot B) = 0$$

which is known as *Kiepert's hyperbola*.

2. Reciprocal Jacobi Triangles.

Given any triangle ABC and angles α, β, γ there is an associated Jacobi triangle PQR. Starting with triangle PQR and angles α', β', γ' another triangle may be constructed. If this third triangle coincides with the first then we say that ABC and PQR are *reciprocal Jacobi triangles*, see Figure 2. In this case, better notation is $A'B'C'$ rather than PQR (and A' may denote a point or an angle).

Theorem 1. *Let the triangle ABC and the angles α, β, γ in order produce the Jacobi triangle PQR and let the Jacobi triangle produced by this triangle with the angles α', β', γ' be the original triangle ABC. Then*

$$\frac{\sin(A + 2\alpha)}{\sin A} = \frac{\sin(B + 2\beta)}{\sin B} = \frac{\sin(C + 2\gamma)}{\sin C} \ (= \mu). \tag{3}$$

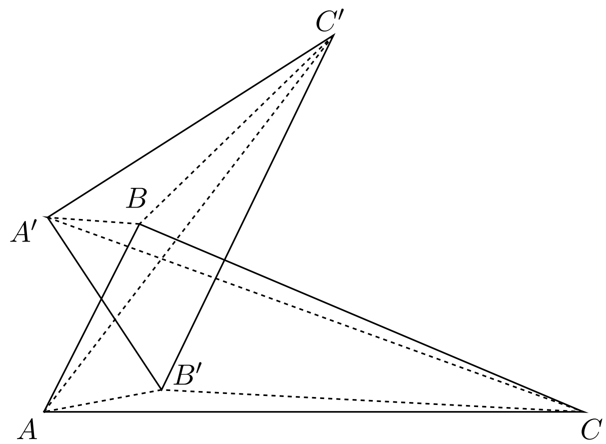

Figure 2. ABC and $A'B'C'$ are reciprocal Jacobi triangles.
There are six pairs of equal angles, e.g. $\angle BAC' = \angle CAB'$.
The lines AA', BB' and CC' are concurrent..

Proof. Since $\angle AQR = \beta'$ and $\angle ARQ = \gamma'$ we have

$$\left.\begin{array}{l}\beta' + \gamma' + 2\alpha + A = \pi \\ \gamma' + \alpha' + 2\beta + B = \pi \\ \alpha' + \beta' + 2\gamma + C = \pi\end{array}\right\} \implies (\alpha' + \beta' + \gamma') + (\alpha + \beta + \gamma) = \pi \qquad (4)$$

giving

$$\left.\begin{array}{l}\alpha' = A + \alpha - \beta - \gamma \\ \beta' = B - \alpha + \beta - \gamma \\ \gamma' = C - \alpha - \beta + \gamma\end{array}\right\} \text{ and } \left.\begin{array}{l}A' = \pi - 2A - 2\alpha + \beta + \gamma \\ B' = \pi - 2B + \alpha - 2\beta + \gamma \\ C' = \pi - 2C + \alpha + \beta - 2\gamma\end{array}\right\}.$$

Hence $(A' + \alpha') + (A + \alpha) = \pi$ and so $AQPB$ is one of many cyclic quadrilaterals in the figure. It is now readily shown that

$$\begin{array}{ll}\angle APR = \angle ACR = \frac{\pi}{2} + \beta - \alpha - A, & \angle APQ = \angle ABQ = \frac{\pi}{2} + \gamma - \alpha - A; \\ \angle BQP = \angle BAP = \frac{\pi}{2} + \gamma - \beta - B, & \angle BQR = \angle BCR = \frac{\pi}{2} + \alpha - \beta - B; \\ \angle CRQ = \angle CBQ = \frac{\pi}{2} + \alpha - \gamma - C, & \angle CRP = \angle CAP = \frac{\pi}{2} + \beta - \gamma - C.\end{array}$$

Thus

$$\begin{array}{rcl}\angle BPA & = & \angle BPR + \angle RPA \\ & = & \alpha' + (\frac{\pi}{2} + \beta - \alpha - A) \\ & = & \frac{\pi}{2} - \gamma.\end{array}$$

Likewise $\angle CPA = \frac{\pi}{2} - \beta$ and so (2) gives

$$\frac{\cos\gamma}{\cos\beta} = \frac{\sin C}{\sin B}\frac{\sin(B+\beta)}{\sin(C+\gamma)}$$

$$\implies \quad \frac{\sin(C+\gamma)\cos\gamma}{\sin C} = \frac{\sin(B+\beta)\cos\beta}{\sin B}$$

$$\implies \quad \frac{\sin(C+2\gamma)}{\sin C} = \frac{\sin(B+2\beta)}{\sin B}$$

as required. □

Although not needed here, it is stated without proof that we also have

$$\frac{\tan\alpha'}{\tan\alpha} = \frac{\tan\beta'}{\tan\beta} = \frac{\tan\gamma'}{\tan\gamma}.$$

3. The Locus of K.

For a given triangle ABC, any value of μ gives a reciprocal triangle and so the point K is parametrized by μ. Figure 3 shows a typical result for its locus and it is this locus which is now investigated.

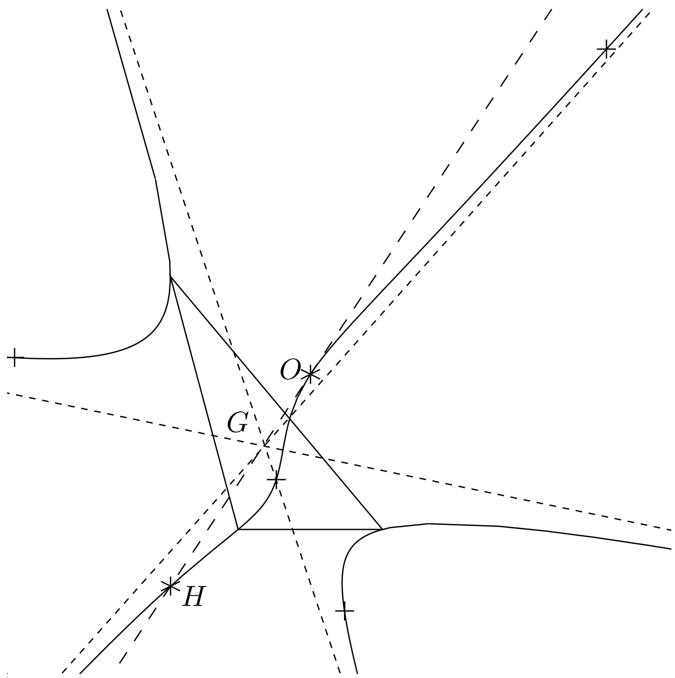

Figure 3. The locus of K (together with its asymptotes, shown as dotted lines) when α, β, γ are constrained by ABC having a reciprocal Jacobi triangle. The dashed line is the Euler line, O the circumcenter, G the centroid and H the orthocenter. Also shown (as crosses) are the incenter and ex-centers.

Using relative areal (a.k.a. barycentric) coordinates with ABC as the triangle of reference, it was shown in Section 1.1 that the coordinates of K (for any Jacobi triangle) are

$$\left(\frac{1}{\cot A + \cot\alpha}, \frac{1}{\cot B + \cot\beta}, \frac{1}{\cot C + \cot\gamma}\right).$$

Now

$$\mu = \frac{\sin(A+2\alpha)}{\sin A} = \cos 2\alpha + \cot A \sin 2\alpha \implies \cot A = \frac{\mu - \cos 2\alpha}{\sin 2\alpha}$$

and so

$$\cot A + \cot \alpha = \frac{\mu + 1}{\sin 2\alpha}.$$

Hence the coordinates of K (when there is a reciprocal Jacobi triangle) can be taken to be

$$(x, y, z) = (\sin 2\alpha, \sin 2\beta, \sin 2\gamma)$$

and it is easily seen that

$$\frac{x^2}{\sin^2 A} - 2\mu x \cot A + \mu^2 - 1 = 0,$$

from which it follows that the locus of K is

$$\frac{x^2}{\sin^2 A}(y \cot B - z \cot C) + \frac{y^2}{\sin^2 B}(z \cot C - x \cot A) + \frac{z^2}{\sin^2 C}(x \cot A - y \cot B) = 0,$$

or, equivalently,

$$a^2(-a^2 + b^2 + c^2)(c^2y^2 - b^2z^2)x + b^2(a^2 - b^2 + c^2)(a^2z^2 - c^2x^2)y + c^2(a^2 + b^2 - c^2)(b^2x^2 - a^2y^2)z = 0.$$

This cubic curve is known as the *McCay cubic* of ABC. Gibert's website [3], together with [2], gives a wealth of information regarding this and other cubic curves in triangle geometry.

References

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