Pairs of Cocentroidal Inscribed and Circumscribed Triangles

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Abstract. Let $\Delta$ be a reference triangle and $P$ a point not on the sidelines. We consider all inscribed and circumscribed triangles $\Delta'$ and $\Delta^*$ with centroid $P$ and remarkable properties as well as relationships between them.

1. Notations

Let $\Delta = ABC$ an arbitrary positively oriented triangle with sidelines $a$, $b$, $c$, centroid $G$ and area $S$. A point $P$ in the plane of $\Delta$ is described by its standardized homogeneous barycentric coordinates $u$, $v$, $w$ in reference to $\Delta$ with

$$u + v + w = 1.$$ (1)

For a triangle given by its vertices we use the matrix notation with vertex coordinates in the columns.

2. Inscribed triangles with centroid $P$

Given a fixed point $P = (u : v : w)$ not on the sidelines of $\Delta$. The reflections of the medians of $\Delta$ in $P$ intersect the respective sidelines at $A'_0, B'_0$ and $C'_0$. These points are the vertices of the inscribed (oriented) triangle $\Delta'_0$ (see Figure 1)

![Figure 1](image_url)

with matrix notation

$$\Delta'_0 = (A'_0 B'_0 C'_0) = \frac{1}{2} \begin{pmatrix} 0 & 1 - 2(w - u) & 1 + 2(u - v) \\ 1 + 2(v - w) & 0 & 1 - 2(u - v) \\ 1 - 2(v - w) & 1 + 2(w - u) & 0 \end{pmatrix}. \quad (2)$$

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By introducing the abbreviations
\begin{align*}
p_0 &:= 2(v-w), \\
q_0 &:= 2(w-u), \\
r_0 &:= 2(u-v),
\end{align*}
the representation of \( \Delta'_0 \) is simplified to
\[ \Delta'_0 = (A'_0B'_0C'_0) = \frac{1}{2} \begin{pmatrix} 0 & 1 - q_0 & 1 + r_0 \\ 1 + p_0 & 0 & 1 - r_0 \\ 1 - p_0 & 1 + q_0 & 0 \end{pmatrix}. \] (4)

**Proposition 1.** The centroid of \( \Delta'_0 \) is \( P \).

**Proof.** The row sums of (4) are barycentric coordinates of \( P \). \( \square \)

We know that \( \Delta'_0 \) is not the only inscribed triangle with centroid \( P \), but there is an infinite family \( \mathcal{D}' = \{ \Delta'(t) \mid t \in \mathbb{R} \} \) of such centroidal triangles.

If \( A' \) is an arbitrary point on the sideline \( a \), then the well-known construction of \( \Delta' \) is the following: Let \( X \) be the point on the line \( A'P \), so that \( P \) divides the line segment \( A'X \) in the ratio \( 2:1 \). Let \( Y \) be the reflection point of \( A \) in \( X \). The parallel of \( c \) through \( Y \) cuts \( b \) at \( B' \); the parallel of \( b \) through \( Y \) cuts \( c \) at \( C' \).

Now we want to choose a parametric representation of \( \Delta' \) with a simple geometrically relevant parameter \( t \). Denote by \( S'_a \) the (oriented) area of the triangle \( AA'_0A' \) (green in Figure 2).

According to (4) the second coordinate of \( A' \) is \( \frac{1}{2}(1+p_0-2S'_a) \), the third coordinate \( \frac{1}{2}(1-p_0+2S'_a) \). With
\begin{align*}
p &:= p_0 - 2t, \\
q &:= q_0 - 2t, \\
r &:= r_0 - 2t,
\end{align*}
and \( t := S'_a \), we obtain \( A' = \frac{1}{2}(0:1+p:1-p) \). Define \( B' := (1-q:0:1+q) \) and \( C' := (1+r:1-r:0) \). It is clear (see proof of Proposition 1) that the
Triangles

\[ \Delta'(t) = (A'B'C') = \frac{1}{2} \begin{pmatrix} 0 & 1-q & 1+r \\ 1+p & 0 & 1-r \\ 1-p & 1+q & 0 \end{pmatrix}, \quad t \in \mathbb{R} \quad (6) \]

have the centroid \( P \) and thus they constitute the family \( \mathcal{D}' \).

Let us now calculate the area \( S' \) of \( \Delta' \):

\[ S' = S \cdot \det \Delta' = \frac{S}{8} ((1+p)(1+q)(1+r) + (1-p)(1-q)(1-r)) \]

\[ = \frac{S}{4} (1+pq+qr+rp). \quad (7) \]

From (1), (3), (5) and the abbreviation

\[ k := 1 - 2(u^2 + v^2 + w^2) \quad (8) \]

follows \(^1\)

\[ S' = \frac{3}{4} \cdot S \cdot (k + 4t^2) = S'_0 + 3S \cdot t^2. \quad (9) \]

This leads to

**Proposition 2.** Among the triangles \( \Delta'(t) \in \mathcal{D}' \), the triangle \( \Delta'(0) = \Delta'_0 \) has minimum area.

A known special case is \( P = G \): \( \Delta'_0 \) is the medial triangle of \( \Delta \) with \( S'_0 = \frac{1}{4} \cdot S \).

3. **Circumscribed triangles with centroid \( P \)**

The above investigation of cocentroidal inscribed triangles \( \Delta' \) with centroid \( P \) naturally suggests an investigation of circumscribed triangles \( \Delta^* \) with the same centroid.

Let \( P_a, P_b, P_c \) be the traces of \( P \). The line \( P_bP_c \) cuts the sideline \( a \) at \( P'_a \). Denote the reflection point of \( P'_a \) in \( a \) at \( P'_a \) by \( P^*_a \) and the line \( AP^*_a \) by \( a^*_a \). The lines \( b^*_a, c^*_a \) can be constructed similarly. These lines form a triangle \( \Delta^*_0 \) with vertices \( A^*_0, B^*_0, C^*_0 \) (see Figure 3).

From this construction it is easy to calculate the standardized barycentric coordinates of the vertices of \( \Delta^*_0 \):

\[ \Delta^*_0 = (A^*_0B^*_0C^*_0) \]

\[ = \frac{1}{k} \begin{pmatrix} (1+q_0)(1-r_0)u & (1-r_0)(1-p_0)u & (1+p_0)(1+q_0)u \\ (1+q_0)(1+r_0)v & -(1+r_0)(1-p_0)v & (1-p_0)(1-q_0)v \\ (1-q_0)(1-r_0)w & (1+r_0)(1+p_0)w & -(1+p_0)(1-q_0)w \end{pmatrix}. \quad (10) \]

**Proposition 3.** \( P \) is the centroid of \( \Delta^*_0 \).

**Proof.** The row sums of (10) are barycentric coordinates of \( P \).

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\(^1\)k is zero, positive, or negative according as \( P \) lies on, inside or outside the Steiner in-ellipse.
In a similar fashion as in §2 (from (4) to (6)) we define from (10) for each \( t \in \mathbb{R} \) the triangle
\[
\Delta^*(t) = (A^*B^*C^*)
\]
\[
= \frac{1}{k + 4t^2} \begin{pmatrix}
-(1 + q)(1 - r)u & (1 - r)(1 - p)u & (1 + p)(1 + q)u \\
(1 + q)(1 + r)v & -(1 + r)(1 - p)v & (1 - p)(1 - q)v \\
(1 - q)(1 - r)w & (1 + r)(1 + p)w & -(1 + p)(1 - q)w
\end{pmatrix}.
\]

(11)

It is not difficult to prove that \( \Delta^*(t) \) is a circumscribed triangle with centroid \( P \) for all \( t \). Thus the triangles \( \Delta^*(t) \) constitute the family \( \mathcal{D}^* \) of cocentroidal circumscribed triangles with centroid \( P \).

3.1. Construction of \( \Delta^* \). Given \( a^* \) as an arbitrary line (sideline of \( \Delta^*(t) \) for a certain \( t \)) through \( A \), we are able to construct \( b^* \) and \( c^* \):

Let \( T \) be the point on \( AP \), so that \( P \) divides the line segment \( AT \) in the ratio 1:2. Construct \( a_T^* \) the parallel to \( a^* \) through \( T \), and \( a_B^*, a_C^* \) parallels to \( a^* \) through \( B, C \) respectively. The line \( TB \) cuts \( a_C^* \) at \( D_B \), and \( TC \) cuts \( a_B^* \) at \( D_C \). The intersection of \( D_BD_C \) with \( a \) is \( X \). Then \( PX \) and \( a_T^* \) intersects at the required point \( A^* \). The line \( A^*C \) cuts \( a^* \) at \( A^*B \) and \( a^* \) intersects at \( C^* \) (see Figure 4).

From (11) we determine the area \( S^* \) of \( \Delta^* \):
\[
S^* = \frac{S}{(k + 4t^2)^2} + (1 + p)(1 + q)(1 + r) + (1 - p)(1 - q)(1 - r))
\]
\[
= 36 \cdot \frac{wvw}{k + 4t^2} S^*.
\]

(12)

From this follows for \( P \) inside the Steiner in-ellipse

**Proposition 4.** Among the triangles \( \Delta^*(t) \in \mathcal{D}^* \), the triangle \( \Delta^*(0) = \Delta_0^* \) has maximum area.
Special case \( P = G \): \( \Delta_0^* \) is the antimedial (anticomplementary) triangle of \( \Delta \) with \( S_0^* = 4S \).

4. Cocentroidal pairs \((\Delta', \Delta^*)\)

The structure of the coordinates of \( A_0^*, B_0^*, C_0^* \) shows that they are the barycentric products (symbol \( \ast_b \)) of \( P \) and the wedge (symbol \( \wedge \)) of two vertices of \( \Delta_0' \), for instance

\[
A_0^* = (B_0' \wedge C_0')^T \ast_b P,
\]

similarly \( B_0^* \) and \( C_0' \) (see [1]). So it is clear that \( \Delta_0^* \) is the unary cofactor triangle with respect to \( P \) of \( \Delta_0' \). We recall ([1], [2]) that the isoconjugate of a point \( U = (l : m : n) \) with respect to pole \( P = (u : v : w) \) is the point \( U^* = (u : \frac{m}{u} : \frac{n}{w}) \).

The unary cofactor triangle of triangle \( T = T_1T_2T_3 \) is the triangle \( U_P(T) =: X = X_1X_2X_3 \) whose vertices \( X_i \) are the isoconjugates of the vertices of the line-polar triangle of the points \( T_i = (\alpha_i : \beta_i : \gamma_i) \), that is

\[
X_i = (T_{i+1} \wedge T_{i+2}) \ast_b P \tag{13}
\]

(subscripts are taken modulo 3). In matrix notation,

\[
U_P(T) = (X_1X_2X_3)
\]

\[
= \begin{pmatrix}
(\beta_2\gamma_3 - \beta_3\gamma_2)u & (\beta_3\gamma_1 - \beta_1\gamma_3)u & (\beta_1\gamma_2 - \beta_2\gamma_1)u \\
(\gamma_2\alpha_3 - \gamma_3\alpha_2)v & (\gamma_3\alpha_1 - \gamma_1\alpha_3)v & (\gamma_1\alpha_2 - \gamma_2\alpha_1)v \\
(\alpha_2\beta_3 - \alpha_3\beta_2)w & (\alpha_3\beta_1 - \alpha_1\beta_3)w & (\alpha_1\beta_2 - \alpha_2\beta_1)w
\end{pmatrix} \tag{14}
\]

This has the following simple properties.

1. \( U_P(U_P(T)) = T \).
2. \( T \) is an inscribed triangle if and only if \( U_P(T) \) is a circumscribed triangle.
3. If \( P \) is the centroid of \( T \), then \( U_P(T) \) has the same centroid as \( T \).

It is not difficult to see that the triangle (11) is the unary cofactor triangle with respect to \( P \) of triangle (6) with centroid \( P \).
If we want to form “natural” pairs \((\Delta', \Delta^*)\) of inscribed and circumscribed triangles with the same centroid \(P\), then the obvious choice is \(\Delta^* = U_P(\Delta')\).

The elimination of \(t\) in (9) and (12) leads to:

**Proposition 5.** The product \(S'(t) \cdot S^*(t) = 27 \cdot uvw \cdot S^2\) is independent on \(t\) for all \(t \in \mathbb{R}\).

**References**


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