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Pairs of Cocentroidal Inscribed and Circumscribed Triangles

Gotthard Weise

Abstract. Let Δ be a reference triangle and P a point not on the sidelines. We consider all inscribed and circumscribed triangles Δ' and Δ^* with centroid P and remarkable properties as well as relationships between them.

1. Notations

Let $\Delta = ABC$ an arbitrary positively oriented triangle with sidelines $a,\ b,\ c$, centroid G and area S. A point P in the plane of Δ is described by its standardized homogeneous barycentric coordinates u, v, w in reference to Δ with

$$u + v + w = 1. \tag{1}$$

For a triangle given by its vertices we use the matrix notation with vertex coordinates in the columns.

2. Inscribed triangles with centroid P

Given a fixed point $P = (u : v : w)$ not on the sidelines of Δ. The reflections of the medians of Δ in P intersect the respective sidelines at A_0', B_0' and C_0'. These points are the vertices of the inscribed (oriented) triangle Δ_0' (see Figure 1)

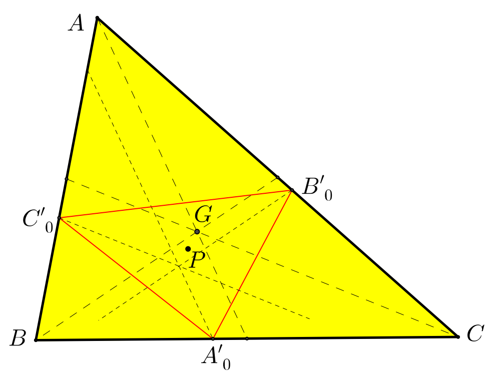

Figure 1.

with matrix notation

$$\Delta_0' = (A_0'B_0'C_0') = \frac{1}{2}\begin{pmatrix} 0 & 1-2(w-u) & 1+2(u-v) \\ 1+2(v-w) & 0 & 1-2(u-v) \\ 1-2(v-w) & 1+2(w-u) & 0 \end{pmatrix}. \tag{2}$$

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By introducing the abbreviations

$$\begin{aligned} p_0 &:= 2(v-w), \\ q_0 &:= 2(w-u), \\ r_0 &:= 2(u-v), \end{aligned} \tag{3}$$

the representation of Δ'_0 is simplified to

$$\Delta'_0 = (A'_0 B'_0 C'_0) = \frac{1}{2}\begin{pmatrix} 0 & 1-q_0 & 1+r_0 \\ 1+p_0 & 0 & 1-r_0 \\ 1-p_0 & 1+q_0 & 0 \end{pmatrix}. \tag{4}$$

Proposition 1. *The centroid of Δ'_0 is P.*

Proof. The row sums of (4) are barycentric coordinates of P. □

We know that Δ'_0 is not the only inscribed triangle with centroid P, but there is an infinite family $\mathcal{D}' = \{\Delta'(t) | \, t \in \mathbb{R}\}$ of such *cocentroidal* triangles.

If A' is an arbitrary point on the sideline a, then the well-known construction of Δ' is the following: Let X be the point on the line $A'P$, so that P divides the line segment $A'X$ in the ratio $2:1$. Let Y be the reflection point of A in X. The parallel of c through Y cuts b at B'; the parallel of b through Y cuts c at C'.

Now we want to choose a parametric representation of Δ' with a simple geometrically relevant parameter t. Denote by S'_a the (oriented) area of the triangle AA'_0A' (green in Figure 2).

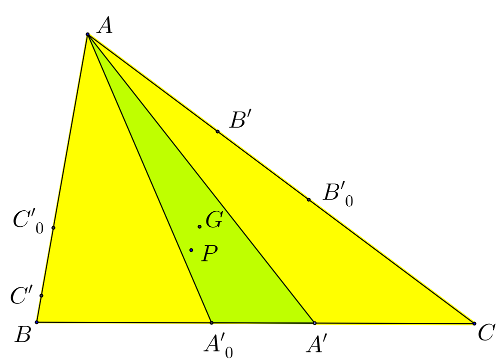

Figure 2.

According to (4) the second coordinate of A' is $\frac{1}{2}(1+p_0-2S'_a)$, the third coordinate $\frac{1}{2}(1-p_0+2S'_a)$. With

$$\begin{aligned} p &:= p_0 - 2t, \\ q &:= q_0 - 2t, \\ r &:= r_0 - 2t, \end{aligned} \tag{5}$$

and $t := S'_a$, we obtain $A' = \frac{1}{2}(0 : 1+p : 1-p)$. Define $B' := (1-q : 0 : 1+q)$ and $C' := (1+r : 1-r : 0)$. It is clear (see proof of Proposition 1) that the triangles

$$\Delta'(t) = (A'B'C') = \frac{1}{2}\begin{pmatrix} 0 & 1-q & 1+r \\ 1+p & 0 & 1-r \\ 1-p & 1+q & 0 \end{pmatrix},\ t \in \mathbb{R} \tag{6}$$

have the centroid P and thus they constitute the family $\mathcal{D}'$.

Let us now calculate the area S' of Δ':

$$\begin{aligned} S' = &\ S \cdot \det \Delta' = \frac{S}{8}\left((1+p)(1+q)(1+r) + (1-p)(1-q)(1-r)\right) \\ = &\ \frac{S}{4}(1 + pq + qr + rp). \end{aligned} \tag{7}$$

From (1), (3), (5) and the abbreviation

$$k := 1 - 2(u^2 + v^2 + w^2) \tag{8}$$

follows [1]

$$S' = \frac{3}{4} \cdot S \cdot (k + 4t^2) = S'_0 + 3S \cdot t^2. \tag{9}$$

This leads to

Proposition 2. *Among the triangles $\Delta'(t) \in \mathcal{D}'$, the triangle $\Delta'(0) = \Delta'_0$ has minimum area.*

A known special case is $P = G$: Δ'_0 is the medial triangle of Δ with $S'_0 = \frac{1}{4} \cdot S$.

3. Circumscribed triangles with centroid P

The above investigation of cocentroidal *inscribed* triangles Δ' with centroid P naturally suggests an investigation of *circumscribed* triangles Δ^* with the same centroid.

Let P_a, P_b, P_c be the traces of P. The line P_bP_c cuts the sideline a at P'_a. Denote the reflection point of P'_a in P by P^*_a and the line AP^*_a by a^*_0. The lines b^*_0, c^*_0 can be constructed similarly. These lines form a triangle Δ^*_0 with vertices A^*_0, B^*_0, C^*_0 (see Figure 3).

From this construction it is easy to calculate the standardized barycentric coordinates of the vertices of Δ^*_0:

$$\begin{aligned} \Delta^*_0 = &\ (A^*_0 B^*_0 C^*_0) \\ = &\ \frac{1}{k}\begin{pmatrix} -(1+q_0)(1-r_0)u & (1-r_0)(1-p_0)u & (1+p_0)(1+q_0)u \\ (1+q_0)(1+r_0)v & -(1+r_0)(1-p_0)v & (1-p_0)(1-q_0)v \\ (1-q_0)(1-r_0)w & (1+r_0)(1+p_0)w & -(1+p_0)(1-q_0)w \end{pmatrix}. \end{aligned} \tag{10}$$

Proposition 3. *P is the centroid of Δ^*_0.*

Proof. The row sums of (10) are barycentric coordinates of P. □

[1] k is zero, positive, or negative according as P lies on, inside or outside the Steiner in-ellipse.

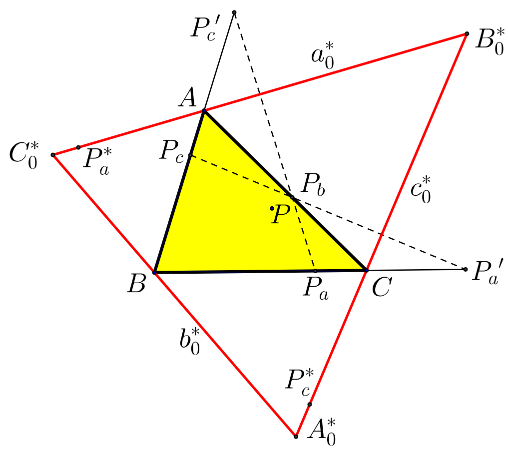

Figure 3.

In a similar fashion as in §2 (from (4) to (6)) we define from (10) for each $t \in \mathbb{R}$ the triangle

$$\Delta^*(t) = (A^*B^*C^*) = \frac{1}{k+4t^2}\begin{pmatrix} -(1+q)(1-r)u & (1-r)(1-p)u & (1+p)(1+q)u \\ (1+q)(1+r)v & -(1+r)(1-p)v & (1-p)(1-q)v \\ (1-q)(1-r)w & (1+r)(1+p)w & -(1+p)(1-q)w \end{pmatrix}. \tag{11}$$

It is not difficult to prove that $\Delta^*(t)$ is a circumscribed triangle with centroid P for all t. Thus the triangles $\Delta^*(t)$ constitute the family $\mathcal{D}^*$ of cocentroidal circumscribed triangles with centroid P.

3.1. *Construction of* Δ^*. Given a^* as an arbitrary line (sideline of $\Delta^*(t)$ for a certain t) through A, we are able to construct b^* and c^* :
Let T be the point on AP, so that P divides the line segment AT in the ratio $1:2$. Construct a_T^* the parallel to a^* through T, and a_B^*, a_C^* parallels to a^* through B, C respectively. The line TB cuts a_C^* at D_B, and TC cuts a_B^* at D_C. The intersection of D_BD_C with a is X. Then PX and a_T^* intersects at the required point A^*. The line A^*C cuts a^* at B^*, A^*B and a^* intersects at C^* (see Figure 4).

From (11) we determine the area S^* of Δ^*:

$$\begin{aligned} S^* &= \frac{S}{(k+4t^2)^3}uvw\left((1+p)(1+q)(1+r)+(1-p)(1-q)(1-r)\right)^2 \\ &= \frac{36\cdot uvw}{k+4t^2}S. \end{aligned} \tag{12}$$

From this follows for P inside the Steiner in-ellipse

Proposition 4. *Among the triangles $\Delta^*(t) \in \mathcal{D}^*$, the triangle $\Delta^*(0) = \Delta_0^*$ has maximum area.*

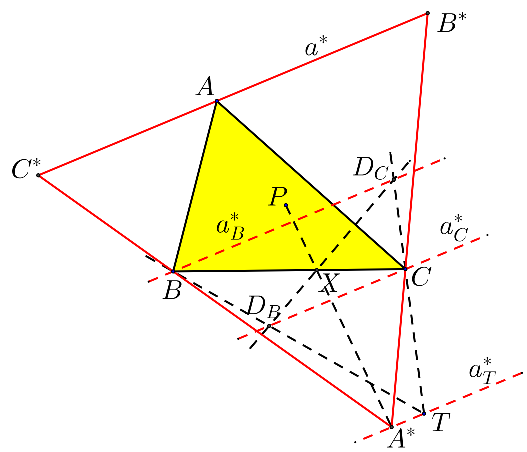

Figure 4.

Special case $P = G$: Δ_0^* is the antimedial (anticomlementary) triangle of Δ with $S_0^* = 4S$.

4. Cocentroidal pairs (Δ', Δ^*)

The structure of the coordinates of A_0^*, B_0^*, C_0^* shows that they are the barycentric products (symbol $*_b$) of P and the wedge (symbol $\wedge$) of two vertices of Δ_0', for instance

$$A_0^* = (B_0' \wedge C_0')^T *_b P,$$

similarly B_0^* and C_0' (see [1]). So it is clear that Δ_0^* is the *unary cofactor triangle* with respect to P of Δ_0'. We recall ([1], [2]) that the isoconjugate of a point $U = (l : m : n)$ with respect to pole $P = (u : v : w)$ is the point $U_P^\bullet = (\frac{u}{l} : \frac{v}{m} : \frac{w}{n})$. The unary cofactor triangle of triangle $T = T_1T_2T_3$ is the triangle $\mathbb{U}_P(T) =: X = X_1X_2X_3$ whose vertices X_i are the isoconjugates of the vertices of the line-polar triangle of the points $T_i = (\alpha_i : \beta_i : \gamma_i)$, that is

$$X_i = (T_{i+1} \wedge T_{i+2}) *_b P \tag{13}$$

(subscripts are taken modulo 3). In matrix notation,

$$\begin{aligned}\mathbb{U}_P(T) =\ & (X_1X_2X_3) \\ =\ & \begin{pmatrix} (\beta_2\gamma_3 - \beta_3\gamma_2)u & (\beta_3\gamma_1 - \beta_1\gamma_3)u & (\beta_1\gamma_2 - \beta_2\gamma_1)u \\ (\gamma_2\alpha_3 - \gamma_3\alpha_2)v & (\gamma_3\alpha_1 - \gamma_1\alpha_3)v & (\gamma_1\alpha_2 - \gamma_2\alpha_1)v \\ (\alpha_2\beta_3 - \alpha_3\beta_2)w & (\alpha_3\beta_1 - \alpha_1\beta_3)w & (\alpha_1\beta_2 - \alpha_2\beta_1)w \end{pmatrix}.\end{aligned} \tag{14}$$

This has the following simple properties.
(1) $\mathbb{U}_P(\mathbb{U}_P(T)) = T$.
(2) T is an inscribed triangle if and only if $\mathbb{U}_P(T)$ is a circumscribed triangle.
(3) If P is the centroid of T, then $\mathbb{U}_P(T)$ has the same centroid as T.

It is not difficult to see that the triangle (11) is the unary cofactor triangle with respect to P of triangle (6) with centroid P.

If we want to form "natural" pairs (Δ', Δ^*) of inscribed and circumscribed triangles with the same centroid P, then the obvious choice is $\Delta^* = \mathbb{U}_P(\Delta')$.

The elimination of t in (9) and (12) leads to:

Proposition 5. *The product $S'(t) \cdot S^*(t) = 27 \cdot uvw \cdot S^2$ is independent on t for all $t \in \mathbb{R}$.*

References

[1] P. L. Douillet, *Translation of the Kimberling's Glossary into Barycentrics*, www.douillet.info/∼douillet/triangle/glossary/glossary.pdf.

[2] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium* 129 (1998) 1–295.

Gotthard Weise: Buchloer Str. 23, D-81475 München, Germany.
E-mail address: gotthard.weise@tele2.de